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About local configurations in arithmetic planes

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Abstract

In Vittone and Chassery (Proc. of DGCI'97, Vol. 1347 of Lecture Notes in Computer Sciences, 1997, pp. 87–98), J.-M. Chassery and J. Vittone studied local configurations of (m,n)-cubes in naive planes in function of the parameters of these naive planes. More precisely, they enumerated the bicubes and the tricubes that appear in a naive hyperplane of parameters (a,b,c). A symmetry about the line $c=a+b$ appears clearly in this enumeration. The aim of this paper is to prove that the configurations of $(n\times n)$-cubes in the plane of parameters (a,b,c) are in one-to-one relation with those in the plane of parameters $(c\text{-}b,\ c\text{-}a,c)$. If we restrict the parameters to the planes such that $a+b\leqslant c$, we note a second symmetry about the line $c=2b$; We also prove this symmetry. We generalize a theorem established by Réveilles and Gérard (Gérard, Proc. of DGCI'99, Vol. 1568 of Lecture Notes in Computer Sciences, 1999, pp. 65–75, Reveilles, Vision Geometry 4, Vol. 2573 of SPIE 95, San Diego, 1995) and these symmetries to the local configurations of planes of given thickness.

Keywords: Digital planes; Arithmetic planes; Local configurations

1. Introduction

In [4] Reveillès defined the *arithmetic plane P of parameters* (a,b,c,r,ω) where a, b, c, and r are integers and ω is a positive integer: It is the set of integer points (x,y,z) such that $0\leqslant ax+by+cz+r<\omega$; ω is called the *thickness* of P. A *naive plane* (resp. *standard plane*) is a plane of parameters (a,b,c,r,ω) such that $\omega=\max(|a|,|b|,|c|)$ (resp. $\omega=|a|+|b|+|c|$).

Using symmetries, the study of arithmetic planes can be reduced to planes such that $0\leqslant a\leqslant b\leqslant c$; Thus the naive planes are the planes of parameters (a,b,c,r,ω) with $\omega=c$. In this way a naive plane is given by the set of integer points (x,y,z) such that

$$0\leqslant ax+by+cz+r<\omega$$

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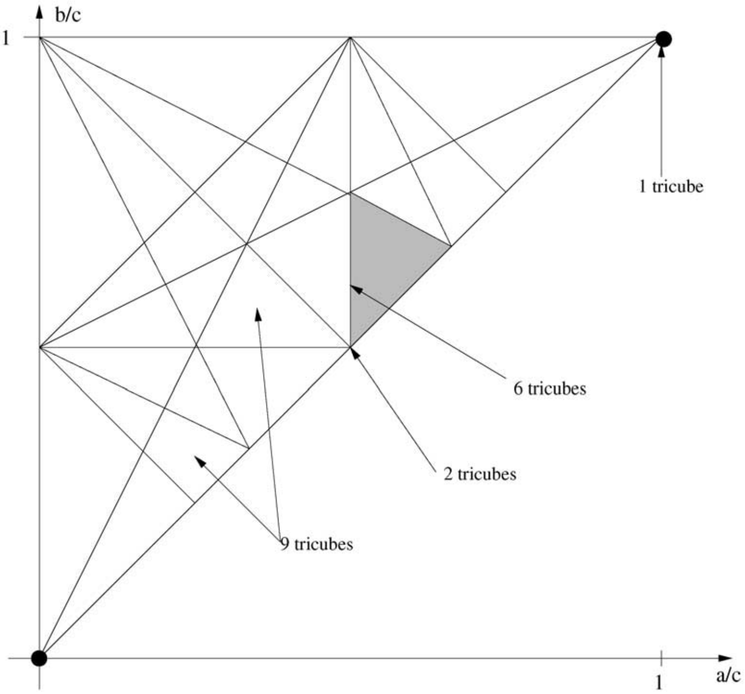

Fig. 1. Number of tricubes that coexist in a digital naive plane.

which is equivalent to

$$\frac{-ax-by-r}{c} \leqslant z < \frac{-ax-by-z}{c}+1,$$

so, the naive plane is the set of points (x, y, z) such that $z=[(-ax-by-r)/c]$ where [ ] denotes the euclidian division (with positive remainder). Many people [5, 1, 3, 2] study the local configurations of naive planes through (m,n)-windows; An (m,n)-*window* is a window composed of the points (i,j) with $0 \leqslant i < m$ and $0 \leqslant j < n$. The *bicubes* and *tricubes* (configurations through (2–2)-windows and (3, 3)-windows) have been introduced by Debled in [1] and studied by some other people [7, 5, 2]. Chassery and Vittone study in [7] the coexistence of tricubes in a digital naive plane (with integer parameters): They give the number of different tricubes that can be found in a digital naive plane of parameters (a,b,c) in function of a/c and b/c (see Fig. 1). In this figure, two symmetries clearly appear. We have a first symmetry about the line $a+b=c$, which means that the number of different tricubes of the plane of parameters (a,b,c) is equal to the number of tricubes of the plane of parameters $(c-b, c-a, c)$. So we can restrict the study to the planes of parameters (a,b,c) such that $a+b \leqslant c$. In this case a second symmetry about the line $c=2b$ appears. In this paper, we prove these two symmetries and more, precisely we establish a one-to-one relation between the (n,n)-cubes of the plane of parameters (a,b,c) and those of the plane of parameters $(c-b,c-a,c)$ (resp. $(a,c-b,c)$ when $a+b\leqslant c$).

In [3] Gérard proved that there exists at most n different configurations in a digital naive plane P with real parameters through a window composed of n points. In the first part of this paper we remind this proof in the particular case of an (m,n)-cube and a digital plane with integer parameters. We generalize this property to arithmetic planes of general thickness. First we give the definition of local configurations through (n,n)-windows in planes of general thickness, we prove that the number of different configurations in a plane of given thickness through a window composed of n points is less than $n(n+1)$. Secondly, we prove that the symmetries about the lines $a+b=c$ and $c=2b$ are also verified for planes of any thickness.

