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Hyperbolic Bridged Graphs

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Given a connected graph G, we take, as usual, the distance xy between any two vertices x, y of G to be the length of some geodesic between x and y. The graph G is said to be δ-*hyperbolic*, for some $\delta \geq 0$, if for all vertices x, y, u, v in G the inequality

$$xy + uv \leq \max\{xu + yv, xv + yu\} + \delta$$

holds, and G is *bridged* if it contains no finite isometric cycles of length four or more. In this paper, we will show that a finite connected bridged graph is 1-hyperbolic if and only if it does not contain any of a list of six graphs as an isometric subgraph.

1. INTRODUCTION

In this paper, all graphs are simple and connected, but they are not necessarily finite. As is well known, a (connected) graph G comes equipped with a natural metric on its vertex set $V(G)$, given by defining the distance xy between any pair of vertices $x, y \in V(G)$ to be the length of some shortest path or *geodesic* between x and y. Given a quartet $x, y, u, v \in V(G)$, define $\delta(x, y, u, v)$ to be the absolute value of the difference between the largest and the second largest of the three sums

$$xu + yv, \qquad xv + yu, \qquad \text{and} \qquad xy + uv.$$

The graph G is called δ-*hyperbolic*, for some $\delta \geq 0$, if we have $\delta(x, y, u, v) \leq \delta$ for all quartets x, y, u, v in $V(G)$, or, equivalently, if

$$xy + uv \leq \max\{xu + yv, xv + yu\} + \delta \tag{1}$$

holds for all quartets x, y, u, v in $V(G)$. The *hyperbolicity*, δ^*, of G is then defined to be the supremum of the values $\delta(x, y, u, v)$ taken over all quartets x, y, u, v in $V(G)$, and G is called *hyperbolic* if its hyperbolicity is finite.

Hyperbolic graphs arise naturally in the area of geometric group theory as Cayley graphs of *hyperbolic groups* [11] (see [10, 11] for more details on such groups). Moreover, the notion of hyperbolicity is of implicit interest in metric graph theory [1, 2], and—due to the fact that hyperbolicity is closely related to concepts arising in the study of trees—also in T-theory [8], classification theory [6], and phylogenetic analysis [13].

In [5], we proposed the study of graphs with low hyperbolicity. Such graphs can have an interesting structure: for example, in [2, Proposition 1] (see also [7]), it is shown that the 0-hyperbolic graphs are precisely the *block graphs*, i.e., graphs in which every 2-connected subgraph is complete, and in [5] that *chordal* graphs, i.e., graphs containing no *induced* cycles of length exceeding three [4, 12], have hyperbolicity strictly bounded by two. As usual, the diameter of a connected graph G is defined to be the maximum distance between any pair of vertices in G. Using Eqn (1) once again, it is straightforward to check that a graph with finite diameter d is $(2\lfloor \frac{d}{2} \rfloor)$-hyperbolic, and, using this fact, it can be seen that a $(4m + i)$-cycle, $m \geq 1$, has hyperbolicity $2m$ for $i = 0, 2, 3$ and, arguing directly, that a $(4m + 1)$-cycle has

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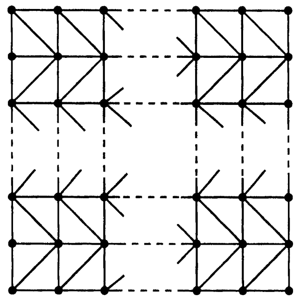

FIGURE 1. The $2m \times 2m$-grid, with chords as indicated in the diagram—a part of the so-called *hexagonal grid*—forms a bridged graph with hyperbolicity at least $2m$ (as can be easily checked by considering the four corner vertices).

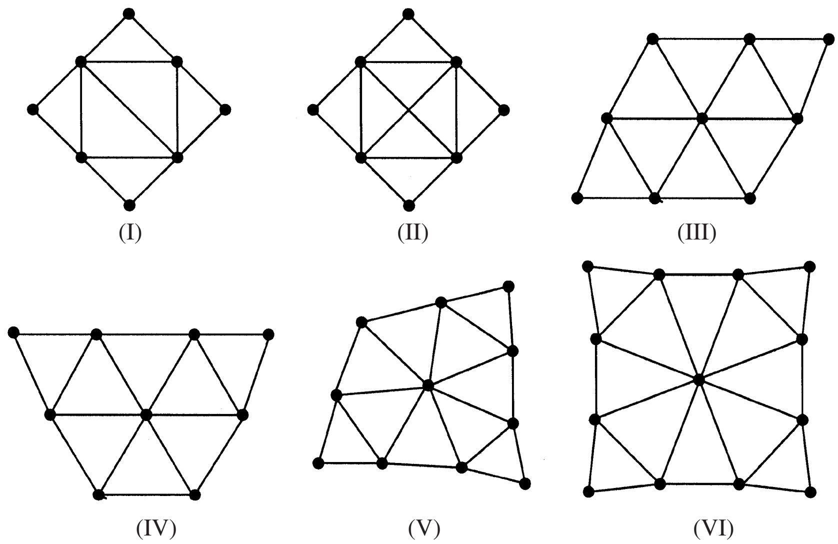

FIGURE 2. Some bridged graphs with hyperbolicity 2.

hyperbolicity $2m - 1$. In consequence, a finite isometric cycle contained in a 1-hyperbolic graph must have length three or five. Hence, it is perhaps a bit surprising to note that the class of so-called *bridged graphs*, consisting of graphs that do not contain finite *isometric*[†] cycles of length larger than three [4, 9, 16], contains graphs with arbitrarily large hyperbolicity (e.g. see Figure 1). In this paper, we classify the finite 1-hyperbolic bridged graphs or—equivalently—the finite 1-hyperbolic graphs that do not contain induced (or, equivalently, isometric) 5-cycles. In particular, we prove that if G is a finite connected bridged graph, then G is 1-hyperbolic if and only if G contains none of the graphs in Figure 2 as an isometric subgraph.

The proof of this result relies on two key properties that a 1-hyperbolic graph enjoys, which we now describe. The first property is related to the concepts of *thin bigons* [15] and the

[†]In general, a subgraph H of a graph G is called isometric if the distance between any pair of vertices in H is the same as that in G. Thus, an isometric cycle is clearly induced, in particular, a chordal graph is always bridged, and, conversely, every induced 5-cycle is isometric.

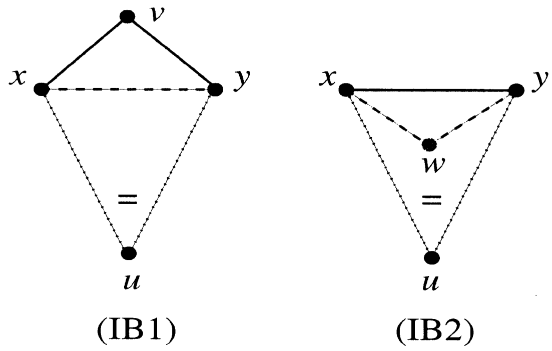

FIGURE 3. Properties (IB1) and (IB2) of an interval-bridged graph.

fellow traveller property [14], both of which are standard tools used in the study of hyperbolic groups.

Given a graph G, we define the *interval* $[x, y]$ between any two vertices $x, y \in V(G)$ to be the set of vertices $z \in V(G)$ that satisfy the equality $xz + zy = xy$. In addition, we define the breadth of the interval between two vertices x and y in $V(G)$ to be the maximum value for uv, taken over all vertices u, v in the interval $[x, y]$ satisfying $xu = xv$, and the *breadth of* G, denoted $\mathrm{br}(G)$, to be the supremum of the interval breadths taken over all intervals in G. If a graph is δ-hyperbolic, then it is straightforward to check using Eqn (1) that its breadth is at most δ. Thus we obtain the first key property that a 1-hyperbolic graph G satisfies.

