ON CURVATURE INTEGRALS AND KNOTS

RÉMI LANGEVIN and HAROLD ROSENBERG

(Received 3 July 1975)

The total curvature of a curve is the integral of the absolute curvature of the curve. I. Fary and J. Milnor proved the total curvature of a knot in $R^3$ is at least $4\pi$ [3, 8]; the total curvature of a standard embedding of $S^1$ is $2\pi$. Many generalisations of this theorem now exist [2, 4, 5, 9], a good survey is [2]. We pursue this study here.

Let $C$ be a knot in $R^3$ which bounds an orientable surface $M$ in $R^3$. Consider functions on $M$ of the form $p(z): x \rightarrow z - x$ (the scalar product of $z$ with $x$, where $z$ is a point of $S^2$). For almost all $z$ of $S^2$ (in the sense of Lebesgue measure), $p(z)$ is a Morse function on $M$. Denote by

- $\mu(z, M)$ the number of critical points of $p(z)/M$,
- $\mu(z, M, C)$ the number of critical points of $p(z)/C + \mu(z, M)$,
- $\mu(M, C)$ the minimum of $\mu(z, M, C)$ for $z$ a point of $S^2$.

We conjectured that:

$$\mu(M, C) \geq 2g + 3,$$

where $g$ is the genus of $M$; that is, the fact $C$ is knotted implies the presence of more critical points than is required by the homology structure of $(M, C)$. This inequality was proved by Jacques Debarbieux. John Milnor proposed we relate this to curvature; the result is:

$$2\int_C |k| \, ds + \int_M |K| \, d\Lambda \geq 2\pi(2g + 3),$$

where $k$ is the curvature of the knot $C$ and $K$ the Gaussian curvature of $M$. We shall prove this inequality when $M$ is immersed in $R^3$ and $\partial M = C$ an embedded knot. We also obtain other relations for $(M, C)$ of a similar nature.

Consider smooth embedding of the torus $S^1 \times S^1$ in $R^3$. If $T$ is the image of the standard embedding (as a surface of revolution), then:

$$\frac{1}{2\pi} \int_T |K| \, d\Lambda = 4.$$ 

Now suppose $T$ is a knotted 2-torus in $R^3$ (one of the two components of $S^3 - T$ is not a solid torus). Then we shall prove:

$$\frac{1}{2\pi} \int_T |K| \, d\Lambda \geq 8. \tag{1.3}$$

Finally we discuss generalisations of these results to higher dimensions.

### II. MORSE THEORY OF $(M, \partial M)$

Let $M$ be a compact orientable $n$ manifold with $\partial M$ homeomorphic to $S^{n-1}$. Let $f: M \rightarrow R^{n+1}$ be an immersion. For each $z$ of $S^{n-1}$, let $p(z): M \rightarrow R$ be the map $p(z): x \rightarrow z - f(x)$ the scalar product of $z$ and $f(x)$. We define $\mu(z, A)$ to be the number of critical points of $p(z)$ in $A \subset M$. If $p(z)$ is a Morse function on $A$ (which is true for almost all $z \in S^{n-1}$), we define $\mu_i(z, A)$ to be the number of critical points of index $i$ of $p(z)$ in $A$. By $\mu(z, \partial M)$ we mean the number of critical points of the function $p(z)$ restricted to $\partial M$. Let $\beta_i$ be the $i$th real Betti number of $D(M)$, the double of $M$. 

405
**Theorem 1.1.** Let \( x \) be any point of \( S^{n-1} \) such that \( p(z) \) is a Morse function on \((M, \partial M)\). If

\[
\mu(z, \text{int } M) + \mu(z, \partial M) = 2 + \sum_{i=1}^{n-1} \nu_i
\]

then \( p(Z)/\partial M \) has exactly two critical points.

**Proof.** Let \( \beta_i = \mu_i(z, \text{int } M) \) and \( \gamma_i = \mu_i(z, \partial M) \).

There is a natural way to define a Morse function \( g \) on \( D(M) \), the double of \( M \), using the function \( p(z) \) on \( M \), cf. Fig. 1. A critical point of index \( j \) of \( p(z) \) on \( M \) becomes a critical point of index \( j \) or \( j + 1 \) of \( g \) on \( D(M) \). Let \( \gamma_j \) be the number of critical points of index \( j \) on \( D(M) \) obtained from critical points on \( \partial M \). Clearly:

\[
\begin{align*}
(a) & \quad \sum_{j=0}^{n-1} \beta_j = \sum_{j=0}^{n-1} \gamma_j \\
(b) & \quad 2 \sum_{j=0}^{n-1} (-1)^j \beta_j + \sum_{j=0}^{n-1} (-1)^j \gamma_j = 2\chi(M) - \chi(\partial M) = 2\epsilon + \sum_{i=1}^{n-1} (-1)^i 2\nu_i \\
(c) & \quad \sum_{j=0}^{n-1} \beta_j + \sum_{j=0}^{n-1} \gamma_j = 2 + \sum_{i=1}^{n-1} \nu_i,
\end{align*}
\]

where \( \epsilon = 1 \) when \( n \) is even, \( \epsilon = 0 \) when \( n \) is odd. By the Morse inequalities for \( g \) on \( D(M) \), we know that, for \( 1 \leq i \leq n - 1 \), \( 2\beta_i + \gamma_i \geq 2\nu_i \). Also there are necessarily a maximum and a minimum which may be on \( \text{int } M \) or \( \partial M \), so \( \beta_0 + \gamma_0 \geq 1 \) and \( \beta_n + \gamma_n \geq 1 \). From (a), (b) and (c), we obtain:

\[
(d) \quad 4(\beta_0 + \beta_1 + \cdots + \beta_n) + 2(\gamma_0 + \gamma_n) + \sum \gamma_i = 4 + 2\epsilon + 4(\gamma_2 + \gamma_4 + \cdots + \gamma_n),
\]

where, if \( n = 2j \) then \( \epsilon = 1 \) and \( k = n - 2 \) and if \( n = 2j + 1 \) then \( \epsilon = 0 \) and \( k = n - 1 \). We can rewrite the left side of (d) as:

\[
4(\beta_0 + \epsilon \beta_n) + 2(\gamma_0 + \epsilon \gamma_n) + \sum \gamma_i = 4 + 2\epsilon + 4(\gamma_2 + \cdots + \gamma_n)
\]

From the inequalities \( 2\beta_i + \gamma_i \geq 2\nu_i \) for \( 1 \leq i \leq n - 1 \), we have

\[
4(\beta_0 + \epsilon \beta_n) + 2(\gamma_0 + \epsilon \gamma_n) + \epsilon \gamma_i \leq 4 + 2\epsilon.
\]

Since \( \beta_0 + \gamma_0 \geq 1 \) and \( \beta_n + \gamma_n \geq 1 \) we conclude \( \Sigma \gamma_i \leq 2 \), hence \( \gamma_0 = \gamma_n = 1 \); \( \gamma_i = 0 \) \( 1 \leq i \leq n - 1 \).