2. Local configurations of (*m*, *n*)-cubes

Let us give some definitions and notations that we will use in this paper.

Let x and y be two integers, $[x/y]$ will denote the quotient of the euclidian division (with positive remainder) of x by y and $\{x/y\}$ the remainder of this division also noted $x \bmod y$.

Definition 1.

- A digital naive plane P of parameters (a,b,c,r) is the set of points (x,y,z) such that $0\leqslant ax+by+cz+r<\max(|a|,|b|,|c|)$ where a,b,c and r are integer parameters such that $\gcd(a,b,c)=1$.
- An (m,n)-window is a window (set of points of $\mathbb{Z}^2$) composed of the points (i,j) with $0\leqslant i<m$ and $0\leqslant j<n$. We will note w_i with $i=1,\dots,m.n$ the points of this window.
- An (m,n)-cube is a set of points of $\mathbb{Z}^3$ whose projection on the two first coordinates on $\mathbb{Z}^2$ is a (m,n)-window.
- An (m,n)-cube at point (i,j) of the naive plane $P(a,b,c,r)$ is the set $\mathscr{C}_P(i,j,m,n)=\{(x,y,z)\in P \mid i\leqslant x\leqslant i+m+1 \textit{ and } j\leqslant y\leqslant j+n\}$.
- Two (m,n)-cubes $C_P(i,j,m,n)$ and $C_P(i',j',m,n)$ are geometrically equal if $C_P(i,j,m,n)$ can be obtained by a translation of $C_P(i',j',m,n)$: The local configuration of $C_P(i,j,m,n)$ equals the local configuration of $C_P(i',j',m,n)$. This local configuration is also the local configuration in the plane $P(a,b,c)$ through an (m,n)-window at point (i,j).
- The local configuration in the plane $P(a,b,c,r)$, through the window $W=(w_1,w_2,\dots,w_n)$ at point $p=(x,y)$ is characterized by the sequence $(h_P(p+w_1)-h_P(p), h_P(p+w_2)-h_P(p),\dots,h_P(p+w_n)-h_P(p))$ with $h_P(x,y)=-[(ax+by+r)/c]$.
- A bicube is a $(2,2)$-cube and a tricube is a $(3,3)$-cube.

Remark. By use of symmetries, we can limit our study to naive planes in the 48th part of space such that $0\leqslant a\leqslant b\leqslant c$. In this way, we always have $\max(|a|,|b|,|c|)$

$= c$. Therefore the points of this plane are given by (x, y, z) such that $z = -[(ax + by + r)/c]$.

As $\gcd(a, b, c) = 1$, there exists $u, v, w \in \mathbb{N}$ such that $au + bv + cw = 1$. If (x, y, z) is a point of the plane of parameters (a, b, c, r), then $0 \leqslant a(x + ur) + b(y + vr) + c(z + wr) < c$. The plane of parameters $(a, b, c, 0)$ can be obtained by translating the plane of parameters (a, b, c, r) by a translation of vector (ur, vr, wr), so that the two planes have the same configurations apart from this translation. Therefore we limit our study to the planes of parameters $(a, b, c, 0)$; In the remainder of this paper P denotes a digital naive plane of parameters (a, b, c).

Notations. Let P be a digital naive plane of parameters (a, b, c) and $p = (x, y)$ a point of $\mathbb{Z}^2$; We will always use the following notations:

$$R_P(p) = \left\{ \frac{ax + by}{c} \right\} \quad \text{and} \quad X_P(p) = c - R_P(p) = c - \left\{ \frac{ax + by}{c} \right\}.$$

Let us remark that: $0 \leqslant R_P(p) < c$ and $0 < X_P(p) \leqslant c$.

The following theorem can be found in [4] and, in a more general case, in [3].

Theorem 1. *The number of different local configurations through an* (m, n)*-window, in a digital naive plane, is smaller than or equals n.m.*

Gérard proved this theorem for general windows and for digital naive planes with real parameters. We will remember this proof, but we will limit us to the particular case of (m, n)-windows and digital naive planes with integer parameters.

Let us first prove the following lemma.

Lemma 1. *Let* P *be a digital naive plane of parameters* (a, b, c) *and* $p = (x, y)$ *and* $w = (u, v)$ *two points of* P; *We then have*

$$h_P(p + w) - h_P(p) = \begin{cases} h_P(w) & \text{if } R_P(w) < X_P(p), \\ h_P(w) - 1 & \text{otherwise.} \end{cases}$$

Proof. We have the following identities:

$$\begin{aligned} h_P(p + w) - h_P(p) &= -\left[\frac{ax + by + au + bv}{c} \right] + \left[\frac{ax + by}{c} \right] \\ &= -\left[\left[\frac{ax + by}{c} \right] + \frac{\{ax + by/c\}}{c} + \left[\frac{au + bv}{c} \right] \right. \\ &\quad \left. + \frac{\{au + bv/c\}}{c} \right] + \left[\frac{ax + by}{c} \right] \\ &= -\left[\frac{ax + by}{c} \right] - \left[\frac{au + bv}{c} \right] \end{aligned}$$

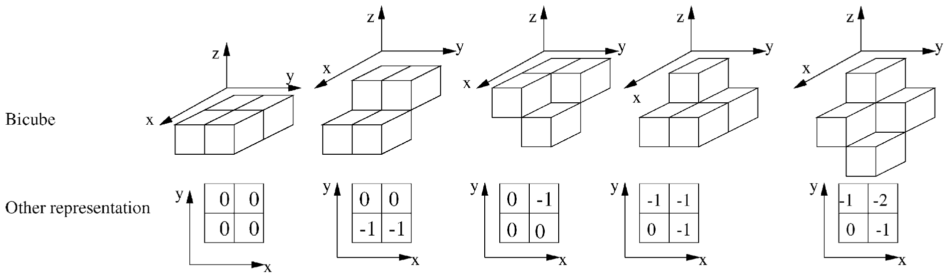

Fig. 2. List of all bicubes that can appear in a digital naive plane.

