On Oscillation of a Second Order Impulsive Linear Delay Differential Equation

L. Berezansky

Ben-Gurion University of the Negev, Department of Mathematics and Computer Science, Beer-Sheva 84105, Israel

and

E. Braverman

Technion-Israel Institute of Technology, Computer Science Department, Haifa 32000, Israel

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For the delay differential equation

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad g_k(t) \leq t, \]

with impulsive conditions

\[ x(\tau_j) = A_j x(\tau_j - 0), \dot{x}(\tau_j) = B_j \dot{x}(\tau_j - 0), \]

a connection between the following properties is established: nonoscillation of the differential equation and the corresponding differential inequality, positiveness of the fundamental function and the existence of a solution of a generalized Riccati inequality.

Explicit conditions for nonoscillation and oscillation and comparison theorems are presented. © 1999 Academic Press

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1. INTRODUCTION

First paper on oscillation of impulsive delay differential equations [1] was published in 1989 and its results were included in monographs [2, 3]. In the recent years impulsive delay differential equations attract attention of many mathematicians and numerous papers have been published on this class of equations.

Most of the publications are devoted to first order differential equations. Among them note the following papers on oscillation problems [4–9]. There are only few papers on high order impulsive differential equations, we also mention two ones on oscillation problems for second order impulsive ordinary differential equations [10, 11].

Our paper is probably one of the first publications concerned with oscillation problems of second order impulsive delay differential equations.

We obtain explicit conditions of oscillation and nonoscillation for sufficiently general class of these equations. For equations without impulses these results coincide with known ones. We present several examples illustrating these conditions. In the first example the impulsive differential equation is nonoscillatory while the corresponding nonimpulsive equation is oscillatory. In the second example the impulsive differential equation is oscillatory and the corresponding nonimpulsive equation is nonoscillatory. In both examples the sequence of values of impulses tends to one. Thus we can “improve” oscillation nature of an equation by a sequence of “disappearing” impulses. Such an example for a second order impulsive ordinary differential equation was constructed in [11]. As it follows from [4], this phenomenon is not possible for first order differential equations.

The paper is organized as follows. Section 2 contains the relevant definitions and notations. In section 3 we present the main result of the paper which is the equivalence of the four properties: nonoscillation of the differential equation and the corresponding differential inequality, positiveness of the fundamental function and the existence of a solution of a generalized Riccati inequality. For nonimpulsive equations this result was obtained in [12]. A similar result for a first order impulsive delay differential equation was obtained in [4].

In section 4 we present comparison theorems. The next section includes explicit conditions for nonoscillation and oscillation. In a partial case when the values of impulses for the solution and its derivative are equal we construct a special nonimpulsive delay differential equation. We establish that the oscillation of an impulsive equation is equivalent to oscillation of the constructed nonimpulsive equation. As a consequence of this theorem we obtain several interesting results for this case of impulsive conditions.

Appendix contains a proof of solution representation formula.
2. PRELIMINARIES

We consider a scalar delay differential equation of the second order

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq 0, \]  

\[ x(\tau_j) = A_jx(\tau_j - 0), \quad \dot{x}(\tau_j) = B_j\dot{x}(\tau_j - 0), \quad j = 1, 2, \ldots, \tag{2} \]

under the following conditions:

(a1) \( 0 = \tau_0 < \tau_1 < \tau_2 < \ldots \) are fixed points, \( \lim \tau_j = \infty \);

(a2) \( a_k, k = 1, \ldots, m, \) are Lebesgue measurable and locally essentially bounded functions on \([0, \infty)\), \( A_j, B_j \in R, \) \( j = 1, 2, \ldots, R \) is a real axis;

(a3) \( g_k: [0, \infty) \rightarrow R \) are Lebesgue measurable functions, \( g_k(t) \leq t, \lim_{t \rightarrow \infty} g_k(t) = \infty, k = 1, \ldots, m. \)

Together with (1), (2) consider for each \( t_0 \geq 0 \) an initial value problem

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = f(t), \quad t \geq t_0; \quad x(t) = \varphi(t), \quad t < t_0; \tag{3} \]

\[ x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \beta_0, \quad x(\tau_j) = A_jx(\tau_j - 0) + \alpha_j, \]

\[ \dot{x}(\tau_j) = B_j\dot{x}(\tau_j - 0) + \beta_j, \quad \tau_j > t_0. \tag{4} \]

We also assume that the following hypothesis holds

(a4) \( f: [t_0, \infty) \rightarrow R \) is a Lebesgue measure locally essentially bounded function, \( \varphi: (-\infty, t_0) \rightarrow R \) is a Borel measurable bounded function.

**Definition.** A function \( x: R \rightarrow R \) with absolutely continuous on each interval \([\tau_j, \tau_{j+1})\) derivative \( \dot{x} \) is called a solution of problem (3), (4) if it satisfies equation (3) for almost every \( t \in [t_0, \infty) \), and equalities (4) hold.

**Definition.** For each \( s \geq 0 \) denote by \( X_0(t, s) \) and \( X(t, s) \) the solutions of the problem

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = 0, \quad t \geq s, \quad x(t) = 0, \quad t < s, \tag{5} \]

\[ x(\tau_j) = A_jx(\tau_j - 0), \quad \dot{x}(\tau_j) = B_j\dot{x}(\tau_j - 0), \quad \tau_j > s \tag{6} \]

with initial conditions \( x(s) = 1, \dot{x}(s) = 0 \) for \( X_0(t, s) \) and \( x(s) = 0, \dot{x}(s) = 1 \) for \( X(t, s) \).

\( X(t, s) \) is called a fundamental function of equation (1), (2). We assume \( X(t, s) = 0, 0 \leq t < s \).
**THEOREM 1.** Let $(a1)-(a4)$ hold. Then there exists one and only one solution of problem $(3),(4)$ that can be presented in the form

\[ x(t) = X_0(t,t_0)\alpha_0 + X(t,t_0)\beta_0 + \int_0^t X(t,s)f(s)\,ds \]

\[ -\sum_{k=1}^m \int_0^t X(t,s)a_k(s)\varphi(g_k(s))\,ds \]

\[ + \sum_{\tau_j > t_0} X_0(t,\tau_j)\alpha_j + \sum_{\tau_j > t_0} X(t,\tau_j)\beta_j, \tag{7} \]

where \( \varphi(g_k(s)) = 0 \), if \( g_k(s) > t_0 \).

The proof of this result is presented in Appendix.

3. NONOSCILLATION CRITERIA

**Definition.** We will say that equation $(1),(2)$ has a positive solution for \( t > t_0 \) if there exist an initial function \( \varphi \) and numbers \( \alpha_0 \) and \( \beta_0 \) such that the solution of initial value problem $(3),(4)$, with \( f = 0 \), \( \alpha_j = \beta_j = 0 \), \( j = 1,2,\ldots \), is positive for \( t > t_0 \).

In this case equation $(1),(2)$ is nonoscillatory. Otherwise the equation is called oscillatory.

Together with equation $(1),(2)$ consider the following second order delay differential inequality

\[ \ddot{y}(t) + \sum_{k=1}^m a_k(t)y(g_k(t)) \leq 0, \quad t \geq 0, \tag{8} \]

\[ y(\tau_j) = A_jy(\tau_j - 0), \dot{y}(\tau_j) = B_j\dot{y}(\tau_j - 0), \quad j = 1,2,\ldots. \tag{9} \]

The following theorem establishes nonoscillation criteria.