**Remark.** We did not really use the fact that \( \partial M \) is a sphere. If a closed manifold admits a Morse function with exactly two critical points then it is homeomorphic to a sphere. Hence if \((M, \partial M)\) admits a Morse function with the total number of critical points on \( M \) and \( \partial M \) equal to \( 2 + \sum_{i=1}^{n-1} \nu_i \), then \( \partial M \) is a sphere and the number of critical points on \( \partial M \) is two.

When \( M \) is a 2-manifold we can say somewhat more.

**Theorem 1.2.** Let \( \Delta_1 = \beta_1 + \gamma_1 - \nu_1 \). Then \( \Delta_1 \geq 0 \) and if \( \Delta_1 = 0 \), the total number of critical points on \( \partial M \) is two.

**Proof.** From the second Morse inequality for \( D(M) \) we know that \( \Delta_1 \geq 0 \). If \( \Delta_1 = 0 \) then \( 2(\beta_0 + \beta_1) - 2\beta_1 + (\gamma_0 + \gamma_1) - \gamma_1 = 2 - 2\nu_1 \) and hence: \( 2(\beta_0 + \beta_1) + (\gamma_0 + \gamma_1 + \gamma_n) = 2 \). Since \( \gamma_0 + \gamma_1 + \gamma_2 = \gamma_0 + \gamma_1 \geq 2 \), we have \( \gamma_0 + \gamma_1 = 2 \).

![Fig. 1. Thickening.](image-url)
ON CURVATURE INTEGRALS AND KNOTS

42. TOTAL CURVATURE OF CURVES AND SURFACES IN \( R^3 \)

Let \( f: S^1 \to R^3 \) be a smooth embedding, the total curvature of \( f \) is defined by:

\[
\tau(f) = \frac{1}{2\pi} \int_{S^1} |k(s)| \, ds,
\]

where \( s \) denotes the arc length parameter and \( k \) is the curvature of the curve. Milnor has shown that:

\[
4\tau(f) = \frac{1}{2\pi} \int_{S^1} \mu(z, f(S^1)) \, ds.
\]

We will derive this relation by a method different from Milnor’s.

Let \( M \) be an orientable 2-manifold in \( R^3 \) with boundary \( C = f(S^1) \). Let \( N(s) \) be the principal normal to \( C \) at \( s \) and let \( e(s) \) be the unit vector in the normal plane to \( C \) at \( s \) which is tangent to \( M \) at \( s \) and pointing into \( M \). Denote by \( \psi \) the angle between \( N(s) \) and \( e(s) \), \( 0 \leq \psi \leq \pi \).

The function \( p(z) \) on \( C \) has a critical point at \( x \in C \), if and only if \( z \) is in the normal plane to \( C \) at \( x \). For \( z \) in this normal plane at \( x \), the function \( p(z) \) has a real or false extremum at \( x \), considered as a function on \( M \). It is clear that if \( z \cdot N(s) \) and \( z \cdot e(s) \) are of the same sign then it is a real extremum, and otherwise it is a false extremum. To measure the subset of \( S^2 \) giving rise to real extremum on a segment \( ds \) of \( C \), we may replace \( C \) by its osculating circle along \( ds \) and take \( \psi \) to be constant. Then the area of the region swept out on \( S^2 \) is easily seen to be:

\[
2|k(s)| \, ds(1 + \cos \psi)
\]

Similarly the vectors \( z \) giving rise to a false extremum along \( ds \) have measure

\[
2|k(s)| \, ds(1 - \cos \psi)
\]

Let \( \tilde{\mu}_i(f, z) \) denote the number of critical points of index \( i \) of the function \( g(z) \), obtained from \( p(z) / dM \), on the double of \( M \). Since the set of \( z \in S^2 \) such that \( p(z) \) has a degenerate critical point is of measure 0, we have

\[
\int_{S^2} \tilde{\mu}_0 + \tilde{\mu}_2 = 2 \int_C |k|(1 + \cos \psi) \, ds,
\]

\[
\int_{S^2} \tilde{\mu}_1 = 2 \int_C |k|(1 - \cos \psi) \, ds.
\]

Since \( \mu(f, z) = \tilde{\mu}_0(f, z) + \tilde{\mu}_1(f, z) + \tilde{\mu}_2(f, z) \), we obtain Milnor’s result upon adding 2.2 and 2.3.

