JOURNAL OF COMBINATORIAL THEORY, Series B 51, 314–328 (1991)

On Nonsymmetric P- and Q-Polynomial Association Schemes

AKIHIRO MUNEMASA

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

Communicated by the Managing Editors

Received March 22, 1989

If a nonsymmetric P-polynomial association scheme, or equivalently, a distance-regular digraph, has diameter d and girth g, then $d = g$ or $d = g - 1$, by Damerell's theorem. The dual of this theorem was proved by Leonard. In this paper, we prove that the diameter of a nonsymmetric P- and Q-polynomial association scheme is one less than its girth and its cogirth. We also give a structure theorem for a nonsymmetric Q-polynomial association scheme whose diameter is equal to its cogirth. We use self-duality and unimodality to show that the eigenvalues of a nontrivial nonsymmetric P- and Q-polynomial association scheme are quadratic over the rationals. The fact that the adjacency algebra becomes a C-algebra gives a necessary condition for the existence of a nonsymmetric P- and Q-polynomial association scheme. As an application, it is shown that the only nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 5 is the directed 5 cycle.

1. INTRODUCTION

Distance-transitive digraphs were first introduced by Lam [3]. Damerell [2] defined the distance regularity for digraphs and proved a fundamental theorem for distance-regular digraphs. There are a few examples of known distance-regular digraphs other than the trivial ones, i.e., the directed cycles. A distance-regular digraph with diameter 2 and girth 3 can be constructed from a skew Hadamard matrix; distance-regular digraphs with diameter 3 and girth 4 have been constructed by Liebler and Mena [6]. These distance-regular digraphs are short digraphs; i.e., the diameter is one less than the girth. One can construct a long digraph, i.e., a distance-regular digraph in which the diameter is equal to the girth, from a short digraph (see [2]). The short digraphs mentioned above are not only P-polynomial but are also Q-polynomial association schemes. Leonard [4, 5] has shown that if $\mathscr{X}$ is a nonsymmetric P- and Q-polynomial association scheme then $\mathscr{X}$ is self-dual and $g = g^* = d + 1$ or $g = g^* = d$,

0095-8956/91 $3.00

where d, g, g^*, are the diameter, the girth, and the cogirth, respectively. However, it seems unnoticed that no long digraph is Q-polynomial. Consequently, the case $g = g^* = d$ in Leonard's theorem does not occur. There does exist a nonsymmetric Q-polynomial association scheme with $g^* = d$, and we prove a structure theorem for such an association scheme. This result is a dual of [2, Theorem 4].

Next we obtain a necessary condition for the existence of a nonsymmetric P- and Q-polynomial association scheme in terms of its eigenvalues, using the fact that the adjacency algebra becomes a C-algebra. This gives a system of equations that the eigenvalues must satisfy. Moreover, the eigenvalues are shown to be at most quadratic over the rationals, unless the association scheme is a directed cycle. If the girth is 5, we are able to solve the system of equations to show the nonexistence of nontrivial nonsymmetric P- and Q-polynomial association schemes.

2. Definitions and Preliminaries

In this paper, we use the notation adopted in [1]. Let $\Gamma = (X, E)$ be a strongly connected digraph and u, v, vertices of Γ. Then $d(u, v)$ denotes the length of a shortest directed path from u to v. Let $R_i = \{(u, v) \in X \times X \mid d(u, v) = i\}$, $i = 0, 1, ..., d$, where d is the diameter of Γ. Γ is called a *distance-regular digraph* if $\mathscr{X}(\Gamma) = (X, \{R_i\}_{i=0}^{d})$ is an association scheme. This is not a commonly used definition of a distance-regular digraph, but is equivalent to those used in the literature. Moreover, if Γ is a distance-regular digraph, then $\mathscr{X}(\Gamma)$ is a P-polynomial association scheme, and every P-polynomial association scheme arises from a distance-regular digraph in this way. We assume the girth of Γ is at least 3, since otherwise Γ is a undirected graph.

THEOREM 2.1 [2]. *Let $\mathscr{X}$ be a nonsymmetric P-polynomial association scheme with diameter d and girth g. Then*

(1) *$g = d + 1$ or*

(2) *$g = d$ and the adjacency matrix of Γ is of the form $A \otimes J$, where J is the all one $m \times m$ matrix for some $m \geqslant 2$, and A is the adjacency matrix of a distance-regular digraph with diameter $d - 1$ and girth g.*

The *cogirth* of a nonsymmetric Q-polynomial association scheme with the primitive idempotents $E_0, E_1, ..., E_d$, is $1 + \hat{1}$, where $E_1^{\mathrm{T}} = E_{\hat{1}}$.

THEOREM 2.2 [4]. *Let $\mathscr{X}$ be a nonsymmetric Q-polynomial association scheme of class d and cogirth g^*. Then $g^* = d + 1$ or $g^* = d$.*

THEOREM 2.3 [4, 5]. *Let $\mathscr{X}$ be a nonsymmetric P- and Q-polynomial association scheme with diameter d, girth g, and cogirth g^*. Then $\mathscr{X}$ is self-dual and either $g = g^* = d + 1$ or $g = g^* = d$.*

In the next section we prove a stronger result than Theorem 2.3 by showing that the case $g = g^* = d$ does not occur. Note that Theorem 2.2 does not assert anything about the structure of the association scheme with $d = g^*$, unlike Theorem 2.1. We supplement this in Theorem 3.3.

3. SOME GENERAL RESULTS

Damerell has shown that every distance-regular digraph with $d = g$ is obtained from a distance-regular digraph with diameter $g - 1$ by "tensoring with the all one matrix." We discuss this construction in more detail. Let $\mathscr{X} = (X, \{R_i\}_{i=0}^{d})$ be an arbitrary association scheme, $A_0, A_1, ..., A_d$, the adjacency matrices, and $E_0, E_1, ..., E_d$, the primitive idempotents. Let J be the $m \times m$ all one matrix, where $m \geqslant 2$ is a positive integer. Then the matrices $\tilde{A}_0 = A_0 \otimes I$, $\tilde{A}_1 = A_1 \otimes J, ..., \tilde{A}_d = A_d \otimes J$, $\tilde{A}_{d+1} = I \otimes (J - I)$ define a commutative association scheme $\tilde{\mathscr{X}}$ of class $d + 1$. To see this, simply check that the linear span of $\tilde{A}_0, ..., \tilde{A}_{d+1}$ is closed under the multiplication. We can also check that the primitive idempotents are $\tilde{E}_0 = E_0 \otimes (1/m)J$, $\tilde{E}_1 = E_1 \otimes (1/m)J, ..., \tilde{E}_d = E_d \otimes (1/m)J$, $\tilde{E}_{d+1} = I \otimes (I - (1/m)J)$. Let $E_j = (1/n) \sum_{i=0}^{d} q_j(i) A_i$, $\tilde{E}_j = (1/nm) \sum_{i=0}^{d+1} \tilde{q}_j(i) \tilde{A}_i$. The matrices $Q = (q_j(i))$, $\tilde{Q} = (\tilde{q}_j(i))$ are called the second eigenmatrices of $\mathscr{X}$, $\tilde{\mathscr{X}}$, respectively. It is easy to check that

