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Deformed commutators on quantum group module-algebras

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Abstract

We construct quantum commutators on module-algebras of quasi-triangular Hopf algebras. These are quantum-group covariant and have generalized antisymmetry and Leibniz properties. If the Hopf algebra is triangular they additionally satisfy a generalized Jacobi identity, turning the module-algebra into a quantum-Lie algebra.

Keywords: Quantum groups; Module-algebras; Commutators; Quantum Lie algebras

The purpose of this short communication is to present a quantum commutator structure which appears naturally on any module algebra A of a quantum group H. In Section 1 we write down the main properties we require from a generalized commutator on a quantum group module-algebra, and we give its definition. In Section 2 we prove a theorem collecting the main properties of this algebraic structure. Finally, in Section 3 we develop an example, showing some explicit calculations for the reduced $SL_q(2,\mathbb{C})$ quantum plane. We refer the reader to Appendix A for notation and some basic facts on quasi-triangular Hopf-algebras.

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1. The q-commutator

Let H be a quasi-triangular Hopf algebra. Take A some H-module-algebra (a left one, say). As usual, we will denote the action of $h \in H$ on $a \in A$ by $h \triangleright a$, and the coproduct using the Sweedler notation $\Delta h = h_1 \otimes h_2$. Being a left-module-algebra, of course $h \triangleright (ab) = (h_1 \triangleright a)(h_2 \triangleright b)$. As our main goal is to define a covariant commutator for which some generalized Leibniz rule holds on both variables, a natural way to start is proposing a deformation of the usual $[a, b] = ab - ba$ structure valid on any associative algebra. The deformation we start with is

$$[a, b]_\chi \equiv m \circ (1 - \chi)(a \otimes b) = ab - m\bigl(\chi(a \otimes b)\bigr). \tag{1}$$

Here m is the product on A and the linear map

$$\chi : A \otimes A \longmapsto A \otimes A,$$

which replaces the standard transposition operator τ, needs to be determined. Later on, we will sometimes use the generic decomposition

$$\chi(a \otimes b) = \sum_i \sigma_a^i(b) \otimes e_i, \quad \{e_i\} \text{ vector space basis of } V. \tag{2}$$

Clearly, the maps σ^i have to be linear in both a and b.

The most basic property we require the commutators to satisfy is some adequate generalization of the Leibniz rule, on both variables. Such a rule means that commuting the first (say) variable a to the right through a product bc must be equivalent to commuting it in two steps, first through b and then through c. Expressed in terms of the map χ, which generalizes and deforms the permutation, this would read

$$\chi \circ (1 \otimes m) = (m \otimes 1) \circ (1 \otimes \chi) \circ (\chi \otimes 1),$$
$$\chi \circ (m \otimes 1) = (1 \otimes m) \circ (\chi \otimes 1) \circ (1 \otimes \chi),$$

where the second relation come from commuting the second variable c to the left through a product ab.

Note now the analogy between the above conditions and the ones required on the braiding [1],

$$\chi_{V,W} : V \otimes W \longmapsto W \otimes V$$

of a braided monoidal category. These are

$$\chi_{V,W\otimes U} = (1 \otimes \chi_{V,U}) \circ (\chi_{V,W} \otimes 1), \qquad \chi_{V\otimes W,U} = (\chi_{V,U} \otimes 1) \circ (1 \otimes \chi_{W,U})$$

and illustrate the fact that moving an element of V to the right through $W \otimes U$ (respectively an element of U to the left through $V \otimes W$) should produce the same result if it is done in one or two steps. Note that V, W, and U are not even vector spaces in the general case, and that our map χ acts on an algebra. However, remembering the standard result that shows that the category of H-modules of a quasi-triangular Hopf-algebra H is braided (see [1], for instance), we take here the same braiding as an Ansatz and we will show in the next section that it satisfies the required conditions.