**Theorem 2.** Suppose \( a_k(t) \geq 0 \), \( k = 1,\ldots,m \), \( A_j > 0 \), \( B_j > 0 \), \( j = 1,2,\ldots \). Then the following statements are equivalent:

1. There exists \( t_1 \geq 0 \) such that the inequality $(8),(9)$ has a positive solution for \( t > t_1 \).
2. There exists \( t_2 \geq 0 \) such that the following inequality

\[ \dot{u}(t) + \prod_{t_2 < \tau_j \leq t} B_j/A_ju^2(t) \]

\[ + \sum_{k=1}^m \prod_{t_2 < \tau_j \leq t} A_j/B_j \]

\[ \times \prod_{\tau_j > t_2} \prod_{\tau_j \leq s} B_j/A_ju(s) \, ds \leq 0 \tag{10} \]
has a locally absolutely continuous solution, where the sum $\Sigma'$ contains only those terms for which $g_k(t) \geq t_2$.

(3) There exists $t_3 \geq 0$ such that $X(t, s) > 0$, $t > s \geq t_3$.

(4) There exists $t_4 \geq 0$ such that equation (1), (2) has a positive solution for $t > t_4$.

Scheme of the Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Let $y(t)$ be a positive solution of inequality (8), (9) for $t > t_2$. Then there exists a point $t$ such that $g_k(t) \geq t_1$ if $t \geq t_2$. We can assume without loss of generality that $y(t_2) = 1$.

Denote $u(t) = \prod_{t_2 < \tau \leq t} B_j/A_j \hat{y}(t)/\hat{y}(t)$, if $t \geq t_2$ and $u(t) = 0$, if $t < t_2$. Then $u$ is a locally absolutely continuous function. Equalities $\hat{y}(t) - \prod_{t_2 < \tau \leq t} B_j/A_j u(t) = 0$, $y(t_2) = 1$ and (9) imply that

$$y(t) = \prod_{t_2 < \tau \leq t} A_j \exp \left[ \int_{t_2}^{t} \prod_{t_2 < \tau \leq s} B_j/A_j u(s) \, ds \right],$$

$$\dot{y}(t) = \prod_{t_2 < \tau \leq t} B_j u(t) \exp \left[ \int_{t_2}^{t} \prod_{t_2 < \tau \leq s} B_j/A_j u(s) \, ds \right],$$

$$\ddot{y}(t) = \prod_{t_2 < \tau \leq t} B_j \dot{u}(t) \exp \left[ \int_{t_2}^{t} \prod_{t_2 < \tau \leq s} B_j/A_j u(s) \, ds \right]$$

$$+ \sum_{t_2 < \tau \leq t} B_j^2/A_j \dot{u}^2(t) \exp \left[ \int_{t_2}^{t} \prod_{t_2 < \tau \leq s} B_j/A_j u(s) \, ds \right], \quad t \geq t_2.$$  \hfill (11)

We substitute (11) into (8) and obtain after carrying the exponential out of the brackets the following inequality

$$\prod_{t_2 < \tau \leq t} B_j \exp \left[ \int_{t_2}^{t} \prod_{t_2 < \tau \leq s} B_j/A_j u(s) \, ds \right] \left[ \dot{u}(t) + \prod_{t_2 < \tau \leq t} B_j/A_j u^2(t) \right]$$

$$+ \sum_{k=1}^{m} \prod_{t_2 < \tau \leq t} A_j/A_{g_k} \prod_{g_k < \tau \leq t} A_j^{-1} \dot{a_k}(t) \exp \left[ - \int_{g_k}^{t} \prod_{g_k < \tau \leq s} B_j/A_j u(s) \, ds \right]$$

$$+ \sum_{k=1}^{m} a_k(t) y(g_k(t)) \leq 0,$$  \hfill (12)

where the sum $\Sigma''$ contains only those terms for which $t_1 \geq g_k(t) < t_2$. Since $y(t) \geq 0$ for $t \geq t_1$ and $a_k(t) \geq 0$ then (12) implies inequality (10).
Consider the initial value problem
\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = f(t), \quad t \geq t_2, \]  
\[ x(t) = 0, \quad t < t_2, \quad x(t_2) = \dot{x}(t_2) = 0 \]  
with impulsive conditions (2). Denote
\[ z(t) = \dot{x}(t) - \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) x(t), \]  
where \( x \) is the solution of (13), (2) and \( u \) is a solution of (10). From (14), (2) we obtain
\[ x(t) = \int_{t_2}^{t} \exp \left( \int_{t_2}^{\tau} \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) \, d\tau \right) \prod_{s < \tau \leq t} A_j z(s) \, ds, \]  
\[ \dot{x}(t) = z(t) + \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) \int_{t_2}^{t} \exp \left( \int_{t_2}^{\tau} \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) \, d\tau \right) \times \prod_{s < \tau \leq t} A_j z(s) \, ds, \]  
\[ \ddot{x}(t) = \ddot{z}(t) + \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) z(t) \]  
\[ + \left( \dot{u}(t) + \prod_{t_2 < t \leq t_\tau} B_j/A_j u^2(t) \right) \prod_{t_2 < t \leq t_\tau} B_j/A_j \]  
\[ \times \int_{t_2}^{t} \exp \left( \int_{t_2}^{\tau} \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) \, d\tau \right) \prod_{s < \tau \leq t} A_j z(s) \, ds. \]  
Substituting \( x, \dddot{x} \) into (13) we obtain
\[ \ddot{x}(t) + \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) z(t) \]  
\[ + \left( \dot{u}(t) + \prod_{t_2 < t \leq t_\tau} B_j/A_j u^2(t) \right) \prod_{t_2 < t \leq t_\tau} B_j/A_j \]  
\[ \times \int_{t_2}^{t} \exp \left( \int_{t_2}^{\tau} \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) \, d\tau \right) \prod_{s < \tau \leq t} A_j z(s) \, ds \]  
\[ + \sum_{k=1}^{m} a_k(t) \int_{t_2}^{t} \exp \left( \int_{s}^{\tau(t)} \prod_{t_2 < t \leq t_\tau} B_j/A_j u(t) \, d\tau \right) \]  
\[ \times \prod_{s < \tau \leq \tau(t)} A_j z(s) \, ds = f(t). \]  
(16)
Equality (14) implies \(z(t_2) = 0\) and
\[
z(\tau) = \dot{x}(\tau) - \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(\tau_i) x(\tau_i)
= B_i \dot{x}(\tau_i - 0) - B_i / A_i \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(\tau_i) A_i x(\tau_i - 0)
= B_i z(\tau_i - 0).
\]
Hence we can rewrite equation (16) in the form
\[
\dot{z}(t) + \prod_{t_2 < \tau \leq t} B_i / A_i u(t) z(t)
= - \left[ \dot{u}(t) + \prod_{t_2 < \tau \leq t} B_i / A_i u^2(t)
+ \sum_{k=1}^{m} \prod_{t_2 < \tau_i \leq \tau} A_i / B_i \prod_{g_i(t) < \tau_i \leq \tau} A_j^{-1} a_k(t)
\times \exp \left\{ - \int_{g_i(t)}^{t} \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(s) \, ds \right\} \right]
\times \prod_{t_2 < \tau \leq t} B_i / A_i \int_{t_2}^{t} \exp \left\{ \int_{t_2}^{\tau} \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(\tau_i) \, d\tau_i \right\} \prod_{t_2 < \tau_i \leq \tau} A_i z(s) \, ds
+ \sum_{k=1}^{m} a_k(t) \int_{g_i(t)}^{t} \exp \left\{ \int_{g_i(t)}^{\tau} \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(\tau_i) \, d\tau_i \right\}
\times \prod_{s < \tau_i \leq g_i(t)} A_j z(s) \, ds + f(t),
\]
\[
z(t_2) = 0, \quad z(\tau_j) = B_j z(\tau_j - 0), \quad \tau_j > t_2. \quad (17)
\]
Then equation (17) is equivalent to the following equation
\[
z = Hz + p, \quad (18)
\]
where
\[
(Hz)(t) = \int_{t_2}^{t} \exp \left\{ - \int_{t_2}^{\tau} \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(\tau_i) \, d\tau_i \right\}
\times \prod_{s < \tau_i \leq t} B_i \left[ - \dot{u}(s) + \prod_{t_2 < \tau_i \leq s} B_i / A_i u^2(s)
+ \sum_{k=1}^{m} \prod_{t_2 < \tau_i \leq s} A_i / B_i \prod_{g_i(s) < \tau_i \leq s} A_j^{-1} a_k(s) \right]
\times \prod_{t_2 < \tau_i \leq \tau} B_i / A_i \prod_{g_i(t) < \tau_i \leq \tau} A_j^{-1} a_k(t)
\times \exp \left\{ - \int_{g_i(t)}^{\tau} \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(s) \, ds \right\}
\times \prod_{t_2 < \tau_i \leq \tau} A_i z(s) \, ds
+ \sum_{k=1}^{m} a_k(t) \int_{g_i(t)}^{\tau} \exp \left\{ \int_{g_i(t)}^{\tau} \prod_{t_2 < \tau_i \leq \tau} B_i / A_i u(\tau_i) \, d\tau_i \right\}
\times \prod_{s < \tau_i \leq g_i(t)} A_j z(s) \, ds + f(t),
\]
\[
\times \exp \left( - \int_{\tau_{t_2} \leq \tau \leq \tau} B_j / A_j u(\tau) \, d\tau \right)
\]
\[
\times \prod_{s \leq \tau \leq \tau} \exp \left( - \int_{s \leq \tau \leq s} B_j / A_j u(\xi) \, d\xi \right)
\]
\[
\times \prod_{s \leq \tau \leq \tau} A_j z(\tau) \, d\tau
\]
\[
+ \sum_{k=1}^{m'} a_k(s) \int_{s \leq \tau \leq s} \exp \left( - \int_{s \leq \tau \leq s} B_j / A_j u(\xi) \, d\xi \right)
\]
\[
\times \prod_{s \leq \tau \leq \tau} A_j z(\tau) \, d\tau \, ds.
\]
(19) 