$$\tilde{Q} = \left[\begin{array}{cccc|c} & & & & n(m-1) \\ & & & & 0 \\ & & Q & & \vdots \\ & & & & 0 \\ q_0(0) & q_1(0) & \cdots & q_d(0) & -n \end{array}\right].$$

Note that no column of $\tilde{Q}$ has all distinct entries, hence $\tilde{\mathscr{X}}$ is not Q-polynomial. According to Theorem 2.1, any distance-regular digraph with $d = g$ is obtained by the way we constructed $\tilde{\mathscr{X}}$ from $\mathscr{X}$. However, $\tilde{\mathscr{X}}$ cannot be Q-polynomial even if $\mathscr{X}$ is. By combining this result with Theorem 2.3, we obtain the following.

THEOREM 3.1. *In any nonsymmetric P- and Q-polynomial association scheme, the diameter is one less than the girth and the cogirth.*

Therefore, a distance-regular digraph with $d = g$ is a nonsymmetric *P-polynomial* association scheme which is not Q-polynomial. Dually, a nonsymmetric Q-polynomial association scheme with $d = g^*$ is not P-polynomial. We describe the structure of such nonsymmetric Q-polynomial association schemes.

Let $\mathscr{X}^{(1)}, \mathscr{X}^{(2)}, ..., \mathscr{X}^{(m)}$ $(m \geqslant 2)$ be commutative association schemes with the same parameters but not necessarily isomorphic. Let $A_i^{(j)}$ be the ith adjacency matrix of $\mathscr{X}^{(j)}$ and $E_i^{(j)}$ the ith primitive idempotent of $\mathscr{X}^{(j)}$, where $0 \leqslant i \leqslant d$, $0 \leqslant j \leqslant m$. Define

$$\tilde{A}_i = \begin{bmatrix} A_i^{(1)} & & 0 \\ & \ddots & \\ 0 & & A_i^{(m)} \end{bmatrix} \qquad (0 \leqslant i \leqslant d)$$

$$\tilde{A}_{d+1} = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}.$$

Then $\tilde{A}_0, \tilde{A}_1, ..., \tilde{A}_{d+1}$ define a commutative association scheme $\tilde{\mathscr{X}}$ of class $d+1$. We write $\tilde{\mathscr{X}} = \mathscr{X}^{(1)} \oplus \mathscr{X}^{(2)} \cdots \oplus \mathscr{X}^{(m)}$. The primitive idempotents of $\tilde{\mathscr{X}}$ are given by

$$\tilde{E}_0 = \frac{1}{nm} J$$

$$\tilde{E}_i = \begin{bmatrix} E_i^{(1)} & & 0 \\ & \ddots & \\ 0 & & E_i^{(m)} \end{bmatrix} \qquad (1 \leqslant i \leqslant d)$$

$$\tilde{E}_{d+1} = \frac{1}{nm} \begin{bmatrix} (m-1)J & & -1 \\ & \ddots & \\ -1 & & (m-1)J \end{bmatrix}.$$

Let $P = (p_j(i))$ be the first eigenmatrix of $\mathscr{X}^{(1)}, \mathscr{X}^{(2)}, ..., \mathscr{X}^{(m)}$, i.e., $A_j^{(k)} = \sum_{i=0}^{d} p_j(i) E_i^{(k)}$. Then the first eigenmatrix $\tilde{P} = (\tilde{p}_j(i))$, where $\tilde{A}_j = \sum_{i=0}^{d+1} \tilde{p}_j(i) \tilde{E}_i$, is given by

$$\tilde{P} = \begin{bmatrix} & & & & n(m-1) \\ & & & & 0 \\ & & P & & \vdots \\ & & & & 0 \\ p_0(0) & p_1(0) & \cdots & p_d(0) & -n \end{bmatrix}.$$

If $\mathscr{X}^{(1)}, \mathscr{X}^{(2)}, ..., \mathscr{X}^{(m)}$ are nonsymmetric Q-polynomial association schemes with cogirth $d+1$, then $\tilde{\mathscr{X}}$ is a nonsymmetric Q-polynomial association scheme of class $d+1$ and cogirth $g^*=d+1$. Indeed, by a result of Leonard [4], $\bar{E}_i^{(j)}=E_i^{(j)T}=E_{\hat{i}}^{(j)}$ with $\hat{i}=d+1-i$ for $1\leqslant i\leqslant d$, $1\leqslant j\leqslant m$. Let q_{ij}^k be the Krein parameter of $\mathscr{X}^{(1)},...,\mathscr{X}^{(m)}$. Then $\tilde{E}_1\circ\tilde{E}_i=(1/n)\sum_{k=1}^{i+1}q_{1i}^k\tilde{E}_k$, $0\leqslant i\leqslant d-1$. Therefore $\tilde{E}_i$ $(0\leqslant i\leqslant d)$ is a polynomial of degree i in $\tilde{E}_1$. Since

$$\begin{aligned}\tilde{E}_1\circ\tilde{E}_d&=\frac{1}{n}\sum_{k=1}^{d}q_{1d}^k\tilde{E}_k+\frac{1}{n}q_{1d}^0\frac{1}{n}J\otimes I\\&=\frac{1}{n}q_{1d}^0\tilde{E}_0+\frac{1}{n}\sum_{k=1}^{d}q_{1d}^k\tilde{E}_k+\frac{1}{n}q_{1d}^0\tilde{E}_{d+1},\end{aligned}$$

$\tilde{E}_{d+1}$ is a polynomial of degree $d+1$ in $\tilde{E}_1$. Thus $\tilde{\mathscr{X}}$ is a Q-polynomial association scheme. Since $\tilde{E}_1^{\mathrm{T}}=\tilde{E}_d$, the cogirth of $\tilde{\mathscr{X}}$ is $d+1$.