The relation 2.1 has been generalised to submanifolds of \( R^n \) by Chern–Lashof and Kuiper [2,5]. Let \( f: \Sigma \to R^3 \) be an immersion as in §1, with \( f \vert \Sigma \) an embedding and \( \Sigma \) of genus \( g \). We have

\[
\int_{\Sigma} \mu(z, \text{int } M) \, d\sigma = 2\int_M |K| \, dA,
\]

where \( K \) is the Gaussian curvature of \( f(\Sigma) = M \) and \( dA. \, d\sigma \) are the standard volume forms on \( M \) and \( S^2 \) respectively. One can see this as follows. Let \( N \) be a unit normal vector field along \( M \) and \( \gamma: M \to S^2 \) the Gauss map, \( \gamma(x) = N(x) \). Then for \( U \) a sufficiently small open set on \( M \) where \( K \neq 0 \), we have:

\[
\int_{\gamma(U)} d\sigma = 2 \int_U |K| \, dA.
\]

The integral on the left side is taken over all vectors \( z \in S^2 \) such that \( p(z) \) has a critical point in \( U \). If \( U \) is sufficiently small, then \( p(z) \) has exactly one critical point in \( U \), which is of index 0 or 2 if \( K > 0 \) and of index 1 if \( K < 0 \). Therefore

\[
\int_{S^2} \mu_0(z, f \vert U) - \mu_1(z, f \vert U) + \mu_2(z, f \vert U) = 2 \int_U |K| \, dA
\]

and

\[
\int_{S^2} \mu_1(z, f \vert U) = 2 \int_U |K| \, dA
\]

The critical points of \( \gamma \) are the points where the Gaussian curvature vanishes. The measure of the image by \( \gamma \) of the critical points is then zero by Sard’s theorem. Thus

\[
\int_{S^2 \setminus \{\text{critical points}\}} \mu_1(z, f \vert \text{int } M) = \int_{S^2} \mu_1(z, f \vert \text{int } M).
\]
The set $\gamma^{-1}[\text{critical points}]$ is composed

1. of points where $K = 0$,
2. of a set of measure zero, because all points of this set are regular points of $\gamma$ and the image of this set by $\gamma$ is by definition contained in a set of measure zero. Also the set of normals to $M$ along $\partial M$ is of measure 0, so we obtain:

$$\int_{S^1} \mu(z, f/\text{int } M) = 2 \int_M |K|.$$

(2.8)

Now by Theorem 1.1 and equations (2.1) and (2.8) we conclude:

**Theorem 2.9.** Let $C$ be a circle embedded in $R^3$ which bounds an orientable immersed surface $M \subset R^3$ of genus $g$. Then $2 \int_C |K| \, ds + \int_M |K| \, dA \geq 2\pi(2g + 2)$. Moreover, if $C$ is knotted then

$$2 \int_C |K| \, ds + \int_M |K| \, dA \geq 2\pi(2g + 3).$$

We can say more by using Theorem 1.2, 2.3:

$$\int_C |k| (1 - \cos \psi) \, ds - \int_M K^+ \, dA \geq 4\pi g.$$  \hspace{1cm} (2.9)

Moreover, if $C$ is knotted, then:

$$\int_C |k| (1 - \cos \psi) \, ds - \int_M K^- \, dA \geq 4\pi g + 2\pi$$

($K^+$ and $K^-$ are the positive and negative parts of the Gaussian curvature, $2g$ is the genus of $D(M)$, the double of $(M, \partial M)$).

**Remark.** By using the Gauss–Bonnet theorem and considerations as above, the reader can prove: $4 \int_C k_\ast \, ds = \int_{S^2} (\xi(x) - \eta(x)) \, d\sigma$ where $k_\ast$ is the geodesic curvature of $C$, $\xi$ and $\eta$ are the number of real and false extremum of $p(z)$ on $C$.

### §3. **KNOTTED TORI IN $R^3$**

Let $f : T^2 \to R^3$ be a smooth embedding and $T = f(T^2)$. We consider $R^3 \subset S^3$ in the standard fashion. It is well known that one of the two connected components of $S^3 - T$ is a solid torus $D^2 \times S^1$. If both components are solid tori then $T$ is isotopic to the standard torus in $R^3$. Therefore we may define a knotted torus in $D^3$ to be a torus which bounds a submanifold of $S^3$ different from $D^2 \times S^1$. An equivalent definition is that the compact component of $R^3 - T$ is either not $D^2 \times S^1$ or if it is a solid torus then its core, $\{0\} \times S^1$, is knotted in $R^3$.†

**Theorem 3.1.** Let $T$ be a torus embedded in $R^3$. Then

$$\frac{1}{2\pi} \int_T |K| \, dA \geq 4.$$  \hspace{1cm} (3.1)

Moreover, if $T$ is knotted then:

$$\frac{1}{2\pi} \int_T |K| \, dA \geq 8.$$  \hspace{1cm} (3.2)

**Proof.** We know that $2 \int_C |K| \, dA = \int_{S^2} \mu(z) \, d\sigma$, where $\mu(\tau)$ is the number of critical points on $T$ of the projection of $T$ along the line through $z$. By elementary Morse theory, we know that $\mu(z) \geq 4$ for almost all $z \in S^2$, hence the first inequality is proved. To prove the second inequality, it suffices to show that if for some $z \in S^2$ with $p(z) : T \to R$ a Morse function, $\mu(z) < 8$, then $T$ is unknotted. Clearly $\mu(z)$ is either 4 or 6 and we can suppose $\mu(z) = 6$.

Without loss of generality we can suppose the projection of $T$ onto the $z$-axis of $R^3$ has exactly six non degenerate critical points $p(i), 1 \leq i \leq 6$, and $z(p(i)) = i$. Let $P(i)$ be the horizontal hyperplane defined by $z = i + \frac{1}{6}$. We have $P(i) \cap T$ a finite union of $n(i)$ circles (smooth submanifolds of $T$) with $N(1) = n(5) = 1$. There are two possibilities for the sequence $n(i):(1, 2, 1, 2, 1)$ and $(1, 2, 3, 2, 1)$.