Breadth property: The breadth of G is at most one.

The second property looks slightly more technical, but also follows in a straightforward fashion from Eqn (1).

Short-cut property: If t_1, t_2, t_3, t_4 is a path in G with $t_1t_3 = t_2t_4 = 2$, and there is some $x \in V(G)$ such that $xt_1 < xt_2 = xt_3 \geq xt_4$ holds, then $t_1t_4 \leq 2$.

We define a graph that satisfies both the breadth and short-cut properties to be *thin*. In Section 2, we prove that a thin graph not containing induced 5-cycles is *interval-bridged*, i.e. the graph satisfies the following two properties (see Figure 3).

(IB1) If $x, y, u, v \in V(G)$ are distinct vertices with $xu = yu$, $xv = yv = 1$ and $x, y \in [u, v]$, then $xy = 1$.

(IB2) If $x, y, u \in V(G)$ are distinct vertices with $xu = yu$ and $xy = 1$, then there exists some vertex $w \in V(G)$ with $xw = yw = 1$ and $w \in [x, u] \cap [y, u]$.

Note that a finite bridged graph satisfies both of these properties (see appendix for a proof of this fact), so that such a graph is, in particular, interval-bridged. In addition, it follows as a straightforward consequence of properties (IB1) and (IB2) that an interval-bridged graph G cannot contain an isometric n-cycle for any $n \geq 4$, $n \neq 5$ and also that if G contains an isometric 5-cycle, then it must also contain a vertex that is adjacent to every vertex in this 5-cycle.

Given an interval-bridged graph G, we show in Section 3, that G satisfies the breadth property if and only if graph (III) in Figure 2 is not an isometric subgraph of G, and also that G satisfies the short-cut property if and only if graph (IV) in Figure 2 is not an isometric subgraph of G (see Corollary 5 and Theorem 2, respectively). Therefore, an interval-bridged graph is thin if and only if it contains neither graph (III) nor graph (IV) as an isometric subgraph.

We now state the key result in this paper, whose proof can be found in Section 4.

THEOREM 1. *Let G be a connected, thin graph that contains no induced* 5*-cycles. Then G is* 2*-hyperbolic and bridged. Moreover, G has hyperbolicity equal to two if and only if G contains at least one of the graphs* (I), (II), (V) *or* (VI) *of Figure* 2 *as an isometric subgraph.*

Note that as a corollary of the above classification of thin interval-bridged graphs and this theorem we obtain the main result of [5], namely:

COROLLARY 1 ([5, THEOREM 1]). *If G is a connected, chordal graph, then G is* 2*-hyperbolic. Moreover G has hyperbolicity equal to two if and only if it contains at least one of the graphs* (I) *or* (II) *in Figure* 2 *as an isometric subgraph.*

As we have already seen, a finite isometric cycle in a 1-hyperbolic graph must have length three or five. Thus, as a consequence of the above classification of thin interval-bridged graphs and Theorem 1, we obtain the following result.

COROLLARY 2. *Let G be a connected graph that contains no induced* 5*-cycles. Then G is* 1*-hyperbolic if and only if G is interval-bridged and does not contain any of the graphs in Figure* 2 *as an isometric subgraph.*

In view of the fact that finite bridged graphs are interval-bridged (see Appendix), we immediately obtain the following result in view of the last corollary.

COROLLARY 3. *Let G be a finite connected bridged graph. Then G is* 1*-hyperbolic if and only if G does not contain any of the graphs in Figure* 2 *as an isometric subgraph.*

To conclude, we discuss in Section 5 the problem of characterizing 1-hyperbolic graphs[†]. In particular, we prove in Proposition 4 that a thin graph is hyperbolic. This indicates that the concept of short-cuts might be a useful tool for the study of hyperbolic graphs in general.

2. THIN GRAPHS

In this section, we present some results concerning thin graphs that will be used throughout the rest of this paper.

PROPOSITION 1. *If G is a connected, thin graph that contains no induced* 5*-cycles, then it is interval-bridged.*

PROOF. We must show that G satisfies properties (IB1) and (IB2).
(IB1): This property follows directly from $\mathrm{br}(G) \leq 1$.
(IB2): Let $x, y, u \in V(G)$ be distinct vertices with $xu = yu$ and $xy = 1$. If $xu = 1$, then (IB2) clearly holds with $w := u$. So suppose $xu \geq 2$, and let x_1, y_1 be on geodesics from x to u and y to u, respectively, with $xx_1 = y_1y = 1$ and $x_1u = y_1u = xu - 1$. Applying the short-cut property to the path x_1, x, y, y_1 and the vertex u, we see that $x_1y_1 \leq 2$ holds. If $x_1 = y_1$, then (IB2) holds with $w := x_1$. If $x_1y_1 = 1$, then x, y, y_1, x_1 is a 4-cycle and since $\mathrm{br}(G) \leq 1$, without loss of generality we have $xy_1 = 1$, and so (IB2) holds with $w := y_1$. Now if $x_1y_1 = 2$, then let v be such that $x_1v = y_1v = 1$. Then x_1, v, y_1, y, x is a 5-cycle, and since (IB2) must hold if either xy_1 or yx_1 equals one, we can assume $xy_1 = yx_1 = x_1y_1 = 2$. Hence, we must have $xv = yv = 1$ as G contains no 5-cycles and $\mathrm{br}(G) \leq 1$. If $uv = xu$, then since $\mathrm{br}(G) \leq 1$, we have $x_1y_1 \leq 1$, which is a contradiction. This implies $uv = xu - 1$, and therefore (IB2) holds with $w := v$. This completes the proof of the proposition. □

[†]We recently discovered through a personal communication that Bandelt and Chepoi appear to have a classification for 1-hyperbolic graphs—see Remark 1 for more details.

We now present a consequence of the short-cut property that holds for thin graphs, which will be used later in the proof of Theorem 1.

PROPOSITION 2. *Suppose that G is a connected, thin graph. Let $t_1, \ldots, t_n$ be a path in G, $n \geq 4$, and $x \in V(G)$. If $xt_1 < xt_2$ and either $xt_n \leq xt_{n-1} \leq xt_{n-2}$ or $xt_n < xt_{n-1}$ holds, then there exists some i, $1 \leq i \leq n-3$, with $t_i t_{i+3} \leq 2$.*

PROOF. First we assume that both $xt_1 < xt_2$ and $xt_n < xt_{n-1}$ hold. In this case, there must clearly exist some i, j with $1 \leq i, j \leq n$ and $i+1 < j$, such that $xt_i = xt_j$ and $xt_p = xt_i + 1$, for all $i < p < j$. Now, if $j = i+2$, then since $\mathrm{br}(G) \leq 1$, we have $t_i t_j \leq 1$, and hence $t_i t_{i+3} \leq 2$, whereas if $j \geq i+3$ then $t_i t_{i+3} \leq 2$ clearly holds by the short-cut property. Thus, in view of these facts, we are reduced to considering the case where both $xt_n = xt_{n-1} = xt_{n-2}$ and $xt_{n-3} < xt_n$ hold. But then $t_{n-3} t_n \leq 2$ by the short-cut property, which completes the proof. □

The following result follows more-or-less immediately from this proposition.

COROLLARY 4. *If G is a connected, thin graph, then any isometric cycle in G must have length three or five.*

REMARK 1. In fact, Proposition 2 implies that cycles in finite, connected, thin graphs must in general satisfy even stronger conditions than the one we have presented in Corollary 4. For example, one can show that every cycle of length at least six has to have at least two essentially different short-cuts. Bandelt and Chepoi appear to have characterized 1-hyperbolic graphs using similar properties for cycles of length at least six, and the exclusion of a finite set of graphs occurring as isometric subgraphs (personal communication).