Now we want to show that every nonsymmetric Q-polynomial association scheme $\mathscr{X}$ of class $d+1$ with cogirth $g^*=d+1$ can be obtained by the above construction. Let $E_0, E_1, ..., E_{g^*}$ be the primitive idempotents of $\mathscr{X}$ such that E_i is a polynomial of degree i in E_1 with respect to the Hadamard product. Let $E_i\circ E_j=(1/n)\sum_{k=0}^{g^*}q_{ij}^kE_k$, where q_{ij}^k is the Krein parameter, and n is the size of $\mathscr{X}$. Then we have $\bar{E}_i=E_i^{\mathrm{T}}=E_{\hat{i}}$ with $\hat{i}=g^*-i$ for $1\leqslant i<g^*$, $\hat{g}^*=g^*$. Moreover, the Krein parameters satisfy a number of relations [1, Chap. III, Proposition 3.7]. We use these relations to obtain the following.

LEMMA 3.2. *If $\mathscr{X}$ is a nonsymmetric Q-polynomial association scheme of class g^*, with cogirth g^*, then the Krein parameters of $\mathscr{X}$ satisfy the following:*

(1) $q_{g^*g^*}^i=0$ *unless* $i=0$ *or* g^*,

(2) $q_{ig^*}^j=0$ *unless* $i=j$, *where* $1\leqslant i\leqslant g^*-1$.

Proof.

Step 1. $q_{1g^*}^j=0$ unless $j=1$ or g^*.

If $2\leqslant j\leqslant g^*-1$, then $1\leqslant\hat{j}=g^*-j\leqslant g^*-2$. So $q_{1\hat{j}}^{g^*}=0$, and $0=q_{1g^*}^{\hat{j}}=q_{1g^*}^j$.

Step 2. $q_{1g^*}^j=0$ unless $j=1$.

By Step 1, $E_1\circ E_{g^*}=(1/n)\sum_{j=0}^{g^*}q_{1g^*}^jE_j=(1/n)(q_{1g^*}^1E_1+q_{1g^*}^{g^*}E_{g^*})$. So $(nE_1-q_{1g^*}^{g^*}J)\circ(nE_{g^*}-q_{1g^*}^1J)=q_{1g^*}^1q_{1g^*}^{g^*}nJ$. If $q_{1g^*}^1q_{1g^*}^{g^*}\neq 0$, then any entry of the right-hand side is nonzero. Thus $(nE_{g^*}-q_{1g^*}^1J)_{ij}\neq 0$, $(nE_1-q_{1g^*}^{g^*}J)\neq 0$, for all i,j. But $nE_1-q_{1g^*}^1J$ is not symmetric and $nE_{g^*}-q_{1g^*}^{g^*}J$ is symmetric, so the left-hand side is not symmetric, a contradiction. Therefore, $q_{1g^*}^1 q_{1g^*}^{g^*}=0$. On the other hand $q_{1g^*}^1 = m_{g^*} q_{11}^{g^*}/m_1 = m_{g^*} q_{g^*-11}^{g^*}/m_1 \neq 0$. Thus $q_{1g^*}^{g^*}=0$.

Step 3. $q_{ig^*}^{g^*}=0$ for $1 \leqslant i \leqslant g^*-1$.

For the rest of the proof, E_1^i denotes the ith power of E_1 with respect to the Hadamard product. By Step 2, $E_1 \circ E_{g^*} = (1/n)\, q_{1g^*}^1 E_1$, so $E_1^i \circ E_{g^*} = (1/n)\, q_{1g^*}^1 E_1^i$ for $1 \leqslant i \leqslant g^*-1$. Since $\mathscr{X}$ is Q-polynomial, there exists a polynomial $v_i^*(x)$ of degree i such that $nE_i = v_i^*(nE_1)$, where n is the size of $\mathscr{X}$. Since the cogirth of $\mathscr{X}$ is g^*, $q_{1i}^0=0$. By induction on i, it follows that $v_i^*(x)$ has no constant term. Therefore E_i is a linear combination of $E_1, E_1^2, ..., E_1^i$, and

$$\begin{aligned} E_i \circ E_{g^*} &\in \langle E_1 \circ E_{g^*}, E_1^2 \circ E_{g^*}, ..., E_1^i \circ E_{g^*} \rangle \\ &= \langle E_1, E_1^2, ..., E_1^i \rangle \\ &= \langle E_1, E_2, ..., E_i \rangle. \end{aligned}$$

Thus $(E_i \circ E_{g^*})\, E_{g^*}=0$, $q_{ig^*}^{g^*}=0$, for $1 \leqslant i \leqslant g^*-1$.

From Step 3, it is immediate to show $q_{g^*g^*}^i=0$ for $1 \leqslant i \leqslant g^*-1$, so (1) is proved.

We show (2) by induction on i. The case $i=1$ was proved in Step 2. Suppose that the assertion is true up to $i-1$, for some $i \geqslant 2$. If $j<i$, then $q_{ig^*}^j = m_i q_{jg^*}^i/m_j = 0$ by induction. If $j>i$, then $i+\hat{j} = i+g^*-j < g^*$, so $0 = q_{i\hat{j}}^{g^*} = m_j q_{ig^*}^{\hat{j}}/m_{g^*} = m_j q_{ig^*}^j/m_{g^*}$. Thus $q_{ig^*}^j=0$. ∎

THEOREM 3.3. *Let $\mathscr{X}=(X, \{R_i\}_{i=0}^{g^*})$ be a nonsymmetric Q-polynomial association scheme of class g^* with cogirth g^*. Then $\mathscr{X}$ is imprimitive. There exist nonsymmetric Q-polynomial association schemes $\mathscr{X}^{(1)}, \mathscr{X}^{(2)}, ..., \mathscr{X}^{(m)}$ $(m \geqslant 2)$ of class g^*-1, with cogirth g^*, and the same parameters such that $\mathscr{X}$ is isomorphic to $\mathscr{X}^{(1)} \oplus \mathscr{X}^{(2)} \oplus \cdots \oplus \mathscr{X}^{(m)}$.*