\[
p(t) = \int_{t_2}^{t} \exp \left( - \int_{t_2}^{t} \prod_{s \leq \tau \leq s} B_j / A_j u(\tau) \, d\tau \right) \prod_{s \leq \tau \leq s} B_j f(s) \, ds.
\]
(20)

Inequality (10) yields that if \( z(t) \geq 0 \) for \( t \geq t_2 \) then \( (Hz)(t) \geq 0 \) for \( t \geq t_2 \) (i.e. operator \( H \) is positive).

Denote
\[
c(t) = \dot{u}(t) + \prod_{t_2 < \tau \leq t} B_j / A_j u^2(t)
\]
\[
+ \sum_{k=1}^{m'} \prod_{t_2 < \tau \leq t} A_j / B_j \prod_{g_k(t) < \tau \leq s} A_j^{-1} a_k(s)
\]
\[
\times \exp \left( - \int_{g_k(t) < \tau \leq s} B_j / A_j u(s) \, ds \right).
\]

Since \( u \) is absolutely continuous in each finite interval, we have \( c \in L_{[t_2, b]} \) for every \( b > t_2 \), where \( L_{[a, b]} \) is the space of all Lebesgue integrable functions on \([a, b]\) with the usual integral norm.

For \( t \in [t_2, b] \) we have
\[
|(Hz)(t)| \leq \exp \left( - \int_{t_2}^{b} \prod_{s \leq \tau \leq t} B_j / A_j u(\tau) \, d\tau \right)
\]
\[
\times \int_{t_2}^{t} \prod_{t_2 < \tau \leq t} B_j \left( \prod_{t_2 < \tau \leq t} B_j / A_j |c(s)| + \sum_{k=1}^{m} |a_k(s)| \right)
\]
\[
\times \int_{t_2}^{s} \prod_{t_2 < \tau \leq s} A_j |z(\tau)| \, d\tau \, ds.
\]
\[
\begin{align*}
&= \exp \left( \int_{t_2}^{b} \prod_{\tau \leq \sigma \leq \tau} B_j A_j \mu(\tau) \, d\tau \right) \\
&\quad \times \int_{t_2}^{l} \left( \int_{\tau}^{l} \prod_{\tau \leq \sigma \leq \tau} B_j \left( \prod_{t_2 \leq \tau \leq s} B_j/A_j \right) c(s) \right) \\
&\quad \quad + \sum_{k=1}^{m} |a_k(s)| \prod_{\tau \leq \sigma \leq \tau} A_j \, ds \| z(\tau) \| d\tau.
\end{align*}
\]

The kernel of the Volterra integral operator \( H \) is bounded in each square \([t_2, b] \times [t_2, b]\), hence [13] \( H: L_{[t_2, b]} \to L_{[t_2, b]} \) is a weakly compact operator and its square is a compact operator. Therefore [14] the spectral radius of such an integral Volterra operator \( r(H) = 0 \).

Thus if in (18) \( p(t) \geq 0 \) for \( t \geq t_2 \) then
\[
z(t) = p(t) + (Hp)(t) + (H^2p)(t) + \ldots \geq 0 \quad \text{for } t \geq t_2.
\]

If \( f(t) \geq 0 \) for \( t \geq t_2 \) then by (20) \( p(t) \geq 0 \) for \( t \geq t_2 \). Hence for equation (16) we have the following: if \( f(t) \geq 0 \) for \( t \geq t_2 \) then \( z(t) \geq 0 \) for \( t \geq t_2 \).

Therefore (15) implies that the solution of (13), (2) is nonnegative for any nonnegative right-hand side \( f \).

The solution of this equation can be written in the form (7),
\[
x(t) = \int_{t_2}^{l} X(t, s) f(s) \, ds.
\] (21)

As was shown above, \( f(t) \geq 0, \ t \geq t_2 \), implies \( x(t) \geq 0, \ t \geq t_2 \). Consequently, the kernel of the integral operator (21) is nonnegative. Therefore \( X(t, s) \geq 0 \) for \( t \geq s \geq t_2 \). A function \( x(t) = X(t, s) \) is a nonnegative solution of (5) for \( t \geq s \). Suppose for certain \( t_3 > s \) \( x(t_3) = 0 \) and \( x(t) > 0, \ s < t < t_3 \). Then \( x(t) < 0 \). By (7) we have for \( t > t_3 \)
\[
x(t) = X(t, t_3) x(t_3) - \sum_{k=1}^{m} \int_{t_3}^{t} X(t, s) a_k(s) \varphi(g_k(s)) \, ds,
\]
where \( \varphi(t) = x(t), \ t < t_3 \). Therefore \( x(t) < 0, \ t > t_3 \) and we get a contradiction.