$$
\begin{aligned}
&-\left\lceil \frac{\{ax+by/c\}+\{au+bv/c\}}{c} \right\rceil + \left\lceil \frac{ax+by}{c} \right\rceil \\
&= -\left\lceil \frac{au+bv}{c} \right\rceil - \left\lceil \frac{\{ax+by/c\}+\{au+bv/c\}}{c} \right\rceil \\
&= h_P(w) - \left\lceil \frac{\{ax+by/c\}+\{au+bv/c\}}{c} \right\rceil \\
&= \begin{cases} h_P(w) & \text{if } \left\{\dfrac{ax+by}{c}\right\} + \left\{\dfrac{au+bv}{c}\right\} < c, \\ h_P(w)-1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Proof of the theorem. Let $W = \{w_1, \ldots, w_q\}$ be an (m,n)-window $(q = m.n)$, and $p = (x, y)$ a point of P. The local configuration at p through the window W, in the plane of parameters (a,b,c) is characterized by the sequence $(h_P(p+w_1) - h_P(p), h_P(p+w_2) - h_P(p), \ldots, h_P(p+w_q) - h_P(p))$. But the above lemma says that

$$
h_P(p+w_i) - h_P(p) = \begin{cases} h_P(w_i) & \text{if } R_P(w_i) < X_P(p), \\ h(w_i)-1 & \text{otherwise.} \end{cases}
$$

If we number the points of the window such that $0 = R_P(w_1) \leqslant R_P(w_2) \leqslant \cdots \leqslant R_P(w_q)$, $X_P(p)$ will belong to one of the q intervals defined by this sequence. To each interval corresponds a configuration so there exists at most q different configurations in a digital naive plane.

Examples. There exists five configurations of bicubes (see [1, 5, 2] which are listed in Fig. 2; The first and the last configuration cannot coexist in a same digital naive plane. If $a+b<c$ (resp. $a+b>c$), the four first (resp. last) configurations of Fig. 2 can appear in the naive plane of parameter (a,b,c). If $a+b<c$, the local configuration at point $(0,0)$ is the first bicube of Fig. 2.

If we assume $a+b<c$, we have: $R_P(0,0) = 0$, $R_P(1,0) = \{a/c\} = a$, $R_P(0,1) = \{b/c\} = b$, $R_P(1,1) = \{(a+b)/c\} = a+b$ and so $R_P(0,0) = 0 \leqslant R_P(1,0) \leqslant R_P(0,1) \leqslant R_P(1,1)$.

Table 1
Configuration in $P(a,b,c)$ such that $a+b<c$, through a $(2,2)$ window

Interval which contains $X_P(p)$	Configuration at point p
$]0,a]$	$\begin{array}{\|r\|r\|}\hline -1 & -1 \\ \hline 0 & -1 \\ \hline\end{array}$
$]a,b]$	$\begin{array}{\|r\|r\|}\hline -1 & -1 \\ \hline 0 & 0 \\ \hline\end{array}$
$]b,a+b]$	$\begin{array}{\|r\|r\|}\hline 0 & -1 \\ \hline 0 & 0 \\ \hline\end{array}$
$]a+b,c]$	$\begin{array}{\|r\|r\|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline\end{array}$

The configuration of a bicube at point $p=(x,y)$ is given in Table 1; This configuration depends on the value of $X_P(p)=c-\{(ax+by)/c\}$.

We do not exhibit the list of tricubes, they can be found in [1, 6, 2]; There exists 40 tricubes but at most 9 tricubes can coexist in a naive plane. Let us consider, for instance, the configurations in the naive plane P of parameters (a,b,c) through a $(3,3)$-window. We assume that $0<a<b<c$, $2a>c$, $2b-c<a$, and $a+2b<2c$ so that $a+b>c$ and $2a+b<2c$. These conditions correspond to the grey polygon in Fig. 1. The configuration at point $p=(x,y)$ is given in Table 2. This configuration depends on the position of $X_P(p)=c-\{(ax+b)/c\}$ in the increasing sequence:

$$R_P(0,0)=0,\ R_P(2,0)=2a-c,\ R_P(1,1)=a+b-c,$$

$$R_P(0,2)=2b-c,\ R_P(2,2)=2a+2b-2c,\ R_P(1,0)=a,$$

$$R_P(0,1)=b,\ R_P(2,1)=2a+b-c,\ R_P(1,2)=a+2b-c.$$

Let us consider a window $W=\{w_1,w_2,\ldots,w_n\}$ and let us note $r_i=R_P(w_i)$ for $i=1,2,\ldots,n$ and $r_{n+1}=c$. As $\gcd(a,b,c)=1$, the number of different configurations through the window W that coexist in a naive plane equals the number of different nonempty intervals defined by $]r_i,r_{i+1}]$ with $i=1,\ldots,n$. So it is equal to n minus the number of integers i such that $1\leqslant i\leqslant n$ and $r_i=r_{i+1}$. If W is a $(3,3)$-window, $r_i=r_{i+1}$ is equivalent to saying that there exists $u,v,u',v'\in[0,2]$ such that $au+bv=(au'+bv') \bmod c$ and $(u,v)\neq(u',v')$. Fig. 1 represents all of the segments defined by these equations, lying in the triangle $0\leqslant a/c\leqslant b/c\leqslant 1$. A digital plane of parameters (a,b,c) such that $(a/c,b/c)$ is inside a polygon delimited by these segments contains exactly 9 different configurations through $(3,3)$-windows, the other planes have less than 9 configurations. See [7] to have more precisions about Fig. 1 and tricubes that coexist in a naive plane of parameters (a,b,c).