Proof. Let $A_0, A_1, ..., A_{g^*}$ be the adjacency matrices of $\mathscr{X}$, $E_0, E_1, ..., E_{g^*}$ the primitive idempotents, and $n=|X|$. Let $\mathscr{U}=\langle A_0, A_1, ..., A_{g^*}\rangle$ be the adjacency algebra of $\mathscr{X}$, $\hat{\mathscr{U}}=\langle nE_0, nE_1, ..., nE_{g^*}\rangle$ the dual of $\mathscr{U}$. Then by Lemma 3.2(1), $\langle nE_0, nE_{g^*}\rangle$ is a C-subalgebra of $\hat{\mathscr{U}}$. Moreover by Lemma 3.2(2), $\langle nE_0+nE_{g^*}, nE_1, ..., nE_{g^*}\rangle$ is the quotient C-algebra $\hat{\mathscr{U}}/\langle nE_0, nE_{g^*}\rangle$. By [1, Theorem 9.9], there exists a C-subalgebra $\mathscr{U}_1$ of $\mathscr{U}$ such that $\hat{\mathscr{U}}_1 \cong \hat{\mathscr{U}}/\langle nE_0, nE_{g^*}\rangle$ and $(\mathscr{U}/\mathscr{U}_1)^\wedge \cong \langle nE_0, nE_{g^*}\rangle$. In particular, $\dim \mathscr{U}_1 = \dim \hat{\mathscr{U}}_1 = g^*$. We may assume $\mathscr{U}_1 = \langle A_0, A_1, ..., A_{g^*-1}\rangle$ with a suitable renumbering of the adjacency matrices. Then by [1, Theorem 9.3], $\bigcup_{i=0}^{g^*-1} R_i$ is an equivalence relation, and every equivalence class has the same size. Let $X_1, X_2, ..., X_m$ be the equivalence classes. Then $A_0|_{X_i}$, $A_1|_{X_i}, ..., A_{g^*-1}|_{X_i}$ define an association scheme on X_i, whose adjacency algebra is isomorphic to $\mathscr{U}_1$. Since $\hat{\mathscr{U}}_1 \cong \hat{\mathscr{U}}/\langle nE_0, nE_{g^*}\rangle = \langle nE_0+nE_{g^*},$

$nE_1, ..., nE_{g^*-1}\rangle$ is a C-algebra of P-polynomial type, the association scheme $\mathscr{X}^{(i)}$ on X_i is Q-polynomial. Since $R_{g^*} = X \times X - \bigcup_{i=0}^{g^*-1} R_i$, it is now easy to see that $\mathscr{X} \cong \mathscr{X}^{(1)} \oplus \mathscr{X}^{(2)} \oplus \cdots \oplus \mathscr{X}^{(m)}$. ∎

Next we investigate nonsymmetric P- and Q-polynomial association schemes in the context of C-algebras. Suppose that there exists a nonsymmetric P- and Q-polynomial association scheme with the adjacency matrices $A_0, A_1, ..., A_d$, $A_i = v_i(A_1)$, $v_i(x)$ is a polynomial of degree i. Let $\theta_0, \theta_1, ..., \theta_d$ be the eigenvalues of A_1, with the multiplicities $m_0 = 1, m_1, ..., m_d$, and $\bar{\theta}_i = \theta_{d+1-i}$. Then $\operatorname{tr} A_1^j = 0$ implies that $\sum_{i=0}^{d} \theta_i^j m_i = n\delta_{j0}$, where n is the size of the association scheme. We may regard this as a system of linear equations with unknowns $m_0, m_1, ..., m_d$. We can actually solve this to find

$$m_i = \frac{n\theta_0 \cdots \hat{\theta}_i \cdots \theta_d}{\prod_{k \neq i} (\theta_k - \theta_i)}.$$

Since $m_0 = 1$, we see that $\theta_1, ..., \theta_d$ are nonzero and

$$m_i = \frac{\theta_0 \prod_{k \neq 0} (\theta_k - \theta_0)}{\theta_i \prod_{k \neq i} (\theta_k - \theta_i)}. \tag{3.1}$$

The polynomials $v_i(x)$, $i = 0, 1, ..., d$, satisfy the following:

$$\deg v_i(x) = i, \qquad i = 0, 1, ..., d, \tag{3.2}$$

$$v_0(x) = 1, \qquad v_1(x) = x, \tag{3.3}$$

$$v_i(0) = 0, \qquad i = 1, 2, ..., d, \tag{3.4}$$

$$\frac{v_i(\theta_j)}{m_i} = \frac{v_j(\theta_i)}{m_j}, \qquad i, j = 0, 1, ..., d. \tag{3.5}$$

Equation (3.4) follows from $\operatorname{tr} A_1^i = 0$, $i = 1, ..., d$, while (3.5) follows from self-duality. By (3.3) and (3.5), we have $\theta_0 = m_1$, hence by (3.1),

$$\theta_1 \prod_{k \neq 1} (\theta_k - \theta_1) = \prod_{k \neq 0} (\theta_k - \theta_0). \tag{3.6}$$

Given nonzero distinct complex numbers $\theta_0, \theta_1, ..., \theta_d$ satisfying (3.6), m_i is determined by (3.1) and there exist unique polynomials $v_i(x)$ of degree at most i satisfying (3.3)–(3.5). Indeed, $v_i(x)$ $(i \geq 2)$ is uniquely determined inductively by $v_i(0) = 0$, $v_i(\theta_0) = m_i$, $v_i(\theta_1) = m_i v_1(\theta_i)/m_1, ..., v_i(\theta_{i-1}) = m_i v_{i-1}(\theta_i)/m_{i-1}$. We then want to show that $v_i(x)$ is of degree exactly i, that is, (3.2) holds. Suppose $\deg v_k(x) < k$ for some k. We may assume that k is the smallest integer such that $\deg v_k(x) < k$. Then $v_0(x), v_1(x), ..., v_k(x)$

are linearly dependent. By (3.4), $v_1(x), ..., v_k(x)$ must be linearly dependent, thus

$$\det\begin{bmatrix} v_1(\theta_0) & \cdots & v_1(\theta_{k-1}) \\ \vdots & & \vdots \\ v_k(\theta_0) & \cdots & v_k(\theta_{k-1}) \end{bmatrix} = 0.$$

Therefore, by (3.5),

$$\det\begin{bmatrix} v_0(\theta_1) & \cdots & v_{k-1}(\theta_1) \\ \vdots & & \vdots \\ v_0(\theta_k) & \cdots & v_{k-1}(\theta_k) \end{bmatrix} = 0,$$

which is a contradiction, since the above is essentially Vandermonde's determinant. Therefore, given nonzero distinct complex numbers $\theta_0, \theta_1, ..., \theta_d$ satisfying (3.6), there exist unique nonzero complex numbers m_i, $i = 0, 1, ..., d$, and unique polynomials $v_i(x)$, $i = 0, 1, ..., d$, satisfying (3.1)–(3.5). The method given in [4] can be carried out to produce the closed form for the parameters; however, to establish the closed form, it is sufficient to assume (3.1)–(3.6).