Hence the strict inequality \( x(t) = X(t, s) > 0, \ t > s \geq t_2 \) holds.

(3) \(\Rightarrow\) (4) The function \( x(t) = X(t, t_2) \) is a positive solution of equation (1), (2).

Implication (4) \(\Rightarrow\) (1) is evident.
Corollary. Equation (1), (2) is nonoscillatory if and only if inequality (8), (9) is nonoscillatory.

Remark. (1) If there exists a nonnegative solution of inequality (10) for \( t \geq t_0 \) then statements (1), (3), and (4) of the theorem are also valid for \( t \geq t_0 \).

(2) A generalized Riccati equation for a delay differential equation without impulses arose for the first time in [12].

(3) If inequality (10) has a nonnegative solution then equation (1), (2) has a positive solution with a nonnegative derivative.

4. COMPARISON THEOREMS

Suppose in this section that \( A_j > 0, B_j > 0 \).

Theorem 2 can be employed for comparison of oscillation properties. To this end, together with equation (1), (2) consider the following equation with impulsive conditions (2):

\[ \ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) = 0, \quad t \geq 0. \quad (22) \]

Suppose (a2) and (a3) hold for equation (22) and denote by \( Y(t, s) \) a fundamental function of this equation.

A proof of the following theorem and its corollary is similar to the proof of the corresponding theorem (Theorem 2) for equations without impulses [12].

Theorem 3. Suppose \( a_j(t) \geq 0, a_j(t) \geq b_j(t) \) for \( t \geq t_0 \) and inequality (8) has a solution for \( t \geq t_0 \). Then equation (22), (2) has a positive solution for \( t > t_0 \) and \( Y(t, s) > 0, t > s \geq t_0 \).

Denote \( a^+ = \max(a, 0) \).

Corollary. (1) If the inequality

\[ \ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t)x(g_k(t)) \leq 0 \quad (23) \]

with impulsive conditions (2) is nonoscillatory then (1), (2) is nonoscillatory.
If the inequality
\[ \dot{u}(t) + \prod_{t_0 \leq \tau_j \leq t} B_j/A_j u^2(t) \]
\[ + \sum_{k=1}^m \prod_{t_0 \leq \tau_j \leq t} A_j/B_j \prod_{g_k(t) \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \]
\[ \times \exp \left\{-\int_{g_k(t)}^t \frac{B_j/A_j u(s) ds}{A_j^{-1}a_k^+(t)} \right\} \leq 0 \] 
(24)
has an absolutely continuous solution for \( t \geq t_0 \), where the sum contains only those terms for which \( g_k(t) \geq t_0 \) then equation (1), (2) has a positive solution for \( t > t_0 \) and \( X(t,s) > 0, t > s \geq t_0 \).

Now let us compare the solutions of problem (3), (4) and the following one
\[ \ddot{y}(t) + \sum_{k=1}^m b_k(t) \dot{y}(g_k(t)) = r(t), \quad t \geq t_0, \quad y(t) = \psi(t), \quad t < t_0 \]
(25)
with impulsive conditions (4). Denote by \( x(t) \) and \( y(t) \) the solution of (3), (4) and (25), (4), respectively.

**Theorem 4.** Suppose there exists a solution of (8) for \( t \geq t_0, x(t) > 0 \) for \( t \geq t_0 \) and \( a_k(t) \geq b_k(t) \geq 0, r(t) \geq f(t) \) for \( t \geq t_0, \phi(t) \geq \psi(t) \) for \( t < t_0 \).

Then \( y(t) \geq x(t) \) for \( t \geq t_0 \).

A proof is similar to the proof of Theorem 4 in [12].

### 5. Explicit Nonoscillation and Oscillation Criteria

We will employ Corollary of Theorem 3 to obtain explicit sufficient conditions for nonoscillation and suppose in this section that \( A_j > 0, \ B_j > 0 \).

**Theorem 5.** Suppose for some \( t_0 > 0, 0 < q < 1, r > -1, m > 0, M > 0 \) at least one of the following conditions holds:
\[ \sup_{t \geq t_0} \left| \prod_{t_0 \leq \tau_j \leq t} B_j/A_j - 1 \right| \leq q, \]
\[ \sup_{t \geq t_0} t^2 \sum_{k=1}^m \prod_{g_k(t) \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \left[ \frac{g_k(t)}{t} \right]^{1-q/2} \leq \frac{(1-q)^2}{4}. \]
\[ m' \leq \sup_{t \geq t_0} \prod_{t_0 < \tau_j \leq t} B_j/A_j \leq M't', \quad (26) \]

\[ \sup_{t \geq t_0} \sum_{k=1}^{m} \prod_{k \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \leq \frac{m(1+r)^2}{4M}. \quad (27) \]

Then equation (1), (2) has a positive solution for \( t > t_0 \) with a nonnegative derivative.

Proof. Suppose (26) holds. We will show that a function \( 1/2t \) is a solution of inequality (24). To this end substitute this function into the left hand side of the inequality and consider a function

\[ h(t) = -\frac{1}{2t^2} + \prod_{t_0 < \tau_j \leq t} B_j/A_j \frac{1}{4t^2} \]

\[ + \sum_k \prod_{k \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \]

\[ \times \exp \left( -\int_{g_k(t)}^{t} \prod_{k \leq \tau_j \leq s} B_j/A_j \frac{1}{2s} \, ds \right). \]

Denote \( \sup_{t \geq t_0} \prod_{t_0 < \tau_j \leq t} B_j/A_j = \alpha \). Then \( 1 - q \leq \alpha \leq 1 + q \). Hence

\[ h(t) \leq -\frac{1}{2t^2} + (1 + q) \frac{1}{4t^2} + \frac{1}{1 - q} \sum_k \prod_{k \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \]

\[ \times \exp \left( -\int_{g_k(t)}^{t} (1 - q) \frac{1}{2s} \, ds \right) \]

\[ = -\frac{1}{4t^2} (1 - q) + \frac{1}{1 - q} \sum_k \prod_{k \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \left( \frac{g_k(t)}{t} \right)^{(1-q)/2} \]

\[ = -\frac{1}{t^2(1 - q)} \]

\[ \times \left[ \frac{(1 - q)^2}{4} - t^2 \sum_k \prod_{k \leq \tau_j \leq t} A_j^{-1}a_k^+(t) \left( \frac{g_k(t)}{t} \right)^{(1-q)/2} \right] \leq 0. \]

Then \( 1/2t \) is a nonnegative solution of the inequality (24), therefore equation (1), (2) has a positive solution with nonnegative derivative.

If (27) holds then \( u = 1 + r/2Mt^{1+r} \) is a nonnegative solution of (24).