Concretely, we take

$$\chi(a \otimes b) \equiv (R_2 \triangleright b) \otimes (R_1 \triangleright a), \tag{3}$$

where

$$R \equiv R_1 \otimes R_2$$

is the R-matrix of H (cf. Appendix A). Of course, a generic sum of the type $R = \sum_k R_1^k \otimes R_2^k$ is understood. Note that we could also use the second quasi-triangular structure $\overline{R}$, obtaining a map $\overline{\chi}$ which will differ from χ unless H is triangular. As it is easy to see from the definition of $\overline{R}$, this second map $\overline{\chi}$ is the inverse of the first one,

$$\overline{\chi} \circ \chi = \chi \circ \overline{\chi} = 1.$$

The properties of R imply now

$$\chi(a \otimes \mathbf{1}) = \mathbf{1} \otimes a, \qquad \chi(\mathbf{1} \otimes a) = a \otimes \mathbf{1}$$

and therefore

$$[\mathbf{1}, a]_\chi = [a, \mathbf{1}]_\chi = 0 \quad \forall a \in A.$$

However, note that in general it will be

$$[a, a]_\chi \neq 0$$

because $\chi(a \otimes a) = (R_2 \triangleright a) \otimes (R_1 \triangleright a)$ is a priori different from $a \otimes a$.

2. Properties of the commutator

2.1. q-Leibniz rules

As was the aim when defining the deformed commutator, we have the following lemma.

Lemma. *The map* $[\,,\,]_\chi$ *has a Leibniz property on the second variable reading*

$$[a, bc]_\chi = [a, b]_\chi c + \sigma_a^i(b)[e_i, c]_\chi \tag{4}$$

or, equivalently,

$$\chi(a \otimes bc) = (m \otimes 1)(1 \otimes \chi)\big(\chi(a \otimes b) \otimes c\big). \tag{5}$$

The corresponding equations for the Leibniz rule on the first variable are

$$[ab, c]_\chi = \big[a, \sigma^i_b(c)\big]_\chi e_i + a[b, c]_\chi \tag{6}$$

or, equivalently,

$$\chi(ab \otimes c) = (1 \otimes m)(\chi \otimes 1)\big(a \otimes \chi(b \otimes c)\big). \tag{7}$$

The equivalency between, say, (4) and (5) is straightforward keeping in mind that $\sigma^i_a(b) \otimes e = \chi(a \otimes b)$ and the definition (1). Using the explicit notation (2), the above properties translate into

$$\sigma^i_a(bc) = \sigma^{i'}_a(b)\sigma^i_{e_{i'}}(c) \quad \text{and} \quad \sigma^j_{ab}(c) \otimes e_j = \sigma^i_a\big(\sigma^{i'}_b(c)\big) \otimes e_i e_{i'},$$

respectively.

We only write down here the proof of (5), the one of (7) corresponds to a trivial alteration of the former. Expand

$$\chi(a \otimes bc) = \big(R_2 \rhd (bc)\big) \otimes (R_1 \rhd a).$$

Considering (18), this gives

$$\chi(a \otimes bc) = (m \otimes 1)(\Delta R_2 \otimes R_1) \rhd (b \otimes c \otimes a) = \tau(1 \otimes m)\big(R_{13}R_{12} \rhd (a \otimes b \otimes c)\big).$$

Rewriting the action of R_{12} in terms of χ, and using the trivial result $\tau(1 \otimes m) = (m \otimes 1)(1 \otimes \tau)(\tau \otimes 1)$, we find

$$\begin{aligned}\chi(a \otimes bc) &= (m \otimes 1)(1 \otimes \tau)(\tau \otimes 1)\big(R_{13} \rhd \big(\tau\big[\chi(a \otimes b)\big] \otimes c\big)\big)\\ &= (m \otimes 1)(1 \otimes \tau)\big(R_{23} \rhd \big(\chi(a \otimes b) \otimes c\big)\big)\\ &= (m \otimes 1)(1 \otimes \chi)\big(\chi(a \otimes b) \otimes c\big)\end{aligned}$$

which is the intended result.

2.2. Covariance

We will now prove the following lemma.

Lemma. *The commutator* $[\,,\,]_\chi$ *is quantum-group covariant, in the sense that*

$$h \rhd [a, b]_\chi = [h_1 \rhd a, h_2 \rhd b]_\chi. \tag{8}$$

Using the definition of the commutator and the quantum group action properties,

$$\begin{aligned} h \triangleright [a,b]_\chi &= h \triangleright \big(ab - (R_2 \triangleright b)(R_1 \triangleright a)\big) \\ &= (h_1 \triangleright a)(h_2 \triangleright b) - m\big[\big(\Delta h R^\tau\big) \triangleright (b \otimes a)\big]. \end{aligned}$$

But according to (17) we see that the last term can be rewritten

$$\begin{aligned} m\big[\big(\Delta h R^\tau\big) \triangleright (b \otimes a)\big] &= m\tau\big[\big(\Delta^{\mathrm{op}} h R\big) \triangleright (a \otimes b)\big] = m\tau\big[(R \Delta h) \triangleright (a \otimes b)\big] \\ &= \chi\big[\Delta h \triangleright (a \otimes b)\big]. \end{aligned}$$

Therefore

$$h \triangleright [a,b]_\chi = m(1-\chi)\big[\Delta h \triangleright (a \otimes b)\big],$$

which coincides with (8).