3. INTERVAL-BRIDGED GRAPHS

We begin this section by characterizing the interval-bridged graphs with breadth at least k, $k \geq 1$. To do this we will need the following result.

PROPOSITION 3. *Suppose that G is a connected, interval-bridged graph, and $x, u, v \in V(G)$. If $t_0 := u, t_1, \ldots, t_n := v$ is a path in G, and we define*

$$m := \max_{i=0,\ldots,n}\{xt_i\} - \max\{xu, xv\},$$

then $uv + m \leq n$ holds.

PROOF. First note that we may assume without loss of generality that $t_i \neq t_j$ holds for $0 \leq i \neq j \leq n$. We prove the proposition using induction on m. Clearly the proposition holds for $m = 0$. Assume $m \geq 1$. Then there exists some j, $1 \leq j \leq n-1$, with

$$xt_j = \max_{i=0,\ldots,n}\{xt_i\}.$$

Hence, there exist $p, q, 0 \leq p < j < q \leq n$, for which $xt_p = xt_q = xt_j - 1$ and $xt_i = xt_j$, $p < i < q$, all hold.

If $q - p = 2$, then by applying (IB1) to t_p, t_{p+1}, t_{p+2} and x, we see that $t_p t_q = 1$ holds. Thus, $t_0, t_1, \ldots, t_p, t_q, \ldots, t_n$ is a path in G of length $n-1$, and since

$$\max_{i=0,\ldots,p,q,\ldots,n}\{xt_i\} = \max_{i=0,\ldots,n}\{xt_i\} - 1,$$

the proposition follows by induction.

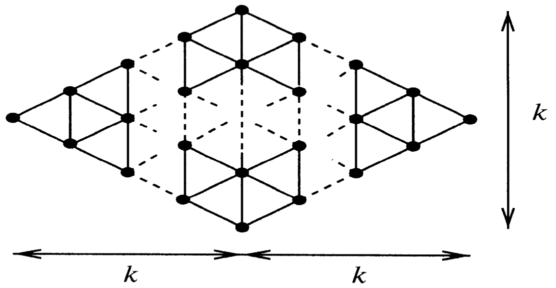

FIGURE 4. The double cardhouse graph of height k.

Thus we may assume $q - p > 2$. Now, by (IB2), for each t_i, t_{i+1}, $p < i < q - 1$, there exists a vertex $w_i \in V(G)$ with $w_i t_i = w_i t_{i+1} = 1$, and $x w_i = x t_i - 1$. Moreover, by (IB1) we have $w_i w_{i+1} \leq 1$ for $p < i < q - 2$, $t_p w_{p+1} \leq 1$, and $t_q w_{q-2} \leq 1$. Hence, once any possible repeats are removed from the sequence of vertices

$$t_0, t_1, \ldots, t_p, w_{p+1}, \ldots, w_{q-2}, t_q, \ldots, t_n,$$

a path in G of length less than or equal to $n - 1$ is obtained, and since

$$\max\left\{\max_{i=0,\ldots,p,q,\ldots,n}\{xt_i\}, \max_{i=p+1,p+2,\ldots,q-2}\{xw_i\}\right\} = \max_{i=0,\ldots,n}\{xt_i\} - 1,$$

the proposition follows by induction. □

COROLLARY 5. *Let G be a connected, interval-bridged graph. Then* $\mathrm{br}(G) \geq k$, $k \geq 1$, *if and only if G contains the graph depicted in Figure* 4 *as an isometric subgraph. In particular,* $\mathrm{br}(G) \leq 1$ *if and only if graph* (III) *in Figure* 2 *is not contained as an isometric subgraph in G.*

PROOF. Clearly $\mathrm{br}(G) < k$ implies that the graph pictured in Figure 4 is not an isometric subgraph of G.

Conversely, suppose $\mathrm{br}(G) \geq k$ holds. We show that this implies that the graph pictured in Figure 4 must be an isometric subgraph of G.

Since $\mathrm{br}(G) \geq k$, we must have vertices $x, y, u, v \in V(G)$ with $xu = xv$, $uv = k$, and $xu + uy = xv + vy = xy$. Note that by Proposition 3, we must have $xu, yu \geq k$. Now let $t_0 := u, t_1, \ldots, t_k := v$ be a geodesic in G. By Proposition 3 we have $xt_i \leq xu$ and $yt_i \leq yu$, for all $0 \leq i \leq k$. Since $xu + uy = xy$, it follows that both $xt_i = xu$ and $yt_i = yu$ hold for all $0 \leq i \leq k$. Now applying (IB2) to $t_i t_{i+1}$ and x for each $0 \leq i \leq k - 1$, we obtain a vertex w_i with $w_i t_i = w_i t_{i+1} = 1$ and $x w_i = x t_1 - 1$, $0 \leq i \leq k - 1$. By (IB1), applied to t_i, w_i, w_{i+1} and x, we have $w_i w_{i+1} = 1$ for each $0 \leq i \leq k - 2$, and hence, by Proposition 3, $w_0, w_1, \ldots, w_{k-1}$ is a geodesic in G. Thus we have constructed the first 'layer' of triangles bordering the central geodesic of length k in the graph pictured in Figure 4. By repeatedly applying (IB1), (IB2) and Proposition 3 in a similar fashion, it is now straightforward to construct the graph in Figure 4 layer by layer, and to check that this graph is indeed an isometric subgraph of G. We leave the details to the reader. □

In general, graphs with bounded breadth are not necessarily hyperbolic, even if they have bounded degree—see Figure 5. Hence, it is interesting to note that in [15] Papasoglu shows

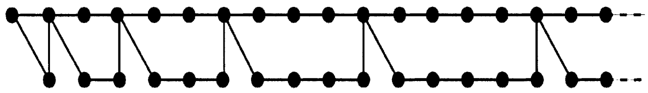

FIGURE 5. An infinite string of odd-length cycles. This graph has breadth zero and bounded degree, but it is not hyperbolic.

that if the Cayley graph associated to a finitely generated group has bounded breadth, then this Cayley graph is necessarily hyperbolic.

Note that a complete graph on four vertices minus an edge clearly has breadth one, and that this is an induced subgraph in every one of the graphs (I)–(VI) pictured in Figure 2. Hence, it immediately follows from Corollary 3 that an interval-bridged graph G with breadth zero is 1-hyperbolic. It is thus natural to ask the following question: suppose that G is an interval-bridged graph with bounded breadth, then is G hyperbolic?

We now classify the interval-bridged graphs that satisfy the short-cut property.

THEOREM 2. *Let G be a connected, interval-bridged graph. Then G contains graph* (IV) *in Figure* 2 *as an isometric subgraph if and only if it does not satisfy the short-cut property.*

PROOF. It is straightforward to see that if G contains graph (IV) as an isometric subgraph, then G does not satisfy the short-cut property.

Conversely, if G does not satisfy the short-cut property, then there must exist some path t_1, t_2, t_3, t_4 in G with $t_1t_4 = 3$, and a vertex $x \in V(G)$ with $xt_2 = xt_3$, $xt_1 < xt_2$ and $xt_4 \leq xt_3$.

Note that by (IB2), there must exist some vertex $w_1 \in V(G)$ with $w_1t_2 = w_1t_3 = 1$, and $w_1x = xt_2 - 1$. Now if $xt_4 < xt_3$, then it follows by (IB1) that $w_1t_1 = w_1t_4 = 1$ holds, and hence $t_1t_4 < 3$, a contradiction. Thus $xt_3 = xt_4$. But then, by (IB2), there must exist a vertex $w_2 \in V(G)$ with $w_2t_3 = w_2t_4 = 1$, and $xw_2 = xt_3 - 1$. Moreover, by (IB1) we must have $w_1w_2 = w_1t_1 = 1$.