Actually, substitute this function into the left hand side of (24) and denote
by \( h(t) \) the expression obtained. We have

\[
h(t) \leq -\frac{(1 + r)^2}{2Mt^{2+r}} + \frac{(1 + r)^2}{4M^2t^{2+2r}} \prod_{t_0 < \tau_j \leq t} B_j/A_j \\
+ \sum_{k=1}^{m} \prod_{g_k(t) < \tau_j \leq t} A_j^{-1} \prod_{t_0 < \tau_j \leq t} A_j/B_j a_k^+(t) \\
\leq -\frac{(1 + r)^2}{2Mt^{2+r}} + \frac{(1 + r)^2}{4M^2t^{2+2r}} + \sum_{k=1}^{m} \prod_{g_k(t) < \tau_j \leq t} A_j^{-1} a_k^+(t) \frac{1}{mt} \\
= -\frac{1}{mt^{2+r}} \left( \frac{m(1 + r)^2}{4M} - t^2 \sum_{k=1}^{m} \prod_{g_k(t) < \tau_j \leq t} A_j^{-1} a_k^+(t) \right) \leq 0.
\]

**Corollary 1.** Suppose

\[ a_k(t) \leq 0, \quad mt^r \leq \sup_{t \geq t_0} \prod_{t_0 < \tau_j \leq t} B_j/A_j \leq Mt^r, \]

\[ r > -1, \quad m > 0, \quad M > 0. \]

Then equation (1), (2) has a positive solution with nonnegative derivative.

**Corollary 2.** Suppose for some \( t_0 \geq 0, \ 0 < q < 1, \ r > -1, \ m > 0, \ M > 0 \) at least one of the following conditions holds:

1. \( \sup_{t \geq t_0} \prod_{t_0 < \tau_j \leq t} B_j/A_j - 1 \leq q, \ sup_{t \geq t_0} t^2a^+(t) \leq (1 - q)^2/4. \)
2. \( mt^r \leq \sup_{t \geq t_0} \prod_{t_0 < \tau_j \leq t} B_j/A_j \leq Mt^r, \ sup_{t \geq t_0} t^2a^+(t) \leq m(1 + r)^2/4M. \)

Then the ordinary differential equation

\[
\ddot{x}(t) + a(t)x(t) = 0
\]

with impulsive conditions (2) has a positive solution with a nonnegative derivative.

**Example 1.** Consider the delay differential equation

\[
\ddot{x}(t) + \frac{1}{2t^2}x(t - \delta) = 0
\]

with impulsive conditions

\[
x(j) = \frac{j}{j + 1}x(j - 0), \quad \dot{x}(j) = \dot{x}(j - 0), \quad j = 1, 2, \ldots; \quad (30)
\]
or

\[ x(j) = x(j - 0), \quad \dot{x}(j) = \frac{j + 1}{j} \dot{x}(j - 0). \tag{31} \]

Then equations (29), (30) and (29), (31) are nonoscillatory.

Actually, for equation (29), (30)

\[ t \leq \prod_{t_0 < \tau_j \leq t} B_j/A_j \leq t + 1 \leq t \left(1 + \frac{1}{t_0}\right). \]

Then the first inequality of (27) holds with \( r = 1, m = 1, M = 1 + 1/t_0 \).

The left-hand side of the second inequality of (27) is less than 0.5 \((1 + (1/t_0 - \delta)^{[\delta]+1})\) and tends to 0.5 as \( t_0 \to \infty \). The right-hand side of this inequality is \( t_0/1 + t_0 \) and tends to 1. Then for sufficiently large \( t_0 \) inequalities (27) hold. Hence equation (29), (30) is nonoscillatory. Similarly, equation (29), (31) is nonoscillatory.

Remark. All the solutions of equation (29) without impulses are oscillatory [15].

Now we will obtain some other nonoscillation conditions.

Denote

\[ b(t) = \sum_{k=1}^{m} \prod_{k \leq \tau_{j}} A_j/B_j \prod_{\kappa(k) < \tau_j \leq t} A_j^{-1}a_k^*(t). \]

Theorem 6. Suppose \( \Pi_{1 \leq j \leq t} B_j/A_j \leq 1 \). If for \( t > t_1 \) there exists positive solution of the nonimpulsive ordinary differential equation

\[ \dot{x}(t) + b(t)x(t) = 0, \]

then for \( t > t_1 \) there exists a positive solution of equation (1), (2).

Proof. Suppose \( u \) is a solution of Riccati inequality

\[ \hat{u}(t) + u^2(t) + b(t) \leq 0, \quad t \geq t_1. \]

The \( u \) is also a solution of inequality (24). Therefore (1), (2) has a positive solution for \( t \geq t_1 \).

Corollary. Suppose

\[ \lim_{t \to \infty} \prod_{\tau_j \leq t} B_j/A_j \leq 1, \quad \lim_{t \to \infty} A_j \geq 1, \tag{32} \]

and
\[ \limsup_{t \to \infty} t^2 \sum_{k=1}^{m} \prod_{\tau_j \leq t} A_j/B_j a_k^*(t) \leq \frac{1}{4}. \tag{33} \]

Then equation (1), (2) is nonoscillatory.

**Example 2.** Consider the delay differential equation
\[ \ddot{x}(t) + \frac{\alpha}{t^2} x(g(t)) = 0 \tag{34} \]
with impulsive conditions
\[ x(j) = \frac{(j+1)^k}{j^k} x(j-0), \quad \dot{x}(j) = \dot{x}(j-0), \quad j = 1, 2, \ldots; k > 0, \tag{35} \]
or
\[ x(j) = x(j-0), \quad \dot{x}(j) = \frac{j^k}{(j+1)^k} \dot{x}(j-0). \tag{36} \]

Here conditions (32) hold, \( \prod_{\tau_j \leq t} A_j/B_j \leq Mt^k \), where \( M = (1 + (1/t_1))^k \to 1 \) as \( t_1 \to \infty \). Hence the left hand side of (33) is less than or equal to \( \alpha M(t^{2+k}/t^2) \). Therefore if \( \beta = 2 + k \) and \( \alpha < 1/4 \) or \( \beta < 2 + k \), then equations (34), (35) and (34), (36) are nonoscillatory.

Now we turn to the oscillation problem.

**Theorem 7.** Suppose \( a_k(t) \geq 0 \) and there exist \( M > 0 \), \( \delta > 0 \) such that
\[ \sup_{t \geq 0} \prod_{\tau_j \leq t} B_j/A_j \leq M, \quad t - g_k(t) \leq \delta. \tag{37} \]

If for some \( k \), \( k = 1, 2, \ldots, m \),
\[ \int_{\tau_j \leq t}^{\infty} A_j/B_j \prod_{g_k(t) < \tau_j \leq t} A_j^{-1} a_k(t) \, dt = \infty \quad \text{and} \quad \int_{\tau_j \leq t}^{\infty} B_j/A_j \, dt = \infty \tag{38} \]
then all the solutions of (1), (2) are oscillatory.

**Proof.** Suppose equation (1), (2) has a positive solution. Then for some \( t_1 \geq 0 \) inequality (10) has a solution \( u(t) \). This function is nonincreasing therefore \( u(t) \leq u(t_1) \) and either there exists a finite limit of \( u(t) \) as \( t \to \infty \) or \( \lim_{t \to \infty} u(t) = -\infty \). We will see that the latter case is impossible.
Inequality (10) implies 
\[ \dot{u}(t) + \prod_{t_{j} \leq t} B_{j} / A_{j} u^{2}(t) \leq 0. \]
Then \(-1/u(t) + 1/u(t_{1}) + \int_{t_{1}}^{t} \prod_{t_{j} \leq s} B_{j} / A_{j} \ ds \leq 0.\) Hence \(\lim_{t \to \infty} u(t)\) is a finite number.