†We are grateful to John Morgan for greatly simplifying our proof of the following theorem.
First we consider $(1, 2, 1, 2, 1)$. Let $C = P(3) \cap T$. Since $C$ is a simple closed curve on $T$, $C$ separates $T$ into two connected components $A$ and $B$. $A = T$-open disc and $B = D^2$ (we use $C = P(3) \cap T$ here). Suppose $p(1) \in A$ and denote by $C_1, C_2$ the connected components of $T \cap P(3)$. Let $\alpha(t)$ be an arc going from a point of $C$ to $p(1)$, intersecting $C_1$, and satisfying $(z: \alpha'(t)) \neq 0$ for all $t$. Similarly let $\beta(t)$ be an arc going from $P(1)$ to $C$, intersecting $C_2$, and satisfying $(z: \beta'(t)) \neq 0$ for all $t$. With $\alpha \cup \beta$ it is easy to construct a smooth embedding $h : S^1 \to T$ such that $h(S^1) \cap C_i = \{\text{one point}\}$ for $i = 1, 2$ and $z \cdot h$ is a Morse function with exactly two critical points. Neither $C_1$ nor $C_2$ bounds a disc on $T$ since there is a simple closed curve meeting each in one point; therefore both $C_1$ and $C_2$ are generators of $\pi_1(T)$. Let $C_1$ bound a disc $D$ on $P(z)$ whose interior does not meet $C_2$. By cutting along $D$, we obtain an embedded 2-sphere in $S^3$ which separates $S^3$ into 3-balls. Hence the connected component of $S^3 - T$ containing $D$ is a solid torus $D^2 \times S^1$. It remains to see the other component $M$ is also solid torus. Now $h(S^1)$ is unknotted in $S^3$ since the projection onto the $z$-axis has exactly two critical points. Also $h(S^1)$ is isotopic to the core of the solid torus bounded by $T$. Therefore $M = D^2 \times S^1$.

Now we consider the case $(1, 2, 3, 2, 1)$. Let $A$ be the part of $T$ above $P(3)$ and $B$ the part below. We can suppose $B$ contains one critical point of index one. Then $B$ also contains a critical point of index 0 or two in addition to $p(1)$. So $B$ has two connected components and $A$ is connected. Let $C_1, C_2, C_3$ be the connected components of $T \cap P(3)$, labelled so that $C_1$ and $C_2$ are generators of $\pi_1(T)$. Let $P$ be the one point compactification of $P(3)$ in $S^3$. One of the circles $C_1, C_2, C_3$ say, bounds a 2-disc $D$ in $P$ whose interior does not meet $C_3 \cup C_2$. Then the connected component of $S^3 - T$ containing $D$ is a solid torus. As before, we construct an embedding $h : S^1 \to T$ such that $h(S^1)$ meets $C_1$ and $C_2$ in one point and $z \cdot h$ has exactly two critical points; $h(S^1)$ passes through $p(1)$ and $p(6)$. This proves $T$ is unknotted in $S^3$.

The situation is quite different when $T$ is only an immersed torus. More precisely we will construct an immersed torus $T$ (with self intersection) which is tight; this is due to N. Kuiper. The main step is the construction of an immersed cylinder with in every point strictly negative curvature, and self intersection.

Let $C_1$ and $C_2$ be two strictly convex curves situated in two distinct parallel planes. Let $\gamma$ be the Gauss map $C_i \to S^1$, and $g$ a smooth diffeomorphism from $C_1$ to $C_2$. Let $g$ be a lifting of $\gamma$. The diagram commutes.

Consider the ruled surface generated by the segments $x, g(x)$ where $x$ is a point of $C_i$. The surface is parametrized by $s, t$:

$$F(s, t) = (1 - t)C_i(s) + tC_i(g(s))$$

(we note also $C_i$ the applications $R \to C_i$). We have: $\partial F/\partial s = (t - 1)C_i(s) + tC_i'(g(s))C_i(g(s))$. A necessary condition to have a degenerate point on the segment $[C_i(s), C_i(g(s))]$ is $g(s) = s + (2k + 1)\pi$. The condition of self intersection is: $(1 - t)(C_i(s_1) - C_i(s_2)) + t(C_i(g(s_1)) - C_i(g(s_2))) = 0$. It implies the vectors $C_i(s_1) - C_i(s_2)$ and $C_i(g(s_1)) - C_i(g(s_2))$ are opposite. When it is the case, and $s_1 - s_2 \neq 2k\pi$, the vectors are different from 0, and there exists $t \in [0, 1]$ such that the point $F(t, s_1) = F(t, s_2)$ is a double point of the surface.

Take for example $g(s) = s + a$, $(\pi/2) < a < \pi$. Choose arbitrary parallel segments in two distinct parallel planes $P_I, P$. Say, they are vertical, on the $y$-axis. Choose two vectors, one in the region $x < 0, y > 0$; the other in the region $x < 0, y < 0$: $N_I, N'$. There exists a strictly convex curve with normals at the two vertices of the first segment $N_I, N'_I$. Say $s_1, s'_1$ are the images on $S^1$ of the two vectors $N_I, N'_I$. There also exists a strictly convex curve with normals at the two vertices of the second segment: $N_2$ and $N'_2$ corresponding to $g(s_I)$ and $g(s'_I)$; see Fig. 2. This constructs the immersed cylinder. We have $K < 0$ on the cylinder we constructed. Consider in $P_1, P_2$ two plane discs $F_1, F_2$ containing $C_i$ and $C_i$ in their interior. Consider the surface obtained by glueing to the cylinder the exterior parts of $F_1 - C_i, F_2 - C_i$; and make it critical is always of saddle type. Then it is possible to round the corners preserving the condition $K \leq 0$. Complete the torus by the positive curvature part of a torus of revolution.
44. EMBEDDINGS OF A SURFACE OF GENUS TWO IN $R^3$

In this section we consider a fixed projection $p(z)$, which we may suppose to be: $(x, y, z) \to z$. We shall call a disc $D$ vertical for the projection if $p(z)$ has no critical points in the interior of $D$ and exactly two on the boundary.

Let $S$ be a surface of genus $g$ and $f : S \to R^3$ an embedding. A system $\Delta = \{D_1, \ldots, D_{2g}\}$ of vertical discs is said to be excellent for $f$ if (cf. Fig. 3):

1. each $D_i$ meets $f(S)$ transversally and $D_i \cap f(S) = \partial D_i$,
2. for $i = 1, \ldots, g$ $D_{2i} \cap D_{2i-1} = \partial D_{2i} \cap \partial D_{2i-1}$ is a critical point of $p(z)$ on $f(S)$,
3. $D_i \cap D_j = \emptyset$ if $j \neq 2i - 1$.

It is clear that if a surface $f(S)$ admits an excellent system $\Delta$ then it is not knotted.