2.3. q-Antisymmetry

Generalizing the classical antisymmetry of a commutator, we now have the following lemma.

Lemma. *The commutator* $[\ ,\ ]_\chi$ *is* q*-antisymmetric, this meaning*

$$[a,b]_\chi = -\big[\sigma_a^i(b), e_i\big]_{\overline{\chi}} = -[\ ,\ ]_{\overline{\chi}}\big(\chi(a \otimes b)\big). \tag{9}$$

Note that in the RHS we have the deformed commutator $[\ ,\ ]_{\overline{\chi}}$ given by the opposite quasi-triangular structure $\overline{R}$. The proof is simply expressing the fact that $\overline{\chi}$ and χ are inverse maps:

$$[a,b]_\chi = m(1-\chi)(a \otimes b) = -m(1-\overline{\chi})\chi(a \otimes b).$$

If the quantum group H is triangular, $\overline{R} = R$ and a same and unique commutator appears in (9).

2.4. Conjugacy properties

Let us now analyze the conjugacy properties of the commutator with respect to a star operation on A. Assume

$$\star_H : H \longmapsto H$$

is a Hopf-star on H, and

$$\star_A : A \longmapsto A$$

is a compatible star [2] on A, in the sense that

$$h \triangleright \left(a^{\star_A}\right) = \left[(Sh)^{\star_H} \triangleright a\right]^{\star_A}. \tag{10}$$

Then we can analyze the conjugacy properties of the commutator. From now on we drop the indexes on $\star$, as there is no confusion possible.

Lemma. *If R is anti-real* [1], *meaning*

$$R^\star = R^{-1}, \tag{11}$$

then

$$[a, b]_\chi^\star = [b^\star, a^\star]_{\overline{\chi}}.$$

For a real R, i.e., such that

$$R^\star = \tau(R), \tag{12}$$

the result is

$$[a, b]_\chi^\star = [b^\star, a^\star]_\chi.$$

The quantum plane example shown in Section 3 corresponds to the first possibility. The proof goes as follows:

$$[a, b]_\chi^\star = b^\star a^\star - (R_1 \triangleright a)^\star (R_2 \triangleright b)^\star.$$

Considering first (10), and using next that $(S \otimes S)R = R$, we obtain

$$\begin{aligned} [a, b]_\chi^\star &= b^\star a^\star - \left((SR_1)^\star \triangleright a^\star\right)\left((SR_2)^\star \triangleright b^\star\right) = b^\star a^\star - m\left[R^\star \triangleright (a^\star \otimes b^\star)\right] \\ &= m\left[1 - \tau \circ \left(\tau(R^\star) \triangleright \cdot\right)\right](b^\star \otimes a^\star). \end{aligned}$$

For a real R (respectively anti-real), $\tau(R^\star) = R$ (respectively $= \overline{R}$) and the lemma follows.

2.5. *Quantum Lie algebra structure and Jacobi identities*

Having defined a generalized commutator with Leibniz and antisymmetry properties, we could now inquire about the relationship between this structure and the one provided by a quantum Lie algebra [3]. Following this reference, a quantum Lie algebra is defined by relations

$$e_i e_j - \sigma_{ij}^{mk} e_m e_k = C_{ij}^k e_k \tag{13}$$

among vector space generators $\{e_i\}$ of the space. The matrix σ_{ij}^{mk} should satisfy a Yang–Baxter equation, and the structure constants C_{ij}^k have to obey Eqs. (2)–(4) of [3], corresponding to generalized Jacobi and Leibniz properties. Comparing (13) with (1), we see that we must take

$$\sigma_{ij}^{mk} e_m \otimes e_k = \chi(e_i \otimes e_j) \quad \text{and} \quad C_{ij}^k e_k = [e_i, e_j]_\chi.$$

Remark also that the Yang–Baxter equation (19) implies for χ the following relation:

$$(1 \otimes \chi)(\chi \otimes 1)(1 \otimes \chi) = (\chi \otimes 1)(1 \otimes \chi)(\chi \otimes 1). \tag{14}$$

The proof is straightforward. Now using our (14), (5), and (7) it is straightforward algebra to see that the conditions (3) and (4) of [3] are satisfied.