Fig. 1 is clearly symmetric about the line $a+b=c$, and, if we consider only the part of the triangle such that $a+b\leqslant c$, we note a second symmetry about the line

Table 2
Configuration in $P(a,b,c)$ such that $0<a<b<c$, $2a>c$, $2b-c<a$ and $a+2b<2c$, through a $(3,3)$-window. Example of parameters verifying these conditions: $a=7$, $b=8$, $c=13$

Interval in which lies $X_P(p)$	Configuration at point p	Interval in which lies $X_P(p)$	Configuration at point p
$]0,2a-c]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -3 \\ \hline -1 & -2 & -2 \\ \hline 0 & -1 & -2 \\ \hline \end{array}$	$]a,b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]2a-c,a+b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -3 \\ \hline -1 & -2 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]b,2a+b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline 0 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]a+b-c,2b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -3 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]2a+b-c,a+2b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]2b-c,2a+2b-2c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -3 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]a+2b-c,c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]2a+2b-2c,a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$		

$c=2b$. In the following section we will study these symmetries not only for tricubes but, more generally, for (n,n)-cubes.

3. Symmetries about the lines $a+b=c$ and $c=2b$

In the remainder of this paper we will consider a window $W=\{w_1,w_2,\dots,w_m\}$ with $m=n^2$ and $w_i=(x,y)$ for $x,y=0,1,\dots,n-1$. The points of this window are numbered such that $w_1=(0,0)$ and $0=R_P(0,0)=R_P(w_1)\leqslant R_P(w_2)\leqslant\cdots\leqslant R_P(w_m)$ and we denote $r_i=R_P(w_i)$ for $i=1,2,\dots,m$ and $r_{m+1}=c$.

3.1. Symmetry about the line $a+b=c$

The following theorem states that the configurations in P are in one-to-one relation with the configurations in P' where P' is the naive plane of parameter $(c-b,c-a,c)$. Let us remark that $0\leqslant c-b\leqslant c-a\leqslant c$: P' belongs to the same 48th part of space as P.

Theorem 2. *Let f be the one-to-one function from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ defined by*

$$f:\mathbb{Z}^3 \mapsto \mathbb{Z}^3$$
$$(x,y,z)\rightarrow(-y,-x,x+y+z).$$

The function f is also a one-to-one function from the set of configurations through W in plane P to the set of configurations through W in P'.

More precisely, we have the following equality: $f(C_P(i,j,n,n)) = C_{P'}(-n-j+1, -n-i+1, n, n)$.

Proof. Let us consider the function f defined by

$$f : \mathbb{Z}^3 \mapsto \mathbb{Z}^3$$
$$(x,y,z) \to (-y, -x, x+y+z).$$

The function f is an isomorphism on $\mathbb{Z}^3$, more precisely, if S is a set of points on $\mathbb{Z}^3$ and v a vector, we have

- $f \circ f = Id$,
- $f(S+v) = f(S) + f(v)$,
- $f(S) + v = f(S + f(v))$.

We deduce that two sets of points S and S' are geometrically equal if and only if $f(S)$ and $f(S')$ are geometrically equal. Moreover, the function f transforms $C_P(i,j,n,n)$ into the set of points defined by

$$\{(-y, -x, x+y+z) \mid (x,y,z) \in P,\ i \leqslant x \leqslant n+i-1,\ j \leqslant y \leqslant n+j-1\}.$$

But we have $(c-b)(-y) + (c-a)(-x) + c(x+y+z) = ax + by + cz$.

It follows that $(x,y,z) \in P \Leftrightarrow f(x,y,z) \in P$.

We also have

$$i \leqslant x \leqslant i+n$$
$$\Leftrightarrow -i-n \leqslant y' \leqslant -i.$$

In the same manner, we have $-j-n+1 \leqslant x' \leqslant -j+1$.

We conclude that $f(C_P(i,j,n,n)) = C_{P'}(-j-n+1, -i-n+1, n, n)$.

It follows that two (m,n)-cubes in P are geometrically equal if and only if their images by f are geometrically equal. This ends the proof of the theorem.

Example. Let us consider the configurations in the naive plane P of parameters (a,b,c) through a $(3,3)$-window. We assume that $0 < a < b < c$, $2a > c$, $2b - c < a$ and $a + 2b < 2c$ so that $a + b > c$ and $2a + b < 2c$. The configuration at point $p = (x,y)$ depends on the position of $X_P(p) = c - \{(ax+b)/c\}$ in the sequence:

$$R_P(0,0) = 0,\ R_P(2,0) = 2a - c,\ R_P(1,1) = a + b - c,\ R_P(1,0) = a,$$

$$R_P(0,2) = 2b - c,\ R_P(2,2) = 2a + 2b - 2c,\ R_P(0,1) = b,$$

$$R_P(2,1) = 2a + b - c,\ R_P(1,2) = a + 2b - c.$$

If we consider the plane P' of parameters $(c-b, c-a, c)$, the configuration at $p' = (-y-2, -x-2)$ depends on the position of $X_{P'}(p') = c - \{(ax+by+2a+2b)/c\}$ in the sequence:

$$R_{P'}(0,0) = 0,\ R_{P'}(2,1) = 2c-a-2b,\ R_{P'}(1,2) = 2c-2a-b,$$

$$R_{P'}(1,0) = c-b,\ R_{P'}(0,1) = c-a,\ R_{P'}(2,2) = 3c-2a-2b,$$

$$R_{P'}(2,0) = 2c-2b,\ R(1,1) = 2c-a-b,\ R'_P(0,2) = 2c-2a.$$

Table 3 presents the configuration at point p in P and the corresponding configuration at point p' in P'.

3.2. Symmetry about the line $c = 2b$

Now we assume that $a+b \leqslant c$ and prove that the configurations in P are in one-to-one relation with the configurations in P' where P' is the naive plane of parameter $(a, c-b, c)$. Let us remark that P' belongs to the same part of space as P.

Theorem 3. *Let g be the one-to-one function from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ defined by*

$$g : \mathbb{Z}^3 \mapsto \mathbb{Z}^3$$
$$(x, y, z) \to (x, -y, y+z).$$

The function g is also a one-to-one function from the set of configurations through W in plane P to the set of configurations through W in P'.