Write

$$xv_j(x) \equiv \sum_{k=0}^{d} p_{1j}^k v_k(x) \qquad (\mathrm{mod}(\varphi(x))),$$

where $\varphi(x) = \prod_{k=0}^{d} (x - \theta_k)$. Let N, N', B' be the square matrices of degree $d+1$ whose (i, j)-entries are $v_i(\theta_j)$, $v_i(\theta_{j+1})$, p_{1j}^{i+1}, respectively, with the convention $\theta_{d+1} = \theta_0$, $p_{1j}^{d+1} = p_{1j}^0$. When considering an LU decomposition (see [4]) of the matrices N, N' we obtain

$$\begin{aligned} &\mathrm{diag}(\theta_0, \theta_1, ..., \theta_d)\, U\, \mathrm{diag}(m_0, m_1, ..., m_d) \\ &\quad = U'\, \mathrm{diag}(m_1, ..., m_d, m_0) B', \end{aligned} \tag{3.7}$$

where $U = ([\theta_j]_i)$, $U' = ([\theta_{j+1}]'_i)$, $[x]_j = (x-\theta_0)(x-\theta_1)\cdots(x-\theta_{j-1})$, $[x]'_j = (x-\theta_1)\cdots(x-\theta_j)$. In comparing the entries of (3.7), we find

$$p_{1i}^{i+1} = \frac{\theta_i [\theta_i]_i m_i}{[\theta_{i+1}]'_i m_{i+1}}, \tag{3.8}$$

$$p_{1i}^{i} = \frac{(\theta_i - \theta_0)(\theta_{i+1}\theta_{i-1} - \theta_i^2)}{(\theta_i - \theta_{i-1})(\theta_{i+1} - \theta_i)}. \tag{3.9}$$

THEOREM 3.4. *Let $\theta_0, \theta_1, ..., \theta_d$ be nonzero distinct complex numbers satisfying $\theta_0 = \bar{\theta}_0$, $\bar{\theta}_i = \theta_{d+1-i}$ $(i = 1, ..., d)$ and (3.6). Let m_i, $v_i(x)$ be* *defined by* (3.1)–(3.5). *Then* $\mathscr{U}=\mathbb{C}[x]/(\prod_{i=0}^{d}(x-\theta_i))=\langle v_i(x)\,|\,0\leqslant i\leqslant d\rangle$ *is a C-algebra if and only if*

$$v_i(x)\in\mathbb{R}[x] \qquad \textit{for} \quad i=0,1,...,d,$$

$$m_i>0 \qquad \textit{for} \quad i=0,1,...,d.$$

Proof. Suppose that $\mathscr{U}$ is a C-algebra. Since for any $i=1,2,...,d-1$,

$$xv_i(x)=\sum_{j=0}^{i+1}p_{1i}^{j}v_j(x)$$

and $p_{1i}^{j}\in\mathbb{R}$, we see that $v_i(x)\in\mathbb{R}[x]$ by induction. Write

$$v_i(x)\,v_j(x)=\sum_{k=0}^{d}p_{ij}^{k}v_k(x)+c_{ij}(x)\prod_{k=0}^{d}(x-\theta_k).$$

Assume $i\neq 0$ and set $x=0$. Then

$$0=p_{ij}^{0}+c_{ij}(0)(-1)^{d+1}\,\theta_0\theta_1\cdots\theta_d,$$
$$\delta_{i'j}k_i=(-1)^{d}\,c_{ij}(0)\,\theta_0\theta_1\cdots\theta_d,$$

with $k_i>0$ by the definition of C-algebra. If we take $j=d+1-i$, then $c_{i,d+1-i}=c_{i,d+1-i}(x)$ is a nonzero constant. This forces $i'=d+1-i$. Indeed, $c_{i,i'}$ is the product of the leading coefficients of $v_i(x)$ and $v_{i'}(x)$. By (3.8), the leading coefficient of $v_i(x)$ is

$$\frac{m_i[\theta_i]_i}{m_1\theta_1\cdots\theta_{i-1}\prod_{j=1}^{i}(\theta_j-\theta_0)},$$

and the leading coefficient of $v_{i'}(x)$ is

$$\frac{\bar{m}_i[\theta_{i'}]_{i'}}{\theta_0\theta_1\cdots\theta_{i'-1}\prod_{j=1}^{i'}(\theta_j-\theta_0)}=\frac{m_i(\theta_i-\theta_0)(\theta_i-\theta_d)\cdots(\theta_i-\theta_{i+1})}{\theta_0\theta_d\cdots\theta_{i+1}\prod_{j=i}^{d}(\theta_j-\theta_0)},$$

since it is real. Thus it follows that $0<k_i=(-1)^{d}\,c_{i,d+1-i}\theta_0\theta_1\cdots\theta_d=m_i$.

Conversely, by assuming $v_i(x)\in\mathbb{R}[x]$, $m_i>0$, we verify that $\mathscr{U}$ is a C-algebra. Clearly $\mathscr{U}$ is a commutative algebra with the identity $v_0(x)=1$. Since $v_i(x)\in\mathbb{R}[x]$, the structure constants p_{ij}^{k} are real. Also $v_i(\bar{\theta}_j)=\overline{v_i(\theta_j)}=m_i\overline{v_j(\theta_i)}/m_j=m_i v_j\overline{(\theta_i)}/m_j=m_{d+1-i}v_j(\theta_{d+1-i})/m_j=v_{d+1-i}(\theta_j)$. Define $0'=0$, $i'=d+1-i$. Then $v_i(x)\to v_{i'}(x)$ induces an automorphism of $\mathscr{U}$. From the first part of the proof, $p_{ii'}^{0}=p_{i,d+1-i}^{0}=m_i$, which is positive. If $0\leqslant i+j\leqslant d$, then set $x=0$ in $v_i(x)\,v_j(x)=\sum_{k=0}^{d}p_{ij}^{k}v_k(x)$. We see that $p_{ij}^{0}=0$ by (3.4). If $i+j>d+1$, then $i'+j'<d+1$ and $p_{ij}^{0}=p_{i'j'}^{0}$. Thus $p_{ij}^{0}=0$ by the previous case. Therefore $p_{ij}^{0}=\delta_{ij'}k_i$ with $k_i>0$. The mapping

$v_i(x) \to k_i$ is the evaluation at $x = \theta_0$, thus it is a linear representation of $\mathscr{U}$. This completes the proof. ∎

The last result of this section is to show that the eigenvalues of nonsymmetric P- and Q-polynomial association scheme are quadratic over $\mathbb{Q}$. Let p_{ij}^k be the intersection numbers of a non-symmetric P- and Q-polynomial association scheme $\mathscr{X}$, with diameter d and $i' = d+1-i$ $(i = 1, ..., d)$, as shown in [2, Theorem 1]. Let $k_i = p_{ii'}^0$ be the ith valency and write $b_i = p_{1'i+1}^i$. As an analogue to the symmetric case (see [1, Chap. III, Propositions 1.2, 1.4]), we have the following.