Now consider the vertices t_1, w_1, w_2, all of which are at distance $xt_2 - 1$ from x. Then by repeated application of (IB1) and (IB2) to these vertices using the vertex x, it is now straightforward to show that graph (IV) can be constructed as an isometric subgraph of G by adding 'layers' to the graph induced on the vertices $t_1, t_2, t_3, t_4, w_1, w_2$ in the same way that was described in the proof of Corollary 5. The details are left to the reader. □

4. PROOF OF THEOREM 1

The proof of Theorem 1 is similar in spirit to the proof of [5, Theorem 1], although it is significantly more complicated. As the proof is quite lengthy, we will break it up into a series of interconnected claims for the sake of clarity.

Let G be a connected, thin graph that does not contain any induced 5-cycles. Note that G is interval-bridged by Proposition 1, and G is bridged by Corollary 4. Now, given any $C \in \mathbb{N}$, define $\delta^* = \delta^*_C$ to be the maximum value of $\delta(x, y, u, v)$ taken over all quartets $x, y, u, v \in V(G)$ satisfying

$$xu + yv, xv + yu \leq xy + uv \leq C \tag{2}$$

(see Section 1 for the definition of $\delta(x, y, u, v)$). Note that $\delta^* \leq C$ clearly holds. We will show that $\delta^* \leq 2$ holds, and that, in the case $\delta^* = 2$ holds, G must contain one of the graphs (I), (II), (V) or (VI) in Figure 2. The theorem follows from this.

Suppose $\delta^* > 1$. Let $x, y, u, v \in V(G)$ be a quartet satisfying Eqn (2) such that

$$xy + uv = \max\{xu + yv, xv + yu\} + \delta^* \tag{3}$$

holds, and assume

($*$) $xy + uv$ is minimal amongst all quartets satisfying Eqn (3).

Note that x, y, u, v must be distinct.

Before proceeding, for the reader's convenience we briefly outline the rest of the proof. We consider the quantity

$$M := \min\{xu, xv, yu, yv\}.$$

As a consequence of Claims 3, 4, and 5 we see that $2 \le M \le 3$ must hold. In Claim 5, we prove that $xu+yv = xv+yu$ must hold. In Claim 6 we show that if $xu = xv = yu = yv = 2$ holds, then either graph (I) or (II) in Figure 2 must be an isometric subgraph of G and $\delta^* = 2$ holds. Using these facts we can then assume, without loss of generality, that $xu, xv \ge 3$ holds. In Claim 8 we show that if in addition $yu = yv = 2$ holds, then $xu = xv = 3$ holds, graph (V) of Figure 2 is an isometric subgraph of G and $\delta^* = 2$ holds. To complete the proof we show that the only other possibility is for $xu = xv = yu = yv = 3$ to hold, and in Claim 9 prove that if this is the case, then graph (VI) of Figure 2 is an isometric subgraph of G and $\delta^* = 2$ holds.

We now proceed with the proof. Let $a_0 := x, a_1, \ldots, a_{xu} := u$, $b_0 := x, b_1, \ldots, b_{xv} := v$, $c_0 := y, c_1, \ldots, c_{yu} := u$, and $d_0 := y, d_1, \ldots, d_{yv} := v$ be four geodesics in G. We assume that the sum

$$a_1b_1 + a_{xu-1}c_{yu-1} + b_{xv-1}d_{yv-1} + c_1d_1$$

is minimal amongst all possible quartets of such geodesics. We now consider some properties that these geodesics must satisfy.

CLAIM 1. (i) *The inequality* $a_1v > b_1v$ *holds* (*and hence, by symmetry, the inequalities* $b_1u > a_1u$, $c_1v > d_1v$, $d_1u > c_1u$, $a_{xu-1}y > c_{yu-1}y$, $c_{yu-1}x > a_{xu-1}x$, $d_{yv-1} > b_{xv-1}x$, *and* $b_{xv-1}y > d_{yv-1}y$ *all hold as well*).

(ii) *If* $ux \ge 2$ *and* $a_2b_1 \le 2$, *then* $a_1b_1 = 1$.

PROOF. (i): Note that $a_1v \ge b_1v$ clearly holds, and hence it suffices to prove that $a_1v = b_1v$ cannot hold. Suppose to the contrary that this were the case. Consider the quartet a_1, u, y, v. Then, as $a_1y \ge xy - 1$, $a_1u = xu - 1$, and $a_1v = xv - 1$ (since $a_1v = b_1v$), we have

$$a_1y + uv \ge \max\{a_1u + yv, a_1v + yu\} + \delta^* + (a_1y - xy + 1).$$

Therefore, $a_1y = xy - 1$, and hence the quartet a_1, u, v, y satisfies

$$a_1y + uv = \max\{a_1u + yv, a_1v + yu\} + \delta^*$$

and $a_1y + uv < xy + uv$ simultaneously, in contradiction to minimality condition ($*$) for x, y, u, v. This completes the proof of (i).

(ii): Clearly $a_2b_1 = 2$ holds. So, there exists a vertex $t \in V(G)$ for which a_2, t, b_1 is a geodesic in G. If $t \ne a_1$, then, as G does not contain a 4-cycle or a 5-cycle as an isometric subgraph, it follows that, as $a_1b_1 \ne 1$, we must have $ta_1 = tx = 1$. But then we have a contradiction, as $a_1b_1 + a_{xu-1}c_{yu-1} + b_{xv-1}d_{yv-1} + c_1d_1$ is by assumption minimal, but $tb_1 < a_1b_1$. This completes the proof of (ii). □

It immediately follows from this claim that $a_i \ne b_j$ holds for all $1 \le i \le xu$ and $1 \le j \le xv$; for if this were not the case, and $a_i = b_j$ were to hold for some $1 \le i \le xu$ and $1 \le j \le xv$, then by Claim 1 (i) we would have $a_ib_1 > a_ia_1$ and $b_ja_1 > b_jb_1$, which is clearly impossible.

We now consider what happens in the case $a_1b_1 = 1$ holds.

CLAIM 2. *If $a_1b_1 = 1$, then the following equalities hold:*

(i) $ux = ub_1$, $vx = va_1$;
(ii) $xu + yv = xv + yu$;
(iii) $a_1y = b_1y = xy - 1$.

PROOF. (i): This follows immediately from Claim 1 (i).
(ii): Suppose that $xu + yv = xv + yu$ does not hold, and therefore, without loss of generality, that $xu + yv > xv + yu$ holds. It follows from Eqn (3), $a_1y \geq xy - 1$, $a_1u = xu - 1$ and $a_1v = xv$, that

$$a_1y + uv = \max\{a_1v + uy, a_1u + vy\} + \delta^* + (a_1y - xy + 1),$$

holds, and therefore, by minimality condition $(*)$, that $a_1y \geq xy$ holds. Hence,

$$a_1y + uv > \max\{a_1v + uy, a_1u + vy\} + \delta^*,$$

contradicting the fact that G is of hyperbolicity δ^*, which completes the proof of (ii).
(iii): We show that $a_1y = xy - 1$ holds; the equality $b_1y = xy - 1$ then holds by symmetry. To this end, first note that $a_1y \geq xy - 1$ clearly holds. Moreover, $a_1v = xv$, $a_1u = xu - 1$ and $xu + yv = xv + yu$ all hold by (i) and (ii). It follows from Eqn (3) that

$$a_1y + uv = \max\{a_1v + uy, a_1u + vy\} + \delta^* - 1 + (a_1y - xy - 1)$$

holds. Therefore, by minimality condition $(*)$ for the quartet x, y, u, v we have $a_1y \leq xy$. Moreover, if $a_1x = xy$ were to hold, then we would have

$$a_1y + uv = \max\{a_1v + uy, a_1u + vy\} + \delta^*.$$

But then a_1, y, u, v would also be minimal in the sense of condition $(*)$, which implies $a_1v + uy = a_1u + vy$ (as can be seen by substituting a_1 for x and applying (ii)). However, this contradicts the fact that both $a_1v + uy = xv + uy$ and $a_1u + vy = xu + vy - 1 = xv + yu - 1$ hold. This completes the proof of (iii). $\square$

We now consider what happens in the case $a_1b_1 = 1$ holds, together with some extra conditions.