More precisely, we have the following equality: $g(C_P(i,j,n,n)) = C_{P'}(i, -n-j+1, n, n)$.

Proof. Let us consider the function g defined by

$$g : \mathbb{Z}^3 \mapsto \mathbb{Z}^3$$
$$(x, y, z) \to (x, -y, y+z).$$

The function g is an isomorphism on $\mathbb{Z}^3$, more precisely, if S is a set of points on $\mathbb{Z}^3$ and v a vector, we have

- $g \circ g = Id$,
- $g(S+v) = g(S) + g(v)$,
- $g(S) + v = g(S + g(v))$.

We deduce that two sets of points S and S' are geometrically equal if and only if $g(S)$ and $g(S')$ are geometrically equal. Moreover, the function g transforms $C_P(i,j,n,n)$ into the set of points defined by

$$\{(x, -y, y+z) \mid (x, y, z) \in P,\ i \leqslant x \leqslant n+i-1,\ j \leqslant y \leqslant n+j-1\}.$$

But we have $ax + (c-b)(-y) + c(y+z) = ax + by + cz$.

Table 3
Correspondences between tricubes in $P(a,b,c)$ and tricubes in $P'(c-b,c-a,c)$

Interval in which lies $X_P(p)$	Configuration in P at point p	Interval in which lies $X_{P'}(p')$	Configuration in P' at point p'
$]0, 2a-c]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -3 \\ \hline -1 & -2 & -2 \\ \hline 0 & -1 & -2 \\ \hline \end{array}$	$]3c-2a-2b, 2c-2b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]2a-c, a+b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -3 \\ \hline -1 & -2 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]2c-2b, 2c-a-b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]a+b-c, 2b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -3 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]2c-a-b, 2c-2a]$	$\begin{array}{\|r\|r\|r\|}\hline 0 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]2b-c, 2a+2b-2c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -3 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]2c-2a, c]$	$\begin{array}{\|r\|r\|r\|}\hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]2a+2b-2c, a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]0, 2c-a-2b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$
$]a, b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]2c-a-2b, 2c-2a-b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]b, 2a+b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline 0 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]2c-2a-b, c-b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline 0 & -1 & -2 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]2a+b-c, a+2b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]c-b, c-a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]a+2b-c, c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]c-a, 3c-2a-2b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$

It follows that $(x,y,z) \in P \Leftrightarrow g(x,y,z) \in P'$.
We also have

$$j \leqslant y \leqslant j+n$$
$$\Leftrightarrow -j-n+1 \leqslant y' \leqslant -j+1.$$

We conclude that $g(C_P(i,j,n,n)) = C_{P'}(i,-j-n+1,n,n)$.

It follows that two (m,n)-cubes in P are geometrically equal if and only if their images by g are geometrically equal. This ends the proof of the theorem.

Example. Let us consider the configurations of the naive plane P of parameters (a,b,c) through a $(3,3)$-window. We assume that $0<a<b<c$, $2a>c$, $2b-c<a$ and $a+2b<2c$ so that $a+b>c$, $2a+b<2c$ and $2a+2b>2c$. The configuration at point $p=(x,y)$ depends on the position of $X_P(p)=c-\{(ax+b)/c\}$ in the sequence

$$R_P(0,0)=0,\ R_P(2,1)=2a+b-c,\ R_P(1,2)=a+2b-c,$$

$$R_P(1,0)=a,\ R_P(0,1)=b,\ R_P(2,2)=2a+2b-c,$$

$$R_P(2,0)=2a,\ R_P(1,1)=a+b,\ R_P(0,2)=2b.$$

If we consider the plane P' of parameters $(c-b,c-a,c)$, the configuration at $p'=(-y-2,-x-2)$ depends on the position of $X_{P'}(p')=c-\{(ax+by+2a+2b)/c\}$ in the sequence

$$R_P(0,0)=0,\ R_P(0,2)=c-2b,\ R_P(2,1)=2a-b,$$

$$R_P(1,0)=a,\ R_P(1,2)=a-2b+c,\ R_P(0,1)=c-b,$$

$$R_P(2,0)=2a,\ R_P(2,2)=2a-2b+c,\ R_P(1,1)=a-b+c.$$

Table 4 presents the configuration at point p in P and the corresponding configuration at point p' in P'.

4. Local configurations in planes of given thickness

Now we study local configurations in planes of arbitrary thickness. At first we remember the definition of thick planes and give the definition of local configurations through (m,n)-windows in these planes. Then we prove that local configurations in the plane P of parameters (a,b,c,ω) through an (n,n)-window are in one-to-one relation with those of plane P' (resp. P'') of parameters $(c-b,c-a,c,\omega)$ (resp. $(a,c-b,c,\omega)$ when $a+b\leqslant c$).

Definition 2. An arithmetic plane P of parameters (a,b,c,r,ω) is the set of integer points (x,y,z) such that $0\leqslant ax+by+cz+r<\omega$ where a,b,c,r and ω are integer parameters such that $\gcd(a,b,c)=1$ and $\omega>0$; ω is called the thickness of P. We limit our study to planes in the 48th part of space such that $0\leqslant a\leqslant b\leqslant c$.

Remark. As $\gcd(a,b,c)=1$, there exists $u,v,w\in\mathbb{N}$ such that $au+bv+cw=1$. If (x,y,z) is a point of the plane of parameters (a,b,c,r,ω), then $0\leqslant a(x+ur)+b(y+vr)+c(z+wr)<\omega$: The plane of parameters $(a,b,c,0,\omega)$ can be obtained by translating the plane of parameters (a,b,c,r,ω) by a translation of vector (ur,vr,wr). So we can reduce our study to the planes of parameters $(a,b,c,0,\omega)$ noted $P(a,b,c,\omega)$.