Further, from inequality (10) we have
\[ \dot{u}(t) + \prod_{t_{j} \leq t} A_{j} / B_{j} \prod_{g_{k}(t) < t_{j} \leq t} A_{j}^{-1} a_{k}(t) \exp\{-\delta Mu(t_{1})\} \leq 0. \]

Hence
\[ u(t) - u(t_{1}) + \exp\{-\delta Mu(t_{1})\} \int_{t_{1}}^{t} \prod_{t_{j} \leq t} A_{j} / B_{j} \prod_{g_{k}(t) < t_{j} \leq t} A_{j}^{-1} a_{k}(t) \ dt \leq 0, \]
which contradicts to (38).

**Corollary 1.** Suppose \(a_{k}(t) \geq 0\), condition (37) holds, there exist \(A > 0\), \(\tau > 0\) such that \(A_{j} \leq A, \ t_{j+1} - t_{j} \geq \tau.\) If for some \(k, k = 1, 2, \ldots, m,\)
\(\int_{t_{1}}^{t} A_{j} / B_{j} a_{k}(t) \ dt = \infty\) and \(\int_{t_{1}}^{t} B_{j} / A_{j} \ dt = \infty\) then all the solutions of (1), (2) are oscillatory.

The inequality
\[ \int_{t_{1}}^{t} \prod_{t_{j} \leq t} A_{j} / B_{j} \prod_{g_{k}(t) < t_{j} \leq t} A_{j}^{-1} a_{k}(t) \ dt \geq A^{(-[k/\tau] + 1)} \int_{t_{1}}^{t} \prod_{t_{j} \leq t} A_{j} / B_{j} a_{k}(t) \ dt = \infty, \]
is the following corollary.

**Corollary 2.** Suppose \(a(t) \geq 0, \sup_{t \geq 0} \prod_{t_{j} \leq t} B_{j} / A_{j} \leq M,\)
\[ \int_{t_{1}}^{t} \prod_{t_{j} \leq t} A_{j} / B_{j} a(t) \ dt = \infty\) and \(\int_{t_{1}}^{t} B_{j} / A_{j} \ dt = \infty.\)

Then all the solutions of ordinary differential equation (28), (2) are oscillatory.

**Example 3.** Consider the delay differential equation
\[ \dot{x}(t) + \frac{1}{4t^{2}} x(t - \delta) = 0 \]  
(39)
with impulsive conditions
\[ x(j) = \frac{j + 1}{j} x(j - 0), \quad \dot{x}(j) = \dot{x}(j - 0), \quad j = 1, 2, \ldots. \]  
(40)
or

\[ x(j) = x(j - 0), \quad \dot{x}(j) = \frac{j}{j + 1} \dot{x}(j - 0). \quad (41) \]

Then all solutions of (39), (40) and (39), (41) are oscillatory.

Actually

\[ \int_{t_1}^{\infty} \prod_{t_1 < j \leq t} A_j/B_j \prod_{t - \delta < j \leq t} A_j^{-1}a(t) \, dt \geq \left( 1 - \frac{1}{t_1 + 1} \right)^{[\delta] + 1} \int_{t_1}^{\infty} \frac{1}{4t^2} \, dt = \infty \]

and

\[ \int_{t_1}^{\infty} \prod_{t_1 < j \leq t} B_j/A_j \, dt \geq \int_{t_1}^{\infty} \frac{1}{t + 1} \, dt = \infty. \]

**Remark.** For ordinary differential equation (\( \delta = 0 \)) with impulsive conditions (41) this result was obtained in [11] in a different way. One can also obtain it by Theorem 2 in [10]. Equation (39) without impulses is nonoscillatory for all \( \delta \geq 0 \) [15, 16].

More interesting results we will be able obtain under the assumption \( A_j = B_j \), it means that impulsive conditions are

\[ x(\tau_j) = A_j x(\tau_j - 0), \quad \dot{x}(\tau_j) = A_j \dot{x}(\tau_j - 0). \quad (42) \]

In this case the Riccati inequality (10) is

\[ \dot{u}(t) + u^2(t) + \sum_{k=1}^{m'} \prod_{g_{k}(t) < j \leq t} A_j^{-1}a_k(t) \exp \left( -\int_{g_{k}(t)}^{t} u(s) \, ds \right) \leq 0. \quad (43) \]

Consider the following delay differential equation without impulses

\[ \ddot{x}(t) + \sum_{k=1}^{m} \prod_{g_{k}(t) < j \leq t} A_j^{-1}a_k(t) x(g_{k}(t)) = 0. \quad (44) \]

**Theorem 8.** Suppose \( a_j(t) \geq 0, \ A_j > 0 \). Then equation (1), (42) is oscillatory (nonoscillatory) if and only if equation (44) is oscillatory (nonoscillatory).

**Proof.** Theorem 2 implies that nonoscillation of (1), (42) is equivalent to the existence of nonnegative solution of inequality (43). However the last condition is equivalent to nonoscillation of nonimpulsive equation (44) (Theorem 1 in [12]).
As an application of the previous theorem and oscillation results, obtained in [12, 16] for nonimpulsive equations we will present the following theorems.

**Theorem 9.** Suppose \( A_j > 0, \)

\[
\limsup_{t \to \infty} \sum_{k=1}^{m} \prod_{g_k(t) < \tau \leq t} A_j^{-1} a_k(t) \sqrt{t^2 g_k(t)} \ln g_k(t) \ln t \leq \frac{1}{4}.
\]

Then equation (1), (42) is nonoscillatory.

**Theorem 10.** Suppose \( a_k(t) \geq 0, A_j > 0 \) and there exists \( \delta > 0 \) such that \( t - g_k(t) < \delta. \) Then equation (1), (42) is oscillatory (nonoscillatory) if and only if the following nonimpulsive ordinary differential equation:

\[
\ddot{x}(t) + \sum_{k=1}^{m} \prod_{g_k(t) < \tau \leq t} A_j^{-1} a_k(t) = 0
\]

is oscillatory (nonoscillatory).

**Theorem 11.** Suppose \( a_k(t) \geq 0, A_j > 0, \) and for some \( c_k, 0 < c_k < 1 \) the following nonimpulsive ordinary differential equation

\[
\ddot{x}(t) + \sum_{k=1}^{m} \prod_{g_k(t) < \tau \leq t} A_j^{-1} a_k(t) c_k \frac{g_k(t)}{t} x(t) = 0
\]

is oscillatory. Then (1), (42) is also oscillatory.

### 6. Appendix: Representation of Solutions

Here we consider a more general \( n \)-dimensional problem and obtain the representation of solutions for it. Let \( R^n \) be the space of \( n \)-dimensional column vectors \( x = \text{col}(x_1, \ldots, x_n) \) with the norm \( \|x\| = \max_{1 \leq i \leq n} |x_i| \) by the same symbol \( \| \cdot \| \) we will denote the corresponding matrix norm. We consider the equation

\[
\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = f(t), \quad t \geq t_0, \quad x(t) = \varphi(t), \quad t < t_0,
\]

(45)
with the following initial and impulsive conditions.

$$
x(t_0) = x_0, \quad \dot{x}(t_0) = \beta_0, \quad x(t_j) = A_j x(t_j - 0) + \alpha_j, \quad \dot{x}(t_j) = B_j \dot{x}(t_j - 0) + \beta_j, \quad t_j > t_0. \quad (46)
$$

Here \(x, \varphi\) and \(f\) are vector valued functions, \(a_k\) are matrix valued functions, \(A_j, B_j\) are matrices. We assume that all the components of vector- or matrix-valued functions satisfy the assumptions described in Section 2. Similarly we introduce the matrix-valued fundamental functions \(X_0(t, s)\) and \(X(t, s)\), only the unit in the initial conditions is changed by the unit matrix, zero is changed by the zero matrix.