LEMMA 3.5. (1) $k_1 = b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{d-1}$.

(2) *$\{k_i\}$ satisfies the unimodality property, i.e.,*

$$1 = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_{[(d+1)/2]} \geqslant \cdots \geqslant k_{d-1} \geqslant k_d,$$

with $k_i = k_{d+1-i}$ for $i = 1, 2, ..., d$.

Proof. (1) Let $\Gamma_j(x)$ be the set of vertices at distance j from x. Suppose that u, v, x are vertices and $d(u, x) = i$, $d(v, x) = i-1$, $d(u, v) = 1$. Then $\Gamma_1(x) \cap \Gamma_{i+1}(u) \subset \Gamma_1(x) \cap \Gamma_i(v)$, so $b_i \leqslant b_{i-1}$.

(2) $k_i b_i = k_i p_{1'i+1}^i = k_{i+1} p_{1i}^{i+1} = k_{i+1} b_{d-i}$. Thus, if $2i \leqslant d$, then by (1), we have $k_i \leqslant k_{i+1}$. ∎

Remark. Lemma 3.5(1) holds for an arbitrary distance-regular digraph. Lemma 3.5(2) holds for an arbitrary distance-regular digraph with $d+1 = g$. For a distance-regular digraph with $d = g$, the following analogue holds:

$$1 = k_0 \leqslant k_1 \leqslant \cdots \leqslant k_{[d/2]} \geqslant \cdots \geqslant k_{d-1}.$$

THEOREM 3.6. *The eigenvalues of a nontrivial nonsymmetric P- and Q-polynomial association scheme are at most quadratic over $\mathbb{Q}$.*

Proof. Let $\theta_0, \theta_1, ..., \theta_d$ be the eigenvalues of a nontrivial nonsymmetric P- and Q-polynomial association scheme $\mathscr{X}$, θ_0 the valency, and $\bar{\theta}_i = \theta_{d+1-i}$. If θ_1 is not quadratic over $\mathbb{Q}$, then θ_1 is algebraic conjugate to θ_i for some $i \neq 1, d$. Let k_i be the ith valency. Then $k_i = k_1$ by self-duality. By Lemma 3.5(2), we must have $k_1 = k_2$. Since $k_1 b_1 = k_2 b_{d-1}$, Lemma 3.5(1) implies $b_1 = b_2 = \cdots = b_{d-1}$. On the other hand, it is shown in [5] that if θ_1 is not quadratic over $\mathbb{Q}$, then $k_1 = b_1$. Thus $k_1 = b_i = p_{1'i+1}^i$ $(i = 0, 1, ..., d-1)$. So $p_{1'j}^i = \delta_{i+1,j} k_1$, if $i < d$. In particular, $p_{1d}^i = p_{1'1}^i = \delta_{i0} k_1$, if $i < d$. Also $p_{1d}^d = p_{1'1}^1 = k_1 - b_1 = 0$. Therefore $xv_d(x) \equiv k_1 v_0(x) \equiv k_1$ (mod $\varphi(x)$). By setting $x = k_1$, we find $k_1^2 = k_1$; hence $k_1 = 1$. Thus $\mathscr{X}$ is a directed cycle, a contradiction.

Since $xv_i(x) = \sum_{j=0}^{i+1} p_{1i}^j v_j(x)$ with p_{1i}^j being an integer, $v_i(x) \in \mathbb{Q}[x]$ for all $i = 0, 1, ..., d$. By (3.5), θ_i is a rational polynomial of degree i in θ_1. Thus all the eigenvalues belong to $\mathbb{Q}(\theta_1)$, and hence are at most quadratic. ∎

Remark. The proof of Theorem 3.6 was suggested by Eiichi Bannai. The proof of Theorem 3.6 shows not only that each eigenvalue is at most quadratic over $\mathbb{Q}$, but also that all eigenvalues belong to the imaginary quadratic extension $\mathbb{Q}(\theta_1)$. Since the eigenvalues belong to the ring of integers of $\mathbb{Q}(\theta_1)$ and the absolute value of the eigenvalues are at most k_1, it follows that the diameter of the non-symmetric P- and Q-polynomial association scheme is bounded by a function of the valency k_1.

4. The Case $d+1=g=5$

In this section we apply Theorem 3.4 together with Theorem 3.6 to show that the only nonsymmetric P- and Q-polynomial association scheme with girth 5 is the directed 5 cycle. By Theorem 2.3 and Theorem 3.1, such an association scheme has diameter 4 and is self-dual. By specializing Theorem 3.4 to the case $d=4$, we obtain the following.

THEOREM 4.1. *For real numbers x_1, x_2, the following are equivalent.*

(1) *There exist nonzero distinct complex numbers $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$, satisfying* (3.6) *with $d=4$, $\theta_0 = \bar{\theta}_0$, $\operatorname{Re}\theta_1 = \operatorname{Re}\theta_4 = x_1$, $\theta_1 = \bar{\theta}_4$, $\operatorname{Re}\theta_2 = \operatorname{Re}\theta_3 = x_2$, $\theta_2 = \bar{\theta}_3$, such that $\mathscr{U} = \mathbb{C}[x]/(\prod_{i=0}^4 (x-\theta_i)) = \langle v_i(x) \mid 0 \leqslant i \leqslant 4\rangle$ is a C-algebra, where $v_i(x)$, $0 \leqslant i \leqslant 4$, are defined by* (3.1)–(3.5).