Table 4
Correspondences between tricubes in $P(a,b,c)$ and tricubes in $P'(a,c-b,c)$

Interval in which lies $X_P(p)$	Configuration in P at point p	Interval in which lies $X_{P'}(p')$	Configuration in P' at point p'
$]0,2a+b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]c-2b,2a-b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$
$]2a+b-c,a+2b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -1 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]2a-b,a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -1 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$
$]a+2b-c,a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline -1 & -1 & -1 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$	$]a,a-2b+c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -2 & -2 \\ \hline -1 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]a,b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline -1 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]a-2b+c,c-b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline -1 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]b,2a+2b-c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]c-b,2a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$
$]2a+2b-c,2a]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -1 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array}$	$]2a,2a-2b+c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]2a,a+b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -1 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$	$]2a-2b+c,a-b+c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -1 \\ \hline 0 & -1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]a+b,2b]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$	$]a-b+c,c]$	$\begin{array}{\|r\|r\|r\|}\hline -1 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$
$]2b,c]$	$\begin{array}{\|r\|r\|r\|}\hline 0 & -1 & -1 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$	$]0,c-2b]$	$\begin{array}{\|r\|r\|r\|}\hline -2 & -2 & -2 \\ \hline -1 & -1 & -2 \\ \hline 0 & -1 & -1 \\ \hline \end{array}$

Let (x,y,z) be a point of $P(a,b,c,\omega)$, we have the following inequalities:

$$0 \leqslant ax+by+cz < \omega$$

$$\Leftrightarrow \frac{-ax-by}{c} \leqslant z < \frac{\omega-ax-by}{c}.$$

Let us note $h_P(x,y)$ the least integer greater than or equal to $(-ax-by)/c$ and $H_P(x,y)$ the greatest integer less than $(\omega-ax-by)/c$.

If u is a real number, we have

$$[-u] = \begin{cases} -[u] & \text{if } u \text{ is an integer,} \\ -[u]-1 & \text{otherwise.} \end{cases}$$

It follows that

$$h_P(x,y) = -\left\lfloor \frac{ax+by}{c} \right\rfloor$$

and

$$H_P(x,y) = \begin{cases} \left\lfloor \dfrac{\omega - ax - by}{c} \right\rfloor - 1 & \text{if } \dfrac{\omega - ax - by}{c} \text{ is an integer,} \\ \left\lfloor \dfrac{\omega - ax - by}{c} \right\rfloor & \text{otherwise.} \end{cases}$$

$$= -\left\lfloor \frac{ax+by-\omega}{c} \right\rfloor - 1.$$

Thus the points of P are given by $\{(x,y,z) \mid x,y \in \mathbb{Z},\ z \in [h_P(x,y), H_P(x,y)]\}$.

Definition 3. The thickness of the plane $P(a,b,c,\omega)$ at point $p=(x,y)$, noted $Th_P(p)$, is the number

$$Th_P(p) = H_P(p) - h_P(p) + 1.$$

Notations. Let P be a digital plane of parameters (a,b,c,ω) and $p=(x,y)$ a point of $\mathbb{Z}^2$; We will always use the following notations:

$$R_P(p) = \left\{ \frac{ax+by}{c} \right\}, \quad X_P(p) = c - \left\{ \frac{ax+by}{c} \right\}$$

and

$$X'_P(p) = c - \left\{ \frac{ax+by-\omega}{c} \right\}.$$

Let us now consider an (n,n)-window $W = \{(x,y),\ 0 \leqslant x,y < n\} = \{w_1, w_2, \ldots, w_m\}$ with $m = n^2$, we are interested in the local configurations in P through W as defined in the following.

Definition 4.

- An (m,n)-cube at point (i,j) of the plane $P(a,b,c,\omega)$ is the set $\mathscr{C}_P(i,j,m,n) = \{(x,y,z) \in P \mid i \leqslant x < i+m \text{ and } j \leqslant y < j+n\} = \{(x,y,z) \in P \mid (x-i, y-j) \in W\}$.
- Two (m,n)-cubes C_1 and C_2 are geometrically equal if C_1 can be obtained by a translation of C_2: They have the same local configuration which is called the local configuration of P through W at point p.
- The local configuration of P through W at point p is characterized by the sequence:

$$Th_P(p), h_P(p+w_1) - h_P(p), h_P(p+w_2) - h_P(p), \ldots, h_P(p+w_m) - h_P(p),$$

$$H_P(p+w_1) - H_P(p), H_P(p+w_2) - H_P(p), \ldots, H_P(p+w_m) - H_P(p).$$

Remark. This definition means that the local configuration at point p is characterized by the thickness at point p, by the configuration at p of the "upper part" of the plane and by the configuration at p of the "lower part" of the plane.

Let us point out that if there exists an integer k such that $\omega = kc$, we have $H_P(x,y) = k - [(ax+by)/c] - 1 = k - 1 + h_P(p)$, and $Th_P(p) = k$, so that the configuration in P through W at point p can be determined by the sequence

$$h_P(p+w_1) - h_P(p), h_P(p+w_2) - h_P(p), \dots, h_P(p+w_m) - h_P(p).$$

We can conclude that in this case, at most m different configurations can coexist in a plane. What can we say about the number of different configurations that coexist in a plane of thickness ω in the general case? The following theorem answers this question.

Theorem 4. *The number of different configurations in a digital plane P through a window $W = \{w_1, w_2, \dots, w_m\}$ is less than or equal to $2m$.*

Proof. We have seen below that if $\{\omega/c\} = 0$ at most m different configurations can coexist in a plane. In the following we will assume that $\{\omega/c\} \neq 0$.