**Lemma 1.** Suppose \((a1), (a3)\) hold and the components of \(a_k\) are Lebesgue measurable locally essentially bounded functions. Then the fundamental matrix \(X(t, s)\) of (45), (46) and its derivative in \(t\) \(X'(t, s)\) are bounded on any square \([t_0, b] \times [t_0, b]\).

**Proof.** Let \(t_{i-1} < s < t_i\). First, consider \(t \in [s, t_i]\). Then the derivative of the solution of the problem \((5), (6), x(s) = 0, \dot{x}(s) = I_n\) where \(I_n\) is a unit \(n \times n\) matrix, can be presented as

$$
X'(t, s) = I_n - \int_s^t \sum_{k=1}^m a_k(\eta) X(g_k(\eta), s) \, d\eta, \quad x(t) = 0, \quad t < s,
$$

therefore

$$
\|X'(t, s)\| \leq 1 + \int_s^t \sum_{k=1}^m \|a_k(\eta)\| \sup_{\xi \in [s, \eta]} \|X(\xi, s)\| \, d\eta. \quad (47)
$$

Denote \(y(t) = \sup_{\xi \in [s, t]} \|X'_\xi(\xi, s)\|\). Then \(X(s, s) = 0\) implies

$$
\sup_{\xi \in [s, \eta]} \|X(\xi, s)\| \leq y(\eta)(\eta - s),
$$

which together with (47) yields the following estimate

$$
y(t) \leq 1 + \int_s^t \sum_{k=1}^m \|a_k(\eta)\| y(\eta)(\eta - s) \, d\eta.
$$

By The Gronwall–Bellman inequality

$$
y(t) \leq \exp \left( \int_s^t \sum_{k=1}^m \|a_k(\eta)\| (\eta - s) \, d\eta \right).
$$
Therefore for the derivative of the fundamental function the same estimate is valid:

$$X'_i(t, s) \leq \exp \left\{ \int_{s}^{t} \sum_{k=1}^{m} \|a_k(\eta)\| (\eta - s) \, d\eta \right\}.$$  (48)

Let $\tau_{i-1} < s < \tau_i \leq t < \tau_{i+1}$. Then

$$X'_i(t, s) = X'_i(\tau_i, s) - \int_{\tau_i}^{t} \sum_{k=1}^{m} a_k(\eta) X(g_k(\eta), s) \, d\eta.$$ 

Hence, the inequality (48) and the impulsive condition $X'_i(\tau_i, s) = B_i X'_i(\tau_i - 0, s)$ imply the estimate.

$$\|X'_i(t, s)\| \leq (1 + \|B_i\|) \exp \left\{ \int_{s}^{\tau_i} \sum_{k=1}^{m} \|a_k(\eta)\| (\eta - s) \, d\eta \right\}$$

$$+ \int_{\tau_i}^{t} \sum_{k=1}^{m} \|a_k(\eta)\| \|X(g_k(\eta), s)\| \, d\eta.$$  (49)

Again we denote $y(t) = \sup_{\zeta \in [s, t]} \|X'_i(\zeta, x)\|$. After rewriting (49) in the form

$$y(t) \leq (1 + \|B_i\|) \exp \left\{ \int_{s}^{\tau_i} \sum_{k=1}^{m} \|a_k(\eta)\| (\eta - s) \, d\eta \right\}$$

$$+ \int_{\tau_i}^{t} \sum_{k=1}^{m} \|a_k(\eta)\| y(\eta)(\eta - s) \, d\eta$$

and applying the Gronwall–Bellman inequality we obtain

$$\|X'_i(t, s)\| \leq y(t)$$

$$\leq (1 + \|B_i\|) \exp \left\{ \int_{s}^{\tau_i} \sum_{k=1}^{m} \|a_k(\eta)\| (\eta - s) \, d\eta \right\}$$

$$\times \exp \left\{ \int_{\tau_i}^{t} \sum_{k=1}^{m} \|a_k(\eta)\| (\eta - s) \, d\eta \right\}$$

$$= (1 + \|B_i\|) \exp \left\{ \int_{s}^{t} \sum_{k=1}^{m} \|a_k(\eta)\| (\eta - s) \, d\eta \right\},$$

$$t \in [\tau_i, \tau_{i+1}).$$
Now let $\tau_{i-1} < s < \tau_{i} < ... < \tau_{j-1} < t < \tau_{j}$. By considering the derivative of the fundamental function $X(t, s)$ in the intervals $[\tau_{i+1}, \tau_{i+2}], ..., [\tau_{j-1}, \tau_{j}]$ and repeating the previous argument we obtain the estimate

$$\|X'(t, s)\| \leq \prod_{s < \tau_{j} \leq t} (1 + \|B_{j}\|) \exp \left( \int_{s}^{t} \sum_{k=1}^{m} \|a_{k}(\eta)\| (\eta - s) \, d\eta \right). \quad (50)$$

Let $z(t)$ be such an absolutely continuous function that $z(s) = 0$ and the norm of its derivative is bounded by the function $u(t)$. Then

$$\|z(t)\| \leq \int_{s}^{t} u(\eta) \, d\eta.$$ 

If $z(\tau_{i}) = A_{j}z(\tau_{i} - 0)$, then for $\tau_{i-1} < s < \tau_{i} < t < \tau_{i+1}$

$$\|z(t)\| \leq \|A_{j}\| \|z(\tau_{i} - 0)\| + \int_{\tau_{i}}^{t} u(\eta) \, d\eta$$

$$\leq (1 + \|A_{j}\|) \left( \int_{s}^{\tau_{i}} u(\eta) \, d\eta + \int_{\tau_{i}}^{t} u(\eta) \, d\eta \right)$$

$$= (1 + \|A_{j}\|) \int_{s}^{t} u(\eta) \, d\eta.$$ 

Similarly by considering $z(t)$ in the intervals $[\tau_{i+1}, \tau_{i+2}], ..., [\tau_{j-1}, \tau_{j}]$ we obtain for $\tau_{i-1} < s < \tau_{i} < ... < \tau_{j-1} < t < \tau_{j}$

$$\|z(t)\| \leq \prod_{s < \tau_{j} \leq t} (1 + \|A_{j}\|) \int_{s}^{t} u(\eta) \, d\eta.$$ 

The fundamental function $X(t, s)$ satisfies the initial condition $X(s, s) = 0$ and satisfies inequality (50). Therefore we have the following estimate:

$$\|X(t, s)\| \leq \prod_{s < \tau_{j} \leq t} (1 + \|B_{j}\|) \prod_{s < \tau_{j} \leq t} (1 + \|A_{j}\|) \int_{s}^{t} d\xi$$

$$\times \exp \left( \int_{s}^{t} \sum_{k=1}^{m} \|a_{k}(\eta)\| (\eta - s) \, d\eta \right)$$

$$\leq \prod_{s < \tau_{j} \leq t} \left[ (1 + \|B_{j}\|) (1 + \|A_{j}\|) \right] (t - s)$$

$$\times \exp \left( \int_{s}^{t} \sum_{k=1}^{m} \sup_{\eta \in [s, t]} \|a_{k}(\eta)\| (\eta - s) \, d\eta \right)$$
Consequently
\[
\|X(t, s)\| \leq \prod_{s < \tau \leq t} \left[ (1 + \|B\|)(1 + \|A\|) \right] (t - s)
\]
\[
\times \exp \left( \int_{t_{j-1}}^{t_j} \sup_{\eta \in [s, t]} \|a_k(\eta)\| (\eta - s) \, d\eta \right)
\]
which completes the proof of the lemma.