CLAIM 3. *If $a_1b_1 = 1$, then the following statements hold:*

(i) *If $ux, vx \geq 2$, then either $a_2y \geq a_1y$ or $b_2y \geq b_1y$ holds;*
(ii) *If $ux \geq 3$, $vx \geq 2$ and $a_2y \geq a_1y$, then $a_{xu-1}c_{yu-1} = 1$ holds;*
(iii) *If $a_{xu-1}c_{yu-1} = 1$ and $a_2y \geq a_1y$, then $ux \leq 3$ holds and, moreover, if $ux = 3$ then $b_1c_{yu-1} = 2$ holds;*
(iv) *If $ux \geq 3$ and $uy \geq 2$, then either $a_{xu-1}c_{yu-1} = 1$ or $c_1d_1 = 1$ holds;*
(v) *If $ux, uy \geq 3$ and $c_1d_1 = 1$, then $a_{xu-1}c_{yu-1} = 1$ holds.*

In particular, it follows from (i) to (iii) that either $ux \leq 3$ or $vx \leq 3$ must hold, and that if $ux \geq 4$ and $vx \geq 3$ both hold, then $vx = 3$ and $b_{xv-1}d_{yv-1} = 1$.

PROOF. (i): Suppose that $a_2y < a_1y$ and $b_2y < b_1y$ both held simultaneously, so that, by Claim 2(iii), $a_2y = b_2y = xy - 2$ holds. Since $\mathrm{br}(G) \leq 1$, we have $a_2b_2 \leq 1$, and therefore as a_2, b_2 are distinct we have $a_2b_2 = 1$. However, this implies that a_1, a_2, b_2, b_1 is a 4-cycle, and therefore either $a_2b_1 = 1$ or $a_1b_2 = 1$. But this immediately leads to a contradiction of Claim 1(i).

(ii): Suppose $a_{xu-1}c_{yu-1} \neq 1$. Consider the path

$$b_1, a_1, a_2, \ldots, a_{xu-1}, u, c_{yu-1}$$

and the vertex y. Since $b_1y = a_1y \leq a_2y$ holds by Claim 2(iii), and $uy > b_{yu-1}$ holds by Proposition 2, we must have $a_{xu-2}c_{yu-1} \leq 2$, and therefore by Claim 1(ii), we obtain $a_{xu-1}c_{yu-1} = 1$ as required.
(iii): Since $yc_{yu-1} < a_{xu-1}y$, by Claim 1, and $a_2y \geq a_1y = b_1y$ by Claim 2(iii), (iii) follows immediately from (ii) and applying Proposition 2 to the vertex y and the path

$$c_{yu-1}, a_{xu-1}, a_{xu-2}, \ldots, a_1, b_1.$$

(iv) and (v): Since $a_1v > b_1v$ and $vc_1 > vd_1$ both hold by Claim 1(i) and clearly $vd_1+1 = vy$, (iv) and (v) immediately follow from Proposition 2 and Claim 1(ii) by considering the vertex v and the paths

$$b_1, a_1, \ldots, a_{xu-1}, u, c_{yu-1}, \ldots, c_1, y, d_1,$$

and

$$b_1, a_1, \ldots, a_{xu-1}, u, c_{yu-1}, \ldots, c_1, d_1,$$

respectively. □

Recall that M is by definition equal to $\min\{xu, xv, yu, yv\}$. We now show that M is bounded below by two.

CLAIM 4. *The inequality $M \geq 2$ holds.*

PROOF. Without loss of generality, assume $xu = 1$. Together with Eqn (3), this implies

$$(yu + 1) + (xv + 1) \geq xy + vu \geq yu + xv + 2,$$

and hence $xv = vu - 1$ and $yu = xy - 1$. Now consider the vertex y path

$$u, x, b_1, \ldots, b_{xv-1}, v, d_{yv-1}.$$

By Proposition 2 and Claim 1(i), we see that $b_{xv-1}d_{yv-1} = 1$ holds, and, by symmetry, i.e., swapping the roles of x and u and using vertex v instead of y, one obtains $c_1d_1 = 1$ as well. This implies that $xv \geq 2$ and $uy \geq 2$ must both hold, and since $xu + vy = xv + uy$ holds by Claim 2(ii), we thus have $yv \geq 3$.

We now show that $yv = 3$ holds: if $c_2x \geq c_1x$, then by Claim 3(iii) (considering $c_1d_1 = b_{xv-1}d_{yv-1} = 1$ instead of $a_1b_1 = a_{xu-1}c_{yu-1} = 1$), we would have $yv \leq 3$, as required. Now, if $c_2x = c_1x - 1$ holds, then we must have $d_2x \geq d_1x$ as otherwise we would have $c_2d_2 = 1$ since $\mathrm{br}(G) \leq 1$, and thus either $c_2d_1 = 1$ or $c_1d_2 = 1$, both of which are impossible. This implies $yu \leq 2$, since otherwise considering the vertex v and the path

$$d_1, c_1, \ldots, c_{uy-1}, u,$$

we would obtain a contradiction using Proposition 2. By symmetry $xv \leq 2$, but this contradicts $xu + vy = xv + uy$. Hence $yv = 3$ as required.

Therefore in view of $xu+yv = xv+yu$, we must have $yu = xv = 2$ and $uv = xy = 3$. But now if we consider the path u, x, b_1, v and the vertex y, then since we clearly have $uy = 2$, and $yx = yb_1 = yv = 3$, by the short-cut property, we must have $uv \leq 2$, which contradicts the fact that $uv = 3$ holds. This completes the proof of the claim. □

We now see that $xu + vy = xv + uy$ must hold.

CLAIM 5. *At least two of $a_1b_1, a_{xu-1}c_{yu-1}, b_{xv-1}d_{yv-1}, c_1d_1$ are equal to one. In particular, it follows from Claim* 2(ii) *that $xu + vy = xv + uy$ must hold.*

PROOF. We will show that at least one of $a_1b_1, a_{xu-1}c_{yu-1}, b_{xv-1}d_{yv-1}$ is equal to one, from which the claim immediately follows.

Since

$$c_{uy-1}, u, a_{xu-1}, \ldots, a_1, x, b_1, \ldots, b_{xv-1}, v, d_{yv-1}$$

is a path in G, using vertex y, Claim 1(i), and Proposition 2, we see that there must exist four consecutive vertices $t_1, \ldots, t_4$ in this path with $t_1t_4 \leq 2$. Note that this can clearly only happen if t_1 equals c_{uy-1}, a_{xu-2}, a_2 or a_1. By symmetry it suffices to consider the case where $t_1 = a_2$ holds. In this case, by Claim 1(ii) $a_1b_1 = 1$ holds, and this completes the proof of the claim. □

Thus, in particular, by Claims 3, 4, and 5, we have $2 \leq M \leq 3$.

We now consider what happens in the case $xv = xu = yu = yv = 2$ holds.