As we have done to prove Theorem 1, we number the points of the window such that $0 = R_P(w_1) \leqslant R_P(w_2) \leqslant \cdots \leqslant R_P(w_m)$. Then we have

$$h_P(p+w_i) - h_P(p) = \begin{cases} h_P(w_i) & \text{if } R_P(w_i) < X_P(p), \\ h_P(w_i) - 1 & \text{otherwise.} \end{cases}$$

Let us note $\beta = [\omega/c]$ and $\alpha = \{\omega/c\}$ so that $\omega = \beta c + \alpha$. The thickness at point p is given by

$$\begin{aligned} Th_P(p) &= H_P(p) - h_P(p) + 1 \\ &= -\left[\frac{ax+by-\omega}{c}\right] - 1 + \left[\frac{ax+by}{c}\right] + 1 \\ &= \beta - \left[\left[\frac{ax+by}{c}\right] + \left[\frac{-\alpha}{c}\right] + \frac{\{ax+by/c\} + \{-\alpha/c\}}{c}\right] + \left[\frac{ax+by}{c}\right] \\ &= \begin{cases} \beta + 1 & \text{if } \left\{\dfrac{-\alpha}{c}\right\} < X_P(p), \\ \beta & \text{otherwise.} \end{cases} \end{aligned}$$

So, the first part of the sequence that defines the configuration at point p depends only on the value of $X_P(p)$; More precisely, it depends on the position of $X_P(p)$ in the sequence $R_P(w_1) \leqslant R_P(w_2) \leqslant \cdots \leqslant R_P(w_{i_0}) \leqslant \{-\alpha/c\} = c - \alpha \leqslant R_P(w_{i_0+1}) \leqslant \cdots \leqslant R_P(w_m) \leqslant c$.

Let us consider the second part of the sequence, we have

$$\begin{aligned}
&H_P(p+w) - H_P(p)\\
&\quad= -\left\lceil \frac{ax+by-\omega+au+bv}{c} \right\rceil + \left\lceil \frac{ax+by-\omega}{c} \right\rceil\\
&\quad= -\left\lfloor \frac{ax+by-\omega}{c} \right\rfloor - \left\lfloor \frac{au+bv}{c} \right\rfloor\\
&\qquad - \left\lceil \left\{ \frac{ax+by-\omega}{c} \right\} + \left\{ \frac{au+bv}{c} \right\} \right\rceil + \left\lceil \frac{ax+by-\omega}{c} \right\rceil\\
&\quad= h(w) - \left\lceil \left\{ \frac{ax+by-\omega}{c} \right\} + R_P(w) \right\rceil\\
&\quad= \begin{cases} h(w) & \text{if } R_P(w) < X'_P(p),\\ h(w)-1 & \text{otherwise.} \end{cases}
\end{aligned}$$

The second part of the sequence that defines the configuration depends on the position of $X'_P(p)$ in the sequence $0 = R_P(w_1) \leqslant R_P(w_2) \leqslant \cdots \leqslant R_P(w_m) \leqslant c$.

But $X'_P(p)$ can be calculated as a function of $X_P(p)$; Indeed we have

$$\begin{aligned}
X'_P(p) &= c - \left\{ \frac{ax+by-\omega}{c} \right\}\\
&= c - \left\{ \frac{ax+by-\alpha}{c} \right\}\\
&= \begin{cases} c - \left\{ \dfrac{ax+by}{c} \right\} - \left\{ \dfrac{-\alpha}{c} \right\} & \text{if } \left\{ \dfrac{ax+by}{c} \right\} + \left\{ \dfrac{-\alpha}{c} \right\} < c,\\ c - \left\{ \dfrac{ax+by}{c} \right\} - \left\{ \dfrac{-\alpha}{c} \right\} + c & \text{otherwise} \end{cases}\\
&= \begin{cases} X_P(p) - \left\{ \dfrac{-\alpha}{c} \right\} & \text{if } \left\{ \dfrac{-\alpha}{c} \right\} < X_P(p),\\ X_P(p) - \left\{ \dfrac{-\alpha}{c} \right\} + c & \text{otherwise.} \end{cases}
\end{aligned}$$

We then have

$$\begin{aligned}
&X'_P(p) \in \,]R_P(w_i); R_P(w_{i+1})]\\
&\quad\Leftrightarrow X_P(p) \in \left] R_P(w_i) + \left\{ \frac{-\alpha}{c} \right\}; R_P(w_{i+1}) + \left\{ \frac{-\alpha}{c} \right\} \right] \cap \left] \left\{ \frac{-\alpha}{c} \right\}; c \right]\\
&\qquad\text{or } X_P(p) \in \left] R_P(w_i) + \left\{ \frac{-\alpha}{c} \right\} - c; R_P(w_{i+1}) + \left\{ \frac{-\alpha}{c} \right\} - c \right] \cap \left] 0; \left\{ \frac{-\alpha}{c} \right\} \right].
\end{aligned}$$

But only one of the two numbers $R_P(w_i)+\{\alpha/c\}$ and $R_P(w_i)+\{\alpha/c\}-c$ belongs to the interval $]0;c]$; Let us note s_i as this number. The position of $X'_P(p)$ in the sequence $0=R_P(w_1)\leqslant R_P(w_2)\leqslant\cdots\leqslant R_P(w_m)\leqslant c$ depends on the position of $X_P(p)$ in the intervals of boundaries given by the numbers s_i. We conclude that the local configuration depends on the position of $X_P(p)$ in the intervals of boundaries given by the numbers $R_P(w_i)$, s_i with $i=1,2,\ldots,m$ and $\{-\alpha/c\}$. But we also have

$$s_0=\begin{cases} R_P(w_0)+\left\{\dfrac{-\alpha}{c}\right\} & \text{if } R_P(w_0)+\left\{\dfrac{-\alpha}{c}\right\}\in\,]0,c],\\ R_P(w_0)+\left\{\dfrac{-\alpha}{c}\right\}-c & \text{otherwise}\end{cases}$$
$$=\left\{\frac{-\alpha}{c}\right\}.$$

The local configuration depends only on the position of $X_P(p)$ in the intervals of boundaries given by the numbers $R_P(w_i)$, s_i with $i=1,2,\ldots,m$. We have $2m$ intervals and so at most $2m$ different configurations.