(2)

$$4(2x_1+1)(x_1-x_2+2x_2^2)(x_2^2+x_2-x_1) = x_2(6x_1x_2+2x_1-2x_2^2+x_2)^2, \tag{4.1}$$

$$x_2 > 0 \quad \textit{and} \quad x_1 < \min\{x_2 - 2x_2^2, -\tfrac{1}{2}\},$$

or

$$x_2 < 0 \quad \textit{and} \quad x_1 > \max\{x_2 + \tfrac{1}{2}, x_2 + x_2^2, -\tfrac{1}{2}\}. \tag{4.2}$$

Proof. Suppose that (1) holds. Write $\theta_0 = k$, $\theta_1 = \bar{\theta}_4 = x_1 + \sqrt{-1}\, y_1$, $\theta_2 = \bar{\theta}_3 = x_2 + \sqrt{-1}\, y_2$, $k, y_1, y_2 \in \mathbb{R}$. By Theorem 3.4, we have $k = m_1 > 0$, $m_2 > 0$. Thus $m_1 = m_4$, $m_2 = m_3$, and

$$k + 2x_1 m_1 + 2x_2 m_2 = 0,$$
$$k^2 + 2(x_1^2 - y_1^2)\, m_1 + 2(x_2^2 - y_2^2)\, m_2 = 0.$$

By eliminating m_2, we obtain

$$k(x_2^2 - y_2^2 - x_2 k) = 2m_1\{x_2(x_1^2 - y_1^2) - x_1(x_2^2 - y_2^2)\}, \tag{4.3}$$

$$x_2^2 - y_2^2 - x_2 k = 2x_2(x_1^2 - y_1^2) - 2x_1(x_2^2 - y_2^2). \tag{4.4}$$

By (3.8) and (3.1), we have

$$\theta_1(\theta_1 - \theta_0)/(\theta_2 - \theta_1) \in \mathbb{R} \tag{4.5}$$

$$\operatorname{Re}(\theta_1 - \theta_2)^2 (\theta_1 - \theta_3) = 0; \tag{4.6}$$

i.e.,

$$(x_1 - k)(x_1 y_2 - x_2 y_1) - y_1\{x_1(x_2 - x_1) + y_1(y_2 - y_1)\} = 0, \tag{4.7}$$

$$(x_1 - x_2)(y_2^2 + 2y_1 y_2 + (x_1 - x_2)^2 - 3y_1^2) = 0. \tag{4.8}$$

Since $p_{11}^1 + p_{12}^2$ is real, we obtain

$$x_2(x_2 - k) - y_2^2 = 2y_1 y_2. \tag{4.9}$$

One can directly check that $x_1 = x_2$ leads to a contradiction. Thus we have

$$y_2^2 + 2y_1 y_2 + (x_1 - x_2)^2 - 3y_1^2 = 0. \tag{4.10}$$

By eliminating k from (4.4), (4.7), (4.9), we have

$$x_1 y_2^2 - y_1 y_2 + x_2(x_1^2 - y_1^2) - x_1 x_2^2 = 0, \tag{4.11}$$

$$\begin{aligned} x_1 y_2^3 + (2x_1 - x_2)\, y_1 y_2^2 + (x_1^2 - x_1 x_2 - 3y_1^2)\, x_2 y_2 \\ + x_2 y_1\{(x_1 - x_2)^2 + y_1^2\} = 0. \end{aligned} \tag{4.12}$$

From (4.10) and (4.12), we have

$$\begin{aligned} \{(3x_1 - x_2)\, y_1^2 - x_1(x_1 - x_2)(x_1 - 2x_2)\}\, y_2 \\ + 2x_2 y_1\{(x_1 - x_2)^2 - y_1^2\} = 0. \end{aligned} \tag{4.13}$$

By eliminating y_2^2 from (4.10), (4.11), we obtain

$$(2x_1 + 1)\, y_1 y_2 - (3x_1 - x_2)\, y_1^2 + x_1(x_1 - x_2)(x_1 - 2x_2) = 0. \tag{4.14}$$

From (4.13) and (4.14), we have

$$(2x_1 + 1)\, y_2^2 + 2x_2\{(x_1 - x_2)^2 - y_1^2\} = 0. \tag{4.15}$$

We can solve (4.10), (4.11), and (4.15) linearly for y_1^2, $y_1 y_2$, y_2^2 to obtain

$$y_1^2 = (x_1 - x_2)(x_1 - x_2 + 2x_2^2)/3, \tag{4.16}$$

$$y_1 y_2 = \frac{x_2(x_1 - x_2)(6x_1 x_2 + 2x_1 - 2x_2^2 + x_2)}{3(2x_1 + 1)}, \tag{4.17}$$

$$y_2^2 = \frac{4x_2(x_1 - x_2)(x_2^2 + x_2 - x_1)}{3(2x_1 + 1)}. \tag{4.18}$$

Note that one can easily check that $2x_1 + 1 \neq 0$. From (4.16), (4.17), (4.18), we find (4.1). Also from (4.9), (4.17), (4.18), we find

$$k = x_2(2x_2 - 2x_1 + 1) > 0. \tag{4.19}$$

Since $k = m_1 > 0$ and $m_2 > 0$, the equality preceding to (4.3) implies

$$x_2(2x_1 + 1) < 0. \tag{4.20}$$

By (4.16), (4.18), (4.20), we have

$$(x_1 - x_2)(x_1 - x_2 + 2x_2^2) > 0, \tag{4.21}$$

$$(x_1 - x_2)(x_1 - x_2 - x_2^2) > 0. \tag{4.22}$$

Now it is straightforward to check that (4.2) is equivalent to the inequalities (4.19)–(4.22).

Conversely, suppose that x_1, x_2 satisfy (4.1), (4.2). Then the inequalities (4.19)–(4.22) hold. In particular, $x_1 \neq -\frac{1}{2}$, $x_2 \neq 0$, $x_1 \neq x_2$. Then the right-hand sides of (4.16), (4.18) are positive, so we can find nonzero real numbers y_1, y_2 satisfying (4.16), (4.18). y_1, y_2 are determined up to sign, but we may choose their signs so that y_1, y_2 satisfy (4.17). This is possible by (4.1). Define k by (4.19). By reversing the argument in the first part of the proof, we can recover (4.4)–(4.18). By (4.5) and (4.6) we see that $m_1 \in \mathbb{R}$. We can directly check that $m_2 \in \mathbb{R}$ by (4.16)–(4.18) and (4.1). Therefore, as in the first part of the proof, we obtain (4.3). Thus $m_1 = k$ by (4.4). Hence (3.6) holds and $m_1 > 0$. Also,

$$m_2 = -\frac{k(2x_1 + 1)}{2x_2} > 0. \tag{4.23}$$

By Theorem 3.4, it now suffices to show $v_i(x) \in \mathbb{R}[x]$, $i = 0, 1, ..., 4$. By the definition, $v_0(x)$, $v_1(x) \in \mathbb{R}[x]$. By (4.5), we have $p_{11}^2 \in \mathbb{R}$, thus the leading coefficient of $v_2(x)$ is real. Since $v_2(0) = 0$, $v_2(\theta_0) = m_2 \in \mathbb{R}$, we see that $v_2(x) \in \mathbb{R}[x]$. By (3.8), $p_{12}^3 \in \mathbb{R}$ since $m_2 \in \mathbb{R}$. Since $xv_1(x) = p_{11}^2 v_2(x) + p_{11}^1 v_1(x)$ and $p_{11}^2 v_2(x) \in \mathbb{R}[x]$, we see that $p_{11}^1 \in \mathbb{R}$. Then by