We call $Y$ a cylindrical component of $f(S)$ if $Y$ is a connected component of the intersection of $f(S)$ with $\{(x, y, z)/z_i < z < z_{i+1}\}$ which is homeomorphic to a cylinder and whose closure contains exactly two critical points at heights $z_i$ and $z_{i+1}$, cf. Fig. 4. We shall prove:

**Theorem 4.1.** If the genus of $S$ is two, and $p(z)$ has six critical points on $f(S)$, then $f(S)$ is not knotted.

**Proof.** First we observe that an embedding of the torus $T$ into $R^3$, with four critical points admits an excellent system of discs. Suppose $g : T \to R^3$ is such an embedding. Let $Y$ be a
cylindrical component of \( g(T) \) which is minimal, i.e. the inside of the cylinder \( Y \) in \( \mathbb{R}^3 \) contains no other points of \( g(T) \). By tilting slightly the top disc of \( Y \), we obtain \( D_1 \). Let \( \alpha \) be a path on \( Y \) from \( z_1 \) to \( z_2 \) (the two critical points of \( p(z) \) on \( g(T) \) of index one) which is monotone, that is, \( p(z) \) has no critical points restricted to \( \alpha \). Let \( \beta \) be a monotone path between \( z_1 \) and \( z_2 \) on \( g(T) - Y \). Then the disc \( D_2 \) can be constructed by choosing arcs in the horizontal planes (the level surfaces of \( p(z) \)) joining a point of \( \alpha \) to a point of \( \beta \); cf. Fig. 5.

Now let \( f:S \to \mathbb{R}^3 \) be an embedding with six critical points, \( S \) of genus two. Let \( Y \) be a minimal cylindrical component of \( f(S) \). \( Y \) is a tubular neighborhood of a monotone arc \( \alpha \) and \( f(S) \) is obtained from an embedding of the torus \( T \) with four critical points by attaching a tubular neighborhood of \( \alpha \). We can suppose that \( \alpha \) is a monotone arc going from the maximum on \( f(S) \) to the minimum, since every orbit of the gradient vector field of \( p(z) \) on \( f(S) \) goes from the minimum to the maximum, perhaps passing through a saddle point on the way.

Let \( \Delta = \{D_1, D_2\} \) be an excellent system for the torus \( T \) which gives \( f(S) \) after adding \( Y \). If \( C \) does not intersect \( \Delta \) then one can find an excellent system for \( f(S) \) as follows. Let \( D_1 \) be the top of \( Y \) tilted slightly. We construct \( D_2 \) by finding a monotone arc \( \beta \) from the maximum on \( f(S) \) to the minimum which does not meet \( \Delta \). Then \( \alpha \cup \beta \) is unknotted and bounds \( D_4 \). To find \( \beta \) we observe that when \( T \) is cut along \( D_2 \) (or \( D_1 \)), we obtain a 2-sphere embedded in \( \mathbb{R}^3 \) and \( p(z) \) has two critical points on this 2-sphere \( S_1 \). The trace of \( D_2 \) on \( S_1 \) is two 2-discs with two critical points each and an arc with one maximum joining them as the trace of \( \partial D_1 \). Clearly, an arc \( \beta \) can be found joining the north and south pole which avoids the trace of \( D_2 \cup \partial D_1 \).

Thus it remains to prove that \( \alpha \) can be made disjoint from \( \Delta \). Let \( \alpha \) meet \( \Delta \) transversally and let \( x \) be the highest point of \( \alpha \cap \Delta \). Let \( \tilde{\alpha} \) be the arc on \( \alpha \) from the maximum to \( x \). Suppose that \( x \in D_2 \), hence \( \alpha \) does not meet \( D_1 \). Let \( y \) be the highest point of \( D_1 \), and let \( \beta_1 \) be a monotone arc from \( y \) to the maximum.

Let \( z \) be the highest point of \( D_2 \) and let \( \beta_1, \beta_2 \) be the monotone arcs joining \( z \) to \( y \) which are on \( \partial D_2 \). Let \( \gamma \) be a monotone arc on \( D_1 \) from \( x \) to \( z \). Suppose \( \beta_2 \) pierces \( D_1 \) in the same sense as \( \tilde{\alpha} \). Consider the closed curve \( C = \tilde{\alpha} + \gamma + \beta_2 + \alpha_1 \). We know \( C \) is unknotted since \( p(z) \) has two critical points on \( C \). Moreover \( C \) bounds a disc in the complement of \( T \). To see this, first push \( C \) off \( T \cup D_2 \) and notice that after thickening \( D_2 \), \( T \cup D_2 \) becomes a 2-sphere \( \tilde{S} \) and \( C \cap \tilde{S} = \emptyset \) so \( C \) bounds a disc in \( \mathbb{R}^3 - \tilde{S} \). Since \( C \) bounds a disc in the complement of \( T \), we know that \( \alpha \) is isotopic to \( \gamma + \beta_1 + \alpha \), across this disc. Next replace \( \gamma + \beta_2 + \alpha_1 \) by \( \gamma + \beta_1 + \alpha \), in the obvious way. This eliminates the intersection \( x \). In case \( \alpha \) intersects \( D_1 \) at \( x \) a similar argument with arcs from the minimum eliminates \( x \). (cf. Fig. 6).

Remark. With more work, the reader can prove that a surface of genus \( g \) with a projection having \( 2g \) critical points is not knotted. We believe this to be true if there are \( 2g + 2 \) critical points.
§5. SOME UNKNOTTED SPHERES

A natural question posed by Theorem 1.1. is what does the conclusion of this theorem mean. More precisely, if \( S^n \) is a sphere, smoothly embedded in \( R^m \) and some height function has exactly two non degenerate critical points then is \( S^n \) the standard embedding? This is clear for \( n = 1, m = 3 \) (hence for \( n = 1 \) and all \( m \)) for, suppose the projection to the \( z \)-axis has two critical points at heights \( 0 \) and \( 1 \). The hyper planes \( z = \text{constant} \) intersect \( S \) in exactly two points for each \( z, 0 < z < 1 \). The union of the line segments in these hyperplanes joining the two points, gives a disc with boundary \( S \).

D. Ferus has proved that \( S \) is always topologically unknotted, and if \( n \geq 5, m = n + 2 \), then \( S \) is differentiably unknotted. We shall consider the case \( n = 2 \).