Condition (2) of [3] corresponds to the Jacobi identity of the quantum Lie algebra, and we have not yet analyzed such a property for the commutators $[\,,\,]_\chi$.

The usual Jacobi identity can, a priori, be generalized in several possible ways. However, in order to maintain the parallel with the q-Lie algebras of [3], we take here the generalization

$$\big[[\cdot,\cdot]_\chi,\cdot\big]_\chi = \big[\cdot,[\cdot,\cdot]_\chi\big]_\chi + \big[[\cdot,\cdot]_\chi,\cdot\big]_\chi \circ (1 \otimes \chi), \tag{15}$$

which corresponds to their Eq. (2). After using the Leibniz properties, (15) translates into

$$\begin{aligned}\big\{1-(\chi\otimes 1)(1\otimes\chi)\big\}\big\{1-(\chi\otimes 1)\big\} &= \big\{1-(1\otimes\chi)(\chi\otimes 1)\big\}\big\{1-(1\otimes\chi)\big\}\\ &\quad + \big\{1-(\chi\otimes 1)(1\otimes\chi)\big\}\big\{1-(\chi\otimes 1)\big\}(1\otimes\chi).\end{aligned}$$

Making use of the Yang–Baxter equation for χ (14), we get:

$$0 = \big\{(\chi\otimes 1) - (1\otimes\chi)(\chi\otimes 1)\big\}\big(1 - \big(1\otimes\chi^2\big)\big).$$

Therefore, the Jacobi identity is satisfied only in the case $\chi^2 = 1$, i.e., if H is a triangular Hopf algebra.

All the above results can be collected in the following theorem.

Theorem 1. *Let A be a left-module-algebra of a quasi-triangular Hopf algebra H, and take $[\,,\,]_\chi$ the quantum commutator on A defined by* (1) *and* (3)*. Then $[\,,\,]_\chi$ is quantum-group covariant and has generalized antisymmetry and Leibniz properties. If the H is triangular, then they additionally satisfy a generalized Jacobi identity, turning the module-algebra into a quantum-Lie algebra.*

3. The quantum plane example

Take the quantum plane algebra A generated by x and y such that

$$xy = qyx, \quad q \in \mathbb{C},\ q \neq 0.$$

On A we have the action [4] of the quantum enveloping algebra $H = U_q(sl(2,\mathbb{C}))$ generated by K, K^{-1}, X_+, X_- with relations

$$KX_\pm = q^{\pm 2}X_\pm K, \qquad [X_+, X_-] = \frac{1}{(q-q^{-1})}\big(K - K^{-1}\big).$$

Additionally one can take the complex parameter q to be a root of unit, $q^N = 1$ for some (odd) integer N. In such a case one can get non-trivial finite dimensional algebras by taking the quotient of the above ones by the following ideals:

$$x^N = \mathbf{1}, \qquad y^N = \mathbf{1} \quad \text{and} \quad K^N = \mathbf{1}, \qquad X_\pm^N = 0.$$

Of course, now $K^{-1} = K^{N-1}$. To be concrete, we take the value $N = 3$, thus $q^3 = 1$. In this case the R-matrix of $U_q(sl(2,\mathbb{C}))$ is given by [4],

$$R = \frac{1}{3}R_K R_X, \tag{16}$$

where

$$\begin{aligned}
R_K &= \mathbf{1}\otimes\mathbf{1} + (\mathbf{1}\otimes K + K\otimes\mathbf{1}) + \big(\mathbf{1}\otimes K^2 + K^2\otimes\mathbf{1}\big) \\
&\quad + q^2\big(K\otimes K^2 + K^2\otimes K\big) + qK\otimes K + qK^2\otimes K^2, \\
R_X &= \mathbf{1}\otimes\mathbf{1} + \big(q - q^{-1}\big)X_-\otimes X_+ + 3qX_-^2\otimes X_+^2.
\end{aligned}$$