**Lemma 2.** Suppose the assumptions of Lemma 1 hold. Then the solution of (45), (46) with \( \alpha_j = \beta_j = 0, \ j = 0, 1, \ldots, \) and \( \varphi = 0 \) can be presented as
\[
x(t) = \int_{t_0}^{t} X(t, s) f(s) \, ds.
\]

*Proof.* By differentiating (52) twice one obtains, first,
\[
\dot{x}(t) = \int_{t_0}^{t} X'(t, s) f(s) \, ds,
\]
and, second,
\[
\ddot{x}(t) = f(t) + \int_{t_0}^{t} X''(t, s) f(s) \, ds,
\]
since \( X'(s, s) = I, \ X(s, s) = 0 \) for each \( s. \) The equality (52) together with \( X(t, s) = 0, \ t \leq s, \) implies
\[
x(g_k(t)) = \int_{t_0}^{\max\{g_k(t), t_0\}} X(g_k(t), s) f(s) \, ds = \int_{t_0}^{t} X(g_k(t), s) f(s) \, ds.
\]
Consequently by the definition of the fundamental function
\[
\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = f(t) + \int_{t_0}^{t} X''(t, s) f(s) \, ds
\]
\[
+ \int_{t_0}^{t} \sum_{k=1}^{m} a_k(t) X(g_k(t), s) f(s) \, ds = f(t).
\]
Therefore (52) is a solution of the equation.

Now we will prove that (52) also satisfies the impulsive conditions. Let \( i \) be a fixed positive integer and \( \{t_i\}_{i=1}^{n} \subset [t_0, \tau_i) \) be a sequence tending to \( \tau_i \) as \( j \to \infty. \) Let us prove the following relation
\[
\lim_{t_i \to \tau_i - 0} \int_{t_0}^{t_i} X(t, s) f(s) \, ds = \int_{t_0}^{\tau_i - 0} X(\tau - 0, s) f(s) \, ds.
\]
By Lemma 1 the functions under the integral are uniformly bounded. The Lebesgue theorem on the limit under the integral yields (55). Similarly the equality for the derivative of $X(t, s)$ in $t$ is obtained

$$\lim_{t_i \to t_i - 0} \int_{t_0}^{t_i} X'_i(t_i, s) f(s) \, ds = \int_{t_0}^{t_i} X'_i(t_i, s) f(s) \, ds. \quad (56)$$

Fundamental function $X(t, s)$ satisfies the impulsive conditions

$$X(\tau_i, s) = A_i X(\tau_i - 0, s), \quad X'_i(\tau_i, s) = B_i X'_i(\tau_i - 0, s).$$

By (55), (56), and (53) we obtain for $x(t)$ defined by (52):

$$A_i x(\tau_i - 0) = A_i \lim_{t_i \to t_i - 0} \int_{t_0}^{t_i} X(t_i, s) f(s) \, ds = \int_{t_0}^{t_i} A_i X(\tau_i - 0, s) f(s) \, ds$$

$$= \int_{t_0}^{t_i} X(\tau_i, s) f(s) \, ds = x(\tau_i),$$

$$B_i \dot{x}(\tau_i - 0) = B_i \lim_{t_i \to t_i - 0} \int_{t_0}^{t_i} X'_i(t_i, s) f(s) \, ds = \int_{t_0}^{t_i} B_i X'_i(\tau_i - 0, s) f(s) \, ds$$

$$= \int_{t_0}^{t_i} X'_i(\tau_i, s) f(s) \, ds = \dot{x}(\tau_i),$$

consequently $x(t)$ satisfies the impulsive conditions which completes the proof.

**Theorem 12.** Suppose assumptions (a1), (a3) of Section 2 hold, $\varphi: (-\infty, t_0) \to R$ is a Borel measurable bounded function, the components of $a_k$, $\varphi$ and $f$ are Lebesgue measurable locally essentially bounded functions. Then there exists one and only one solution of (45), (46) that can be presented on the form

$$x(t) = X_0(t, t_0) \alpha_0 + X(t, t_0) \beta_0 + \int_{t_0}^{t} X(t, s) f(s) \, ds$$

$$- \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s) a_k(s) \varphi(g_k(s)) \, ds + \sum_{t_0 < \tau_j \leq t} X_0(t, \tau_j) \alpha_j$$

$$+ \sum_{t_0 < \tau_j \leq t} X(t, \tau_j) \beta_j, \quad (57)$$

where $\varphi(g_k(s)) = 0$, if $g_k(s) > t_0$.

**Proof.** By considering the solution of (45), (46) successively on $[t_0, \tau_1)$, $[\tau_1, \tau_2)$, ..., we obtain that there exists a unique solution of the initial value problem. We claim that it coincides with (57).
The solution of the problem
\[ \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = f(t) - \sum_{k=1}^{m} a_k(t)\varphi(g_k(t)), \]
\[ x(\xi) = 0, \quad \xi < t_0, \quad \varphi(\xi) = 0, \quad \xi \geq t_0, \quad (58) \]
also solves (45), (46), with \( \alpha_j = \beta_j = 0 \) for every \( j \). Therefore by Lemma 2 the solution of (58) can be presented as
\[ x_0(t) = \int_{t_0}^{t} X(t, s)f(s) \, ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)a_k(s)\varphi(g_k(s)) \, ds. \]

Observe that the matrices \( X_0(t, s) \) and \( X(t, s) \) satisfy homogeneous equation (45) \( f = 0, \varphi = 0 \), therefore
\[ x_1(t) = X_0(t, t_0)\alpha_0 + X(t, t_0)\beta_0 + \sum_{t_0 < \tau_j \leq t} X_0(t, \tau_j)\alpha_j + \sum_{t_0 < \tau_j \leq t} X(t, \tau_j)\beta_j \]
also satisfies the homogeneous equation.

Therefore \( x(t) = x_0(t) + x_1(t) \) satisfies equation (58). It is easily checked that it also satisfies impulsive conditions (46).

Indeed, for example,
\[ x_1(\tau_i) = X_0(\tau_i, t_0)\alpha_0 + X(\tau_i, t_0)\beta_0 \]
\[ + \sum_{t_0 < \tau_j \leq \tau_i} X_0(\tau_i, \tau_j)\alpha_j + \sum_{t_0 < \tau_j \leq \tau_i} X(\tau_i, \tau_j)\beta_j \]
\[ = A_i X_0(\tau_i - 0, t_0)\alpha_0 + A_i X(\tau_i - 0, t_0)\beta_0 \]
\[ + A_i \sum_{t_0 < \tau_j \leq \tau_i} X_0(\tau_i - 0, \tau_j)\alpha_j + \alpha_i + A_i \sum_{t_0 < \tau_j \leq \tau_i} X(\tau_i - 0, \tau_j)\beta_j \]
\[ = A_i x_1(\tau_i - 0) + \alpha_i, \]
since \( X_0(\tau_i - 0, \tau_i) = X(\tau_i - 0, \tau_i) = 0. \)

Therefore \( x(t) = x_0(t) + x_1(t) \) satisfies both equation (58) and impulsive conditions (46). Consequently, (57) is a solution of (45), (46), which completes the proof.

REFERENCES