CLAIM 6. *If $xv = xu = yu = yv = 2$, then either graph* (I) *or graph* (II) *of Figure* 2 *is an isometric subgraph of G and $\delta^* = 2$ holds.*

PROOF. If $xy = 4$, then as $\mathrm{br}(G) \leq 1$, we must have $uv = 1$, so that $xy + uv = 5$ which contradicts Eqn (3). Hence $xy \leq 3$. By symmetry, it follows that $uv \leq 3$. Hence, by Eqn (3), it follows that $xy = uv = 3$ holds. Now applying Proposition 2 to u, a_1, x, b_1, v, and vertex y we see that $a_1b_1 = 1$ holds, and, hence, by symmetry, that $c_1d_1 = 1$ holds as well. But then $va_1 = vc_1 = 2$, and as $uv = 3$, we see by (IB1), that $a_1c_1 = 1$. By symmetry $b_1d_1 = 1$, and thus we obtain either graph (I) or graph (II) of Figure 2 as an isometric subgraph of G and hence it also follows that $\delta^* = 2$ holds, as required. □

In view of this claim, and the fact that by Claim 5 we have $xu + yv = xv + yu$, to complete the proof of Theorem 1 it suffices to assume from now on that $xu, xv \geq 3$ holds. We now show that if this is the case, then a_1b_1 must equal one.

CLAIM 7. *If $xu, xv \geq 3$, then $a_1b_1 = 1$, and hence at least one of ux and vx is equal to three.*

PROOF. Suppose $a_1b_1 \neq 1$. By Claim 5 we may assume without loss of generality that $a_{xu-1}c_{yu-1} = 1$ holds. Applying Claim 3(iv) (with a_1, b_1 replaced by a_{xu-1}, c_{yu-1}) we have $b_{xv-1}d_{yv-1} = 1$, and so by Claim 3 (v) (replacing c_1, d_1 by b_{xv-1}, d_{yv-1}) we have $a_1b_1 = 1$, a contradiction. But now by Claim 3(i)–(iii) at least one of ux and vx must be less than or equal to three, as required. □

We now consider what happens in the case $yu = yv = 2$.

CLAIM 8. *If $xu, xv \geq 3$ and $yu = yv = 2$, then $xu = xv = 3$, graph* (V) *of Figure* 2 *is an isometric subgraph of G and $\delta^* = 2$ holds.*

PROOF. By Claim 7 we have $a_1b_1 = 1$, therefore by Claim 3(i)–(iii), without loss of generality we have $a_{xu-1}c_{yu-1} = 1$ and $ux = 3$. Moreover, by Claim 3(iv) at least one of c_1d_1 and $b_{xv-1}d_{yv-1}$ is equal to one.

If $c_1d_1 = 1$, then $uv \leq 3$ and $xy \leq 4$, and since $xy + uv \geq xu + yv + 2 = 7$, we have $uv = 3$ and therefore $a_2v = 2$. As $va_1 = 3$ it follows by $\mathrm{br}(G) \leq 1$ that $b_1a_2 \leq 1$ holds, which is a contradiction.

Thus, $c_1d_1 \neq 1$ and therefore $b_{xv-1}d_{yv-1} = 1$. Hence $xy = 3$ and therefore $uv = 4$. By the short-cut property, we have $b_1c_1 \leq 2$, and, as $b_1c_1 = 1$ is impossible, there exists a vertex w with $c_1w = b_1w = 1$. Now $a_1w = b_1w = 1$ since a_1, a_2, c_1, w, b_1 is a 5-cycle in G. Moreover, if $wb_2 = 1$, then the graph induced on $x, y, u, v, a_1, a_2, b_1, b_2c_1, d_1, w$ in G is graph (V) of Figure 2, so that this graph is an isometric subgraph of G and hence $\delta^* = 2$ holds. Therefore we may assume $wb_2 \neq 1$.

As $c_1x = yx = d_1x = 3$ and $b_2x = 2$ it follows that $c_1b_2 \leq 2$ by the short-cut property, and since $c_1b_2 = 1$ is impossible (otherwise c_1, y, d_1, b_2 would be an induced 4-cycle), we have $c_1b_2 = 2$. Thus, there exists some $w' \in V(G)$ with $c_1w' = b_2w' = 1$. Now we have $w'b_1 = 2$, as otherwise we may replace w by w'. But then looking at the 5-cycle w, c_1, w', b_1, b_2 we have to have $w'b_1 = 1$ or $wb_2 = 1$, either of which leads to a contradiction. This completes the proof of the claim. □

In light of this claim, we may assume from now on that $yu \geq 3$ holds. Now, assume that $M = 2$ holds, so that $yv = 2$ holds. Then as $xu + yv = xv + yu$, we have $xu \geq 4$. Hence, by Claim 7, we have $yu = xv = 3$, $xu = 4$, and $a_1b_1 = a_{xu-1}c_{yu-1} = 1$ (where we replace x by u, to get $yu = 3$ and $a_{xu-1}c_{yu-1} = 1$). By Claim 3(iii) we also get $c_1d_1 = b_{xv-1}d_{yv-1} = 1$. Now $uv \leq 4$ and $xy \leq 4$ and therefore, by Eqn (3), $uv = xy = 4$ holds. Applying Proposition 2 to the vertex y and the path c_2, a_3, a_2, a_1, b_1, (noting that $yc_2 < a_2y$ and $b_1y < a_2y$ both hold), we have $b_1a_3 \leq 2$ or $c_2a_1 \leq 2$, both of which are impossible.

Hence we may assume $yv \geq 3$ holds, so that $M \geq 3$ holds. The proof of Theorem 1 will thus be complete once we have proven the following claim:

CLAIM 9. *If $M \geq 3$, then $xu = xv = yv = yu = 3$, graph* (VI) *in Figure* 2 *is an isometric subgraph of G and $\delta^* = 2$ holds.*

PROOF. By Claim 7 we have $a_1b_1 = c_1d_1 = a_{xu-1}c_{yu-1} = b_{xv-1}d_{yv-1} = 1$. By Claim 3(iii) at least three of xu, xv, yv, yu are equal to three and therefore $xu = xv = yv = yu = 3$, as $xu + yv = xv + yu$ holds. This implies $xy, uv \leq 5$. If $xy = 5$ holds, then $a_2x + a_2y = xy = xb_2 + b_2y$ and $a_2x = b_2x = 2$ both hold. Therefore, since $\mathrm{br}(G) \leq 1$, it follows that $a_2b_2 \leq 1$ holds, and therefore without loss of generality $a_1b_2 = 1$ holds also, which contradicts Claim 1(i). Therefore $xy \leq 4$, and by symmetry $uv \leq 4$. Since $xy + uv \geq 2 + xu + yv$ holds, we thus see that $xy = uv = 4$ holds.

Now $b_1c_2 = 2$ as $yc_2 = 2$, $ya_2 = ya_1 = yb_1 = 3$, and $c_2a_1 = a_2b_1 = 2$. By symmetry $b_2c_1, a_1d_2, a_2d_1 \leq 2$. Moreover, as $b_1c_2 = 2$, there must exist some vertex $w \in V(G)$ so that c_2, a_2, a_1, b_1, w is a 5-cycle in G. Hence, as G is bridged, by Claim 1 neither $a_2b_1 = 1$ nor $a_1, c_{yu-1} = 1$ can hold, so we must have $wa_1 = wa_2 = 1$. By (IB1) it also follows that $wc_2 = wb_1 = 1$ holds. Applying the short-cut property to w, b_1, b_2, d_2 and the vertex y, we also see that $wd_2 \leq 2$ holds.