Example. Let us consider the plane P of parameters $(3,5,13,17)$ and the window $W=\{(0,0),(1,0),(0,1),(1,1)\}$, according to the theorem, there exists at most 8 different configurations in P through W. We have $R_P(0,0)=0$, $R_P(1,0)=3$, $R_P(0,1)=5$, $R_P(1,1)=8$, and $\{-\omega/c\}=\{\frac{-17}{13}\}=9$. The configuration in P at point $p=(x,y)$ depends on the position of $X_P(x,y)=13-\{3x+5y/13\}$ in the increasing sequence $0,3,5,8,9,13$ and on the position of $X'_P(x,y)=13-\{3x+5y-17/13\}$ in the increasing sequence $0,3,5,8,13$. According to the previous remark, we have

$$X'_P(p)=\begin{cases} X_P(p)-9 & \text{if } 9<X_P(p),\\ X_P(p)+4 & \text{otherwise.}\end{cases}$$

For each interval which contains $X_P(p)$ we can the determine the interval which contains $X'_P(p)$ and the number of different configurations corresponding to these intervals, indeed:

$$X_P(x,y)\in\,]0,3]\Leftrightarrow X'_P(x,y)\in\,]4,7]\quad 2\text{ configurations},$$
$$X_P(x,y)\in\,]3,5]\Leftrightarrow X'_P(x,y)\in\,]7,9]\quad 2\text{ configurations},$$
$$X_P(x,y)\in\,]5,8]\Leftrightarrow X'_P(x,y)\in\,]9,12]\quad 1\text{ configuration},$$
$$X_P(x,y)\in\,]8,9]\Leftrightarrow X'_P(x,y)\in\,]12,13]\quad 1\text{ configuration},$$
$$X_P(x,y)\in\,]9,13]\Leftrightarrow X'_P(x,y)\in\,]0,4]\quad 2\text{ configurations}.$$

There exists 8 different configurations: They are listed in Table 5.

Table 5
Correspondences between bicubes in $P(a,b,c,\omega)$ and bicubes in $P'(c-b,c-a,c,\omega)$ with $a=3,\ b=5,\ c=13$ and $\omega=17$

Configuration in P			Configuration in P'		
Upper part	Lower part	Thickness	Upper part	Lower part	Thickness
$\begin{bmatrix}-1&-1\\0&-1\end{bmatrix}$	$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	1	$\begin{bmatrix}-1&-1\\0&-1\end{bmatrix}$	$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	1
$\begin{bmatrix}-1&-1\\0&-1\end{bmatrix}$	$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$	1	$\begin{bmatrix}-1&-1\\0&-1\end{bmatrix}$	$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$	1
$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$	1	$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$	1
$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	$\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$	2
$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	$\begin{bmatrix}0&-1\\0&0\end{bmatrix}$	$\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$	2
$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	$\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$	$\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$	1
$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}-1&-1\\0&-1\end{bmatrix}$	2	$\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$	$\begin{bmatrix}-1&-1\\0&-1\end{bmatrix}$	1
$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	2	$\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$	$\begin{bmatrix}-1&-1\\0&0\end{bmatrix}$	1

5. Symmetries about the lines $a + b = c$ and $c = 2b$ for local configurations of thick planes

In this section we always use the following notations.

- We consider three planes P, P', and P''; P is the plane of parameters (a,b,c,ω), P' the plane of parameters $(c-b, c-a, c, \omega)$ and P'' the plane of parameters $(a, c-b, c, \omega)$.
- We will consider a window $W=(w_1,w_2,\dots,w_m)$ with $m=n^2$ and $w_i=(x,y)$ for $x,y=0,1,\dots,n-1$. The points of this window are numbered such that $0=R_P(0,0)=R_P(w_1)\leqslant R_P(w_2)\leqslant\cdots\leqslant R_P(w_m)$.

5.1. Planes of thickness ω such that $\omega = 0 \bmod c$

Let us consider the digital plane P of parameters (a,b,c,ω) such that $\omega=0 \bmod c$. We have seen in the previous section that the local configuration through a window $W=\{w_1,w_2,\dots,w_m\}$ at point p is determined by the sequence

$$h_P(p+w_1)-h_P(p),\ h_P(p+w_2)-h_P(p),\dots,h_P(p+w_m)-h_P(p).$$

This sequence depends only on the position of $X_P(p)$ in the sequence $0=R_P(w_1)\leqslant R_P(w_2)\leqslant\cdots\leqslant R_P(w_m)\leqslant c$. This does not depend on ω, and so, everything that was proved for naive planes holds for any arbitrary ω such that $\omega=0 \bmod c$.

In the remainder of the paper we will assume that $\omega\neq 0 \bmod c$.

5.2. Planes of thickness ω such that $\omega\neq 0 \bmod c$

The following theorem states that the configurations in P are in one-to-one relation with the configurations in P' (resp. P'').

Theorem 5.

- *Let f be the one-to-one function from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ defined by*

$$\begin{aligned} f:\mathbb{Z}^3 &\mapsto \mathbb{Z}^3 \\ (x,y,z) &\to (-y,-x,x+y+z). \end{aligned}$$

  The function f is also a one-to-one function from the set of configurations through W in plane P to the set of configurations through W in P'.
  More precisely, we have the following equality: $f(C_P(i,j,n,n))=C_{P'}(-n-j+1,-n-i+1,n,n)$.

- *Let g be the one-to-one function from $\mathbb{Z}^3$ to $\mathbb{Z}^3$ defined by*

$$\begin{aligned} g:\mathbb{Z}^3 &\mapsto \mathbb{Z}^3 \\ (x,y,z) &\to (x,-y,y+z). \end{aligned}$$

  The function g is also a one-to-one function from the set of configurations through W in plane P to the set of configurations through W in P''.
  More precisely, we have the following equality: $g(C_P(i,j,n,n))=C_{P''}(i,-n-j+1,n,n)$.

The proof of this theorem is exactly the same as for Theorems 2 and 3.

Example. In Table 5 are listed the local configurations of bicubes ("upper part", "lower part" and thickness) in plane $P(3,5,13,17)$ and the corresponding configurations in plane $P'(10,8,13,17)$.

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