**Theorem 5.1.** Let \( f: S^2 \rightarrow R^4 \) be a smooth embedding such that \( z \cdot f \) has exactly two non degenerate critical points. Then \( S = f(S^2) \) is differentiably unknotted.

**Proof.** We shall prove that \( S \) bounds a smooth 3-ball \( B \) in \( R^4 \). Denote by \( P(c) = \{ u \in R^4 | z(u) - c \} \) and \( E(c) = S \cap P(c) \) (here \( z(u) \) denotes the last coordinate of \( u \) in \( R^4 \)). We may assume \( z(f(S^2)) = [0, 1] \). For \( 0 < c < 1, E(c) \) is a smooth simple closed curve in \( P(c) \). Let \( Z \) denote the vector field \( (0, 0, 0, 1) \) in \( R^4 \). For \( c \) sufficiently small, the integral curves of \( Z \) define an embedding \( \varphi \) of \( \Sigma = \cup \{ E(t) | 0 \leq t \leq 1 \} \) into \( P(0) \); for each \( x \in \Sigma \), the orbit of \( Z \) by \( x \) meets \( P(0) \) in exactly one point, which we call \( \varphi(x) \). Let \( \Sigma' = \varphi(\Sigma) \); \( \Sigma' \) is a smooth surface in \( P(0) - R^3 \) and for each \( t, 0 < t < 1 \), we have \( D(t) = \varphi(\{ E(s) | 0 \leq s \leq t \}) \) is a smooth 2-disc in \( \Sigma' \) with boundary \( \varphi(E(t)) \). By lifting \( D(t) \) back to \( P(t) \) by the orbit of \( Z \), we obtain a smooth 2-disc \( F(t) \) in \( P(t) \) with boundary \( E(t) \). This shows \( E(t) \) is unknotted in \( P(t) \) for \( 0 < t < 1 \). Let \( B_0 = \cup \{ F(t) | 0 \leq t \leq c \}, B_0 \) is a smooth 3-ball except for a corner along \( E(c) \). A similar construction near \( P(1) \) gives a smooth 3-ball \( B_1 \) in \( R^4 \), with a corner along \( E(1 - c) \).

Let \( M = \{ P(t) | c \leq t \leq 1 - c \} \). We define a smooth vector field \( X \) on \( S \cap M \) by \( X = \frac{\nabla g}{\| \nabla g \|^2} \), where \( g = z \cdot f \). If \( \varphi(t) \) denotes the integral curves of \( X \), we have \( \varphi(E(s)) = E(s + t) \) when this makes sense. Clearly we can extend \( X \) to all of \( M \) to obtain a vector field, also called \( X \) which is transverse to each \( E(t) \). This shows \( E(t) \) is unknotted in \( P(t) \) for \( 0 < t < 1 \). Let \( B_0 = \cup \{ F(t) | 0 \leq t \leq c \}, B_0 \) is a smooth 3-ball except for a corner along \( E(c) \). A similar construction near \( P(1) \) gives a smooth 3-ball \( B_1 \) in \( R^4 \), with a corner along \( E(1 - c) \).

Let \( M = \{ P(t) | c \leq t \leq 1 - c \} \). We define a smooth vector field \( X \) on \( S \cap M \) by \( X = \frac{\nabla g}{\| \nabla g \|^2} \), where \( g = z \cdot f \). If \( \varphi(t) \) denotes the integral curves of \( X \), we have \( \varphi(E(s)) = E(s + t) \) when this makes sense. Clearly we can extend \( X \) to all of \( M \) to obtain a vector field, also called \( X \) which is transverse to each \( E(t) \). This shows \( E(t) \) is unknotted in \( P(t) \) for \( 0 < t < 1 \). Let \( B_0 = \cup \{ F(t) | 0 \leq t \leq c \}, B_0 \) is a smooth 3-ball except for a corner along \( E(c) \). A similar construction near \( P(1) \) gives a smooth 3-ball \( B_1 \) in \( R^4 \), with a corner along \( E(1 - c) \).

We can suppose \( H(1 - c) \) is the standard \( D^2 \subset R^3 \) and \( F(1 - c) \) is some smooth 2-disc \( D \) with \( \partial D = \partial D^2 \). By a small isotopy of \( D \), relative to \( \partial D \), we can make \( D \) and \( D^2 \) in general position. Then \( \text{(int } D \text{) } \cap D^2 \) is a finite union of connected disjoint one-manifolds, the compact components are circles and the non compact components are arcs whose closure is in \( \partial D \) and meets \( \partial D \) transversally. We shall isotope \( D \) so that \( \text{(int } D \text{) } \cap D^2 = \emptyset \). Then \( D \cup D^2 \) bounds a 3-ball in \( R^3 \) and \( D \) is isotopic to \( D^2 \) across this ball. First we remove the compact components of \( \text{(int } D \text{) } \cap D^2 = A \). Let \( C \) be a circle in \( A \) which bounds a 2-disc \( E \subset D^2 \) with \( \text{(int } E \text{) } \cap D = \emptyset \). Let \( F \) be the 2-disc on \( D \) bounded by \( C \). Then \( E \cup F \) bounds a 3-ball \( B \) in \( R^3 \), and by isotoping \( F \) across \( B \) to \( E \) we decrease the number of compact components of \( A \). So we can suppose we have no compact components in \( A \). Let \( \alpha \) be an arc in \( A \) whose closure meets \( \partial D^2 \) in \( p \) and \( q \). If we choose \( \alpha \) properly, we can suppose one of the 2-arcs on \( \partial D^2 \) between \( p \) and \( q \), \( \beta \) say, has the property that \( \alpha \cup \beta \) bounds a 2-disc \( D \) on \( D^2 \) whose interior is disjoint from \( D \). Let \( F \) be the 2-disc on \( D \) bounded by \( \alpha \cup \beta \). Then \( E \cup F \) bounds a 3-ball in \( R^3 \) and we remove the intersection \( \alpha \) by isotoping \( F \) across this ball.

**Remark.** The arguments in the proof of Theorem 5.1 are well known; we include this proof for completeness. It would be interesting to know if an embedding of the 2 torus in \( R^4 \) with four critical points, is standard.