In the case $wd_2 = 1$, it is easy to see that we obtain graph (VI) as an isometric subgraph of G (and hence obtain $\delta^* = 2$), since clearly $wb_2 = 1$ as w, b_1, b_2, d_2 is a 4-cycle, and also $wc_1 = wd_1 = 1$, as w, d_2, d_1, c_1, c_2 is a 5-cycle.

In the case $wd_2 = 2$, then without loss of generality we may assume $wd_1 = 2$ also. Now, as $wd_1 = wd_2 = 2$, there must exist some vertex w' with w, w', d_1 and w, w', d_2 both geodesics in G by (IB2). Consider the 5-cycle w, b_1, b_2, d_2, w'. Since $b_1d_2 \neq 1$ and $wd_2 \neq 1$, we must have $w'b_2 = 1$. By symmetry we have $w'c_1 = 1$ also.

Consider now the 4-cycle w, w', b_1, b_2. Then without loss of generality we can assume $w'b_1 = 1$. If $w'c_2 = 1$, then a_1, a_2, c_2, w', b_1 is a 5-cycle, and hence without loss of generality, we have $w'a_1 = 1$. But then we can construct graph (VI) (and hence obtain $\delta^* = 2$) in the same way as described earlier. So suppose $wc_1 = 1$. Then, as $xw' = 2$ and $xc_1 = xc_2 =$

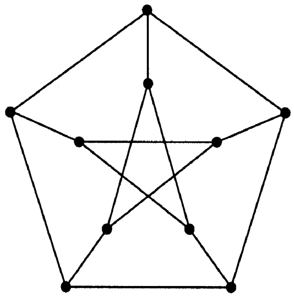

FIGURE 6. An old friend with hyperbolicity one.

$xu = 3$, it follows from the short-cut property that $uw' \leq 2$ must hold. Thus, since $uv = 4$ we must have $uw' = 2$. Now, suppose that u, t, w' is a geodesic in G. Then u, t, w, w', a_2 is a 5-cycle. If $uw = 1$, then we can construct graph (VI) (and hence obtain $\delta^* = 2$) as earlier replacing a_2 by w. Thus we can assume $uw = 2$, from which $ta_2 = tw = 1$ follows. But now since t, w', b_1, a_1, a_2 is a 5-cycle, and we can assume $a_1 w', a_2 w' > 1$, it follows that $tb_1 = 1$ must hold, which implies $ub_1 = 2$. This is a contradiction, and hence the proof of the claim is complete. □

REMARK 2. If, rather than assuming that G contains no induced 5-cycles in Theorem 1, we assumed that for every induced 5-cycle in G there exists a vertex in $V(G)$ that is adjacent to every vertex in the 5-cycle, then the conclusions stated in Theorem 1 would still be valid, with 'bridged' replaced by 'the only finite isometric cycles in G are either 3- or 5-cycles'. This also shows that the conclusions of Corollary 3 would still hold if we replaced 'bridged' by 'interval-bridged'.

5. CONCLUDING REMARKS

In Corollary 2, we gave a classification of the 1-hyperbolic graphs that do not contain induced 5-cycles. In general, 1-hyperbolic graphs appear to have a rich structure. This is indicated by the fact that a graph with diameter two containing no induced 4-cycles is 1-hyperbolic, so that, in particular, geodetic graphs [3] (such as the Petersen graph—see Figure 6—and the Hoffman–Singleton graph) are 1-hyperbolic. Moreover, 1-hyperbolic graphs can be constructed, for example, from graphs not containing 4-cycles through adjoining a vertex which is adjacent to all vertices, or by gluing together pairs of 1-hyperbolic graphs at a vertex (since, in general, the hyperbolicity of a graph is the maximal hyperbolicity of its 2-connected components).

In connection with the problem of classifying 1-hyperbolic graphs the following result is of interest.

PROPOSITION 4. *If G is a connected thin graph, then G is hyperbolic.*

PROOF. We are going to prove that G must be 10-hyperbolic. Suppose to the contrary that G is a connected graph with $\mathrm{br}(G) \leq 1$ which satisfies the short-cut property, and that G is not 10-hyperbolic. Let $x, y, u, v \in V(G)$ be a quartet for which $\delta^* := \delta(x, y, u, v)$ is minimal, so that

$$xy + uv = \max\{xu + yv, xv + yu\} + \delta^* \tag{4}$$

and $\delta^* > 10$ both hold. Assume also that

($\Diamond$) $xy+uv$ is minimal amongst all quartets satisfying Eqn (4), and $\min\{xu+yv, xv+yu\}$ is in addition minimal amongst such quartets.

Note that x, y, u, v must be distinct, and that by assumption $\delta^* > 10$. Let $a_0 := x, a_1, \ldots, a_{xu} := u$, $b_0 := x, b_1, \ldots, b_{xv} := v$, $c_0 := y, c_1, \ldots, c_{yu} := u$, and $d_0 := y, d_1, \ldots, d_{yv} := v$, be four geodesics in G.

We first show that the quantity $M := \min\{xu, xy, yu, yv\}$ is less than or equal to five. Suppose to the contrary that $M \geq 6$ holds. Considering the vertex y and the path

$$c_{yu-1}, u, a_{xu-1}, \ldots, a_1, x, b_1, \ldots, b_{xv-1}, v, d_{yv-1},$$

we see by Proposition 2 that, without loss of generality, we can assume $a_1b_2 \leq 2$ and therefore that $a_1b_2 = 2$ holds. By Claim 1(i) in the proof of Theorem 1, which holds given the assumptions we have made earlier, we have $a_1v > b_1v$, and hence $a_1v \geq xv$. Hence, as $a_1b_2 \leq 2$, we have $a_1v = xv$. Since in addition we have $a_1u = xu - 1$ and $a_1x \geq xy - 1$, it is straightforward to check that $a_1y = xy - 1$ must hold using Eqn (4) and minimality condition ($\Diamond$).

Now let $w \in V(G)$ be such that a_1, w, b_2 is a geodesic in G, and consider the vertex y together with the path

$$a_1, w, b_2, \ldots, b_{xv-1}, v, d_{yv-1}.$$

If $a_1b_1 \neq 1$, then since $\mathrm{br}(G) \leq 1$ holds, we must have $b_1y \geq xy$, and therefore $b_2y \geq xy-1$. Moreover, if $a_1b_1 = 1$ holds, then since Claim 3(i) of Theorem 1 holds under the assumptions that we have made, we can still assume without loss of generality that $b_2y \geq xy - 1$ holds. Thus, by Claim 1(i) and Proposition 2 we see that $b_{xv-2}d_{yv-1} = 2$ must hold.

Let w' be such that $b_{xv-2}w' = d_{yv-1}w' = 1$. Consider the vertex y and the path γ to be

$$a_1, w, b_2, \ldots, b_{xv-2}, w', d_{yv-1}.$$

Since $a_1, w, b_2, \ldots, b_{xv-2}$ and $b_2, \ldots, b_{xv-2}, w', d_{yv-1}$ are both geodesics, γ does not contain a short-cut (i.e. a sequence of four consecutive vertices whose first and last vertices are at distance less than three from one another). Hence by Proposition 2 we see that w' and b_{xv-1} must be distinct, and therefore $b_{xv-1}d_{yv-1} \neq 1$ holds. By symmetry, we also see that $a_1b_1 \neq 1$ must hold. Now since $b_{xv-2}y \geq d_{yv-1}y$ and $b_2y \geq a_1y$ both hold, it follows from $\mathrm{br}(G) \leq 1$ and Proposition 2, that $yt_1 = yt_2$ must hold for all t_1, t_2 of γ. Hence, we have $a_1y = d_{yv-1}y$ so that $xy = yv$ holds. By symmetry $xu = uv$ holds as well. But this implies $xy + uv = xu + yv$, which contradicts Eqn (4). Therefore $M \leq 5$ as claimed.