Applying formula (3), we can calculate the following elementary χ's:

$$\chi(x\otimes x) = q^2x\otimes x, \qquad \chi(y\otimes x) = qx\otimes y,$$
$$\chi(x\otimes y) = qy\otimes x + \big(q^2-1\big)x\otimes y, \qquad \chi(y\otimes y) = q^2y\otimes y.$$

The quantum plane algebra can be extended in a covariant way introducing derivative operators ∂_x and ∂_y [6]. We refer the reader to [4] for the complete algebraic structure. Including these derivatives, the braiding is

$$\chi(\partial_x\otimes x) = qx\otimes\partial_x, \qquad \chi(\partial_y\otimes x) = q^2x\otimes\partial_y, \qquad \chi(\partial_x\otimes y) = q^2y\otimes\partial_x,$$
$$\chi(\partial_y\otimes y) = (q-1)x\otimes\partial_x + qy\otimes\partial_y,$$
$$\chi(x\otimes\partial_x) = q\partial_x\otimes x + \big(q^2-q\big)\partial_y\otimes y, \qquad \chi(y\otimes\partial_x) = q^2\partial_x\otimes y,$$

$$\chi(\partial_x \otimes \partial_x) = q^2 \partial_x \otimes \partial_x, \qquad \chi(\partial_y \otimes \partial_x) = (q^2 - 1)\partial_y \otimes \partial_x + q\partial_x \otimes \partial_y,$$
$$\chi(x \otimes \partial_y) = q^2 \partial_y \otimes x, \qquad \chi(y \otimes \partial_y) = q\partial_y \otimes y, \qquad \chi(\partial_x \otimes \partial_y) = q\partial_y \otimes \partial_x,$$
$$\chi(\partial_y \otimes \partial_y) = q^2 \partial_y \otimes \partial_y.$$

Using the relations between derivatives and coordinates found in [4], we can now display a few non-trivial commutators. We have, for instance,

$$[x, x]_\chi = x^2 - m\big(\chi(x \otimes x)\big) = \big(1 - q^2\big)x^2,$$
$$[\partial_x, x]_\chi = \partial_x x - m\big(\chi(\partial_x \otimes x)\big) = \mathbf{1} + \big(q^2 - q\big)x\partial_x + \big(q^2 - 1\big)y\partial_y,$$
$$[x, \partial_x]_\chi = x\partial_x - m\big(\chi(x \otimes \partial_x)\big) = -q^2\mathbf{1}.$$

Remark. Note that one could think about using the matrix representation of the reduced quantum plane at $q^3 = 1$ as a way to define commutators. Taking the explicit 3×3 matrices [4,5],

$$\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

we see that our above deformed commutator has nothing to do with the commutator of these matrices. In fact $[\mathbf{x}, \mathbf{x}] = 0$ (as matrices), whereas $[x, x]_\chi = (1 - q^2)x^2$, as we saw above. Of course, the point is that the commutator defined using these matrices does not have the covariance property of our deformed commutator.

4. Concluding remarks

The main results of this communication are collected in Theorem 1, involving the existence of a covariant commutator structure on any module-algebra of a quasi-triangular Hopf algebra. This commutator turns the module-algebra into a quantum Lie algebra in the case that the quantum group acting on it is triangular. The fact that the deformed Jacobi identity (15) is obeyed only for a *triangular* Hopf algebra seems to be independent of the way we choose to generalize the Jacobi identity.

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Appendix A. Quasi-triangular Hopf algebras

We remember here that a *quasi-triangular* Hopf algebra H [1] has, by definition, an element $R \in H \otimes H$ with the following properties:

$$\Delta^{\mathrm{op}} h = R \Delta h R^{-1}, \tag{17}$$

$$(\Delta \otimes 1)R = R_{13}R_{23}, \qquad (1 \otimes \Delta)R = R_{13}R_{12}. \tag{18}$$

It follows that R satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{19}$$

The algebra H automatically has a second quasi-triangular structure given by the related element

$$\overline{R} = \tau\left(R^{-1}\right), \tag{20}$$

where τ is the permutation of tensor product factors. If both R and $\overline{R}$ coincide one says that the Hopf algebra H is in fact *triangular*. Two additional basic properties of the R-matrix which we need in our proofs are

$$(\varepsilon \otimes 1)R = (1 \otimes \varepsilon)R = \mathbf{1}, \qquad (S \otimes S)R = R.$$

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