§6. GENERALISATIONS

Let \( M^n \) be a compact manifold, with possibly non empty boundary, and let \( f: M^n \rightarrow R^n \) be an immersion. Denote by \( E \) the total space of the unit normal bundle of \( f \). The total curvature of \( f \) is
ON CURVATURE INTEGRALS AND KNOTS

\[ \tau(f) = \frac{1}{c_{n-1}} \int_{S^{n-1}} \mu(z, M) \, d\sigma, \]

(6.1)

defined by.

where \( c_{n-1} \) is the volume of \( S^{n-1} \) and \( \sigma \) is the standard volume form on \( S^{n-1} \); \( c_{n-1} = \text{vol} S^{n-1} \).

Let \( dv \) be the volume form on \( M \) and \( d\rho \) a \( n-k \) form on \( E \) whose restriction to each fibre is a volume form on the sphere. Then \( d\tau = d\rho \wedge dv \) is a volume form on \( E \) (cf. §1). Denote by \( \gamma : E \to S^{n-1} \), the Gauss map: \( \gamma(x, v) = v \). The \( k \)th Weyl polynomial of \( E \) is defined by

\[ \gamma^*(d\sigma) = W_k(x, v) \, d\tau. \]

(6.2)

When \( k = n-1 \), \( W_k \) is the Gauss–Kronecker curvature of \( M \) (also called the Lipshitz–Killing curvature). The relationship between total curvature and the Weyl polynomial is:

\[ \int_E |W_k| \, d\tau = \int_{S^{n-1}} \mu(z, M) \, d\sigma. \]

(6.3)

We can see this as follows. Define: \( C = \{ z \in S^{n-1} | z \cdot f \) is not a Morse function\}. Clearly \( W_k^{-1}(0) \) is the set of critical points of \( \gamma \), hence \( C = \gamma(W_k^{-1}(0)) \) is of measure 0 by Sard’s theorem. Also \( \gamma^{-1}(C) \sim W_k^{-1}(0) \) has measure zero because it is a set of regular points of \( \gamma \) whose image by \( \gamma \) is contained in \( C \). Writing \( E = (E - y^{-1}(C)) \cup (W_k^{-1}(0)), \) we conclude \( f_0 |W_k| \, d\tau = f_{E-y^{-1}(O)} |W_k| \, d\tau. \) Let \( A_i = \{ z \in S^{n-1} \sim C | \mu(z, M) = i \} \) for \( i > 0 \). Then

\[ \int_{E-y^{-1}(O)} |W_k| \, d\tau = \sum_{i=0}^{\infty} i \int_{y^{-1}(A_i)} |W_k| \, d\tau = \sum_{i=0}^{\infty} i \int_{A_i} \mu(z, M) \, d\sigma = \int_{S^{n-1}} \mu(z, M) \, d\sigma. \]

We remark that the set of normals to \( M \) along \( \partial M \) is of measure zero in \( S^{n-1} \), hence \( f_{S^{n-1}} \mu(z, \int M) \, d\sigma = f_{S^{n-1}} \mu(z, M) \, d\sigma. \) Since for almost all \( z \in S^{n-1} \), \( z \cdot f \) is a Morse function, we have \( \mu(z, M) \) at least as big as the sum of the Betti numbers of \( M \) for almost all \( z \). This gives the result of Chern–Lashof [2]:

**Theorem 6.**

Let \( M \) be a closed manifold immersed in \( R^n \), then:

\[ \tau(f) \geq \beta(M). \]

(6.4)

where \( \beta(M) \) is the sum of the Betti numbers of \( M \); with respect to some field.

Let \( f : M \to R^n \) be an immersion with \( M \) a closed manifold. If \( \tau(f) < 3 \) then there must exist \( z \in S^{n-1} \) such that \( \mu(z, M) = 2 \). This gives:

**Theorem 6.2.** (Ferus [4]). Let \( f : M \to R^n \) be an embedding with \( \tau(f) < 3 \). Then \( M \) is homeomorphic to \( S^k \) and is topologically unknotted. If \( k \geq 5 \) and \( n = k + 2 \), then it is differentiably unknotted.

We remark that 5.1 shows an embedding of \( S^2 \) in \( R^4 \) is differentiably unknotted when \( \tau(f) < 4 \).

Now suppose \( \partial M \) is homeomorphic to \( S^{k-1} \) and \( f : M \to R^{k+1} \) is an immersion. Then using Theorem 1.1 we have:

**Theorem 6.3.** Let \( 2v_1 = \text{dim} H_i(DM) \). Then

\[ \int_{E(M)} |W_k| \, d\tau \geq \int_{E(M)} |W_k| \, d\tau \geq C_{n-1}(2 + \sum_{i=1}^{k} v_i). \]

(6.5)

Moreover, if \( f/\partial M \) is an embedding with \( f(\partial M) \) knotted in \( R^{k+1} \), then this is a strict inequality. In fact we have: \( f_{E(M)} |W_k| \, d\tau + f_{E(M)} |W_k| \, d\tau \geq C_{n-1}(3 + \sum_{i=1}^{k} v_i) \). Examples of such embeddings are the fibrations arising from the Brieskorn polynomials.

**Remarks.** Let \( K = f(S^k) \) and \( h(u_1, \ldots, u_k) \) be a local parametrization of \( K \). Let \( (\cdot)^T \) denote the orthogonal projection of a vector on the tangent space to \( K \). \( A \) denotes the canonical volume form on \( K \). Let \( (x, v) = h(u_1, \ldots, u_k) \) be a local parametrization of \( E \) the total space of the unit normal bundle of \( K \) such that \( (t_1, t_2) \) be the coordinate of a point in an orthonormal basis which depends smoothly on \( x : (e_{k+1}, e_{k+2}) \). The Weyl polynomial \( W_k \) is then the sum of the degree

\[ \text{Ferus has proved the same result with } \tau(f) < 4 \text{ and } \gamma(M) \text{ even.} \]
$k$ terms of the polynomial $G$:
\[ G(x, v) = \frac{A(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_4})}{\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_4}\right)} \]