Now, without loss of generality, we assume $ux = M$. Thus $xy \leq yu + ux \leq uy + M$, and $uv \leq ux + vx \leq vx + M$ both hold. Hence, by Eqn (4) $\delta^* \leq 2M$, and since $M \leq 5$, we see that $\delta^* \leq 10$ holds, a contradiction which completes the proof of the proposition. $\square$

In the proof of this proposition we showed that G was 10-hyperbolic, although we suspect that the bound of 10 can be improved upon. In fact, we believe that the sum $xy + uv$ in the proof of Proposition 4 can be bounded above by 10. This would imply that only finitely many graphs would have to be excluded as isometric subgraphs—in addition to assuming the breadth and short-cut properties—to assure that G would be 1-hyperbolic. However, perhaps more importantly, this proposition indicates that the concept of short-cuts together with the implicitly well-known concept of breadth could be useful for both determining the structure and finding good bounds on the hyperbolicity of hyperbolic graphs.

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APPENDIX

In this appendix, we prove that a finite, bridged graph must satisfy the defining properties (IB1) and (IB2) of an interval-bridged graph.

PROPOSITION 5. *If G is a finite, connected, bridged graph, then G is interval-bridged.*

PROOF. We will show that the following two statements hold:

(A) There does not exist a quadruple of distinct vertices $x, y, v, u \in V(G)$ with $xy = 2$, $xv = yv = 1$, and $uv = ux + xv = uy + yv$ all holding simultaneously.

(B) There does not exist a quintuple of distinct vertices $x, y, u, v, w \in V(G)$ with $xy = 2$, $xv = yv = uw = 1$, and $uv = vw = ux + xv = vy + yw$ all holding simultaneously.

This will complete the proof, since (IB1) is simply a reformulation of (A) whereas, (IB2) is a consequence of (B), which we see as follows: let $x, y, u \in V(G)$ be as in (IB2), so that $xu = yu$ and $xy = 1$. We show that (IB2) holds using induction on xu. When $xu = 1$, (IB2) clearly holds. Assume $xu \geq 2$. Considering geodesics, there exist $x_1, y_1, x_2 \in V(G)$ with $ux_1 = x_1x_2 = uy_1 = 1$ an $xx_1 = xx_2 + 1 = yy_1 = xu - 1$. So, by (B), $x_1y_1 \leq 1$. If $xx_1 = yx_1$, then (IB2) holds by induction. So we may assume $yx_1 = yu$. Now since $yx_2 = yy_1 = yu - 1$, we see that $x_2y_1 = 1$ must hold by applying (IB1) to x_2, x_1, y_1 and y, and therefore $xy_1 = yy_1$ holds, from which (IB2) follows by induction.

We show that statements (A) and (B) hold using induction on uv. If $uv \leq 2$ both these statements are easily seen to hold, since a bridged graph does not contain either a 4-cycle or a 5-cycle as an isometric subgraph.

Now suppose that both (A) and (B) hold for $uv \geq 3$, and suppose also that G contains a quartet x, y, u, v satisfying the conditions stated in (A).

Without loss of generality, as G is finite, we may assume that for all vertices $z \in V(G)$ either the induced subgraph on $V(G) - \{z\}$ does not contain any quadruple of vertices satisfying the conditions in (A), or that if this is the case, then the induced subgraph on $V(G) - \{z\}$ is not an isometric subgraph of G. Moreover, we may assume that G is minimal in the sense that there is no $z \in V(G)$ with z distinct from each of x, y, u and v and for which the induced subgraph on $V(G) - \{z\}$ is an isometric subgraph of G.

Now let x', y' in $V(G)$ be vertices on some geodesics from u to x or y, respectively, so that $x'u = y'u = 1$, $ux = xx' + x'u$, $uy = yy' + y'u$ all hold. Note that we can assume $x' \neq y'$, otherwise (A) holds for the quadruple x, y, v, x', which by induction is a contradiction. Moreover, if $x'y' = 1$, then (B) holds for the quintuple x, y, v, x', y' which, by induction, is a contradiction. Thus $x'y' = 2$ holds.

We now see that without loss of generality there must exist some vertex $w \in V(G)$ with $xw = yw = vw = 1$ and $uw = xu$ all holding. To see this we consider two possibilities (which are all we need to consider, as we can clearly interchange the roles of x and y and, also the roles of u and v can be interchanged since $x'y' = 2$):

(1) The induced graph on $V(G) - \{x\}$ is an isometric subgraph of G. In this case clearly there is some $w \in V(G)$ with $vw = 1$ and $wx' = xx'$. Moreover, we can assume $w \neq y$ since $xy = 2$ and $xw = 1$ by minimality. We also have $yw = 1$, since if $yw = 2$, then we would contradict the minimality assumption (as we could replace x by w).

(2) The induced subgraph on $V(G) - \{v\}$ is an isometric subgraph of G. This implies that there is some $w \in V(G)$ with $xw = yw = 1$. Note that we must have $uw < uv$. Suppose $uw = uv$, then we could interchange the roles of v and w, and the induced subgraph on $V(G) - \{w\}$ would then be an isometric subgraph of G in which the quadruple x, y, u, v satisfied (A), contradicting our minimality assumption for G. Therefore $uw < uv$. Moreover, $vw = 1$ as otherwise x, y, v, w is an isometric 4-cycle in G and therefore $xu = uw$.

We now show that there must exist some $w' \in V(G)$ with $xx' = ww'$ and $w'x' = w'y' = uw' = 1$. Clearly, there must exist some $w' \in V(G)$ with $uw' = 1$ and $ww' + 1 = uw$. Moreover, if $w' = x'$, then the quintuple x', u, y', y, w would satisfy (B) which is a contradiction to the inductive hypothesis. Thus $x' \neq w'$ and $y' \neq w'$. In addition, considering the quintuple x', u, w', x, w, we see that $x'w' = 1$ must hold using (B) and induction. Thus w' exists as claimed.

To complete the proof of (A), take $w'' \in V(G)$ with $w'w'' = 1$ and $ww'' = ww' - 1$. Then we have $w'x > x'x$, as otherwise by induction and (B) applied to x, v, y, w', y' we would have $xy = 1$, a contradiction. By symmetry $w'y > y'y$. Thus applying (A) to x, x', w', w''

together with induction, we see that we must have $x'w'' = 1$. Therefore by symmetry we also have $y'w'' = 1$. But then x', y', w'', u is an isometric 4-cycle in G, which contradicts the fact that G is bridged. This completes the proof of (A).

The proof of (B) is similar, and we only outline it. Let $x, y, u, v, w \in V(G)$ be vertices satisfying the conditions given in (B). Take them to be minimal as in the proof of (A). As described in the proof of (A), we can assume that one of the vertices x, y, u, v, w can be removed yielding an isometric subgraph. If this vertex is either u or w, then it can be seen, using the same reasoning as in the proof of (A), that we must have $xy = 1$ which is a contradiction. Therefore, we may assume this vertex is one of x, y or v. Hence, there must exist some vertex $z \in V(G)$ with $xz = yz = vz = 1$ and, without loss of generality, $zu = xu$. Let x_1, y_1 be such that $x_1x = 1$, $y_1y = 1$, $x_1u = xu - 1$ and $y_1w = yw - 1$. Then considering the vertices u, y, z and y_1 we see that $y_1z = 1$ must hold by (A). Similarly we must have $x_1z = 1$. But then we must also have $x_1, y_1 = 1$, so that x_1, x, v, y_1, y is an induced 5-cycle, a contradiction that completes the proof of (B). □

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