(4.9), we have $p_{12}^2 \in \mathbb{R}$. By setting $x=k$ in $p_{12}^3 v_3(x) = xv_2(x) - p_{12}^2 v_2(x) - p_{12}^1 v_1(x)$, we find $p_{12}^1 \in \mathbb{R}$, hence $v_3(x) \in \mathbb{R}[x]$. Since $v_0(x)+v_1(x)+\cdots+v_4(x) = (\theta_1\theta_2\theta_3\theta_4)^{-1}(x-\theta_1)(x-\theta_2)(x-\theta_3)(x-\theta_4) \in \mathbb{R}[x]$, $v_4(x)$ also belongs to $\mathbb{R}[x]$. Therefore $\mathscr{U}$ is a C-algebra by Theorem 3.4. ∎

Remark. The set of (x_1, x_2) satisfying (4.1), (4.2) is not empty. For example, one can take $x_1 = -(21+5\sqrt{33})/32$, $x_2 = \frac{1}{2}$. However, the C-algebra obtained cannot be the adjacency algebra of a nonsymmetric P- and Q-polynomial association scheme, since θ_1 is not an algebraic integer for the above x_1.

By Theorem 3.6 and Theorem 4.1, if there exists a nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 5, then there exists a solution (x_1, x_2) of (4.1) such that $2x_1$, $2x_2$ are rational integers. Write $X_1 = 2x_1$, $Y = 2x_2$, and rewrite (4.1) as

$$8X^3 + (9Y^3 + 16Y^2 - 12Y + 8)X^2 - (10Y^4 + 2Y^3 - 16Y^2 + 16Y)X + Y^5 - 6Y^4 - 3Y^3 + 8Y^2 = 0. \tag{4.24}$$

It turns out that Eq. (4.24) has only finitely many integer solutions, and none of them corresponds to a nonsymmetric P- and Q-polynomial association scheme. We omit the details of the determination of all integer solutions of (4.24), which is rather complicated. Instead, in the proof of Theorem 4.2, we use the integrality of the intersection numbers to obtain a bound for Y.

THEOREM 4.2. *The only nonsymmetric P- and Q-polynomial association scheme with girth 5 is the directed 5 cycle.*

Proof. If there exists a nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 5, there exists an integer solution (X, Y) of (4.24) such that $x_1 = X/2$, $x_2 = Y/2$ satisfy the conditions of Theorem 4.1. By (3.8), (4.16), (4.19), (4.23), we find

$$p_{11}^2 = -\frac{Y(Y^2 - Y - 3XY - 2X)}{6(X+1)}.$$

Since $\mathscr{U}$ is the adjacency algebra of a nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 5, p_{11}^2 is an integer. Thus $Y(Y^2+2Y+2)/(X+1)$ is an integer. Since $Y = 2x_2 \neq 0$ and $Y^2+2Y+2>0$, we have

$$|Y|(Y^2+2Y+2) \geqslant |X+1|. \tag{4.25}$$

Moreover,

$$k = p^1_{10} + p^1_{11} + p^1_{12} + p^1_{13} + p^1_{14} = 1 + 2p^1_{11} + 2p^1_{12},$$

so k is odd. The trace of the intersection matrix $B_1 = (p^j_{1i})$ is $\theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 = k + X + Y$. Also $\operatorname{tr} B_1 = p^0_{10} + p^1_{11} + p^2_{12} + p^3_{13} + p^4_{14} = 2p^1_{11} + 2p^2_{12}$, which is even. Thus $X + Y$ is odd. By (4.19), Y must be odd. Let $f(X, Y)$ be the left-hand side of (4.24).

Case 1. $Y > 0$.

By (4.2), (4.25), we have $-1 > X \geqslant -Y^3 - 2Y^2 - 2Y - 1$. Then $f(X, Y) \geqslant X^2(Y^3 - 28Y) - 2XY(5Y^3 + Y^2 - 8Y + 8) + Y^2(Y^3 - 6Y^2 - 3Y + 8)$. If $Y \geqslant 7$, then $f(X, Y) > 0$, a contradiction. Thus $Y \leqslant 5$.

Case 2. $Y < 0$.

By (4.2), (4.27), we have $0 \leqslant X \leqslant -Y^3 - 2Y^2 - 2Y - 1$. Then $f(X, Y) \leqslant X^2(Y^3 - 28Y) - 2XY(5Y^3 + Y^2 - 8Y + Y) + Y^2(Y^3 - 6Y^2 - 3Y + 8)$. If $Y \leqslant -7$, then $f(X, Y) < 0$, a contradiction. Thus $Y \geqslant -5$.

Now we conclude $Y \in \{\pm 1, \pm 3, \pm 5\}$. The only integer solutions of (4.24) with $Y \in \{\pm 1, \pm 3, \pm 5\}$ are $(X, Y) = (-2, -1)$, $(0, 1)$. However, neither satisfies (4.2). This is a contradiction. ∎

Acknowledgment

The author thanks Eiichi Bannai for many valuable suggestions, especially on the proof of Theorem 3.6.

References

1. E. Bannai and T. Ito, "Algebraic Combinatorics, I," Benjamin/Cummings, Menlo Park, CA, 1984.
2. R. M. Damerell, Distance-transitive and distance-regular digraphs, *J. Combin. Theory Ser. B* **31** (1981), 46–53.
3. C. W. Lam, Distance-transitive digraphs, *Discrete Math.* **29** (1980), 265–274.
4. D. Leonard, Directed distance regular graphs with the Q-polynomial property, *J. Combin. Theory Ser. B* **48** (1990), 191–196.
5. D. Leonard, Non-symmetric, metric, cometric association schemes are self-dual, *J. Combin. Theory Ser. B* **51** (1991), 244–247.
6. R. A. Liebler and R. A. Mena, Certain distance-regular digraphs and related rings of characteristic 4, *J. Combin. Theory Ser. A* **47** (1988), 111–123.