We shall verify that the formula (6.6) gives Milnor's theorem for $k = 1$. Let $e_1$ be the unit tangent vector of $K$, $e_2$ the unit vector of the principal normal, $e_3$ the unit vector of the binormal (Frenet coordinates). We have:
\[ W_i(t_1, t_2) = A \left[ e_1, \left(\frac{de_2}{ds}\right)^T, \left(\frac{de_3}{ds}\right)^T \right], \tag{6.7} \]
where $s$ is the arc length of a point of $K$. And:
\[ (de_2/ds) = (1/p)e_1 + (1/\tau)e_3; \quad (de_3/ds) = -(1/\tau)e_2, \]
where $p$ and $\tau$ are the curvature and torsion radii of $K$. Then:
\[ |W_i(t_1, t_2)| = |t_1/p| = |kt_1| \]
and
\[ \int_E |W_i(t_1, t_2)| = 4 \int |k|. \tag{6.8} \]

Formula 6.8 and Theorem 6.2 give Milnor's theorem.

§7. SOME INVARIANTS FOR KNOTTED SPHERES

For each isotopy type $K$ of a knot, we define:
\[ T(K) = \inf_{k \in K} \inf_{z \in M} \mu(z, k), \quad \Delta_e = \inf_{k \in K} \inf_{M \text{ of genus } e} (\mu(z, M) - 2g). \]

Clearly $T(K) \geq 2$ and $\Delta_e \geq 2$. Moreover, $\Delta_e$ decreases when $g$ increases by 1. For it is possible to build from the surface of genus $g$ a surface of genus $g + 1$ such that the height function has gained exactly two saddle points: attach a handle to the first surface on the neighbourhood of a regular point as in Fig. 7. The following definitions are now natural:

\[ \Delta_0^{1/2}(K) = \inf_{g = 21} \Delta_e(K) = \lim_{g \to \infty} \Delta_e(K), \]
\[ \Delta_e(K) = \inf_{g = 21} \Delta_e(K) = \lim_{g \to \infty} \Delta_e(K), \]
\[ \Delta_e(K) = \inf [\Delta_e(K), \Delta_0^{1/2}(K)] = \inf \Delta_e(K). \]

Of course these invariants can be related to curvature:

**Theorem 7.**

\[ \Delta_e(K) - \frac{1}{2\pi} \inf_{z \in K} \inf_{M} \left[ 2 \int_{k \in K} |k| + \int_{M} |K| - 2g \right], \]

$M$ is a surface of genus $g$. And
\[ \Delta_e(K) = \frac{1}{2\pi} \inf_{z \in K} \inf_{M} \left[ (2 \int_{k \in K} |k| + \int_{M} |K|) - 2g \right]. \]

**Proof.** Choose $S$ and $z$ such that: $\mu(z, S) = 2 + 2g + \Delta_e(K)$. Choose a coordinate system such that $z$ be on the $z$-axis, and let $H_\gamma$ be the isotopy: $H_\gamma(x, y, z) = (ux, uy, z)$. We divide $S$ into two:
\[ A = \{ p \mid (N_p, z) > q \}, \quad B = \{ p \mid (N_p, z) \leq q \}; \quad N_p = \gamma(p). \]

There exists $u$ such that: $P \in H_\gamma(B) \Rightarrow [(N_p, z) - \pi/2] \leq \epsilon$. Then $\int_{H_\gamma(A)} |K|$ can be made arbitrary small. Then, for $q$ sufficiently small $\int_{H_\gamma(A)} |K| \to 2\pi$, where $A_t$ is a connected component of $A$ which contains one critical point of $p(z)$. The analogous result for $\partial S = K$ has already been proved by Milnor[8].
ON CURVATURE INTEGRALS AND KNOTS

Fig. 7. Attaching a handle with two critical points.

Fig. 8. Generating links and "silly link".

**THEOREM 7'.** For an isotopy type $K$ of a knotted sphere $f : S^d \to \mathbb{R}^{d+2}$

$$
\tau(K) = \frac{1}{\alpha^{d+2}} \inf_{k \in K} \int_{\text{Tub}(k)} |W_d|,
$$

$$
\Delta(K) = \frac{1}{C_{d+1}} \inf_{k \in K} \inf_{M} \left[ \int E \cdots \int E |W_d| + 2 \int_M \cdots \int |K| - \sum \nu_i - 2 \right].
$$

**Remark.** $\Delta$ is not an interesting invariant for links. Let $L = \partial \mathcal{M}$ be a link of circles with $c$ components. Of course $\Delta(L) \geq 2$ but there exist non trivial $c$-components links such that $\Delta(L) = 2$.

**PROPOSITION.** For any four Morse link (c.f. [7]) we have $\Delta(L) = 2$.

**Proof.** It is proven in [7] that such links are obtained by connected sum from two generators $a$ and $b$ (Fig. 8) except the "silly link" (Fig. 8). $a$ and $b$ are the edges, with different orientations of cylinders (Fig. 9) and the "silly link" is the edge of the cylinder (Fig. 9).

**Fig. 9. Generators and "silly link" edge of cylinders.**
The contribution of $L$ to the integral $\int_\gamma \mu$ can become as close as we wish to $16\pi$ using the isotopy $H_\nu$. Let us show that there exists $B$ such that

$$\int \int_{H_\nu(A)} |K| = 0 \quad \int \int_{H_\nu(B)} |K| \to 0 \quad \text{when } \nu \to 0.$$

It is enough to choose $B$ such that $B$ be a real cylinder (of equation $f(x, y) = 0$) near the extrema of $p(z)$. At any point of $H_\nu(M)$ belonging to the region $B$ the normal vector tends towards a vector normal to $z$. On the other hand, in the region $A$, $M$ is a cylinder so $K = 0$.

Let us conclude with a problem.

The invariant $\tau$ probably does not allow to calculate $\Delta_\nu$. It would be interesting to know whether for $\tau$ given $\Delta_\nu$ may take any integer value verifying $\Delta_\nu = f(\tau)$ where $f$ is an (increasing?) function to be calculated.

REFERENCES

7. R. LANGEVIN: Links à 4 points de Morse. Preprint.

Université de Paris-Sud,
Orsay