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FUNCTORIAL UNIFORMIZATION OF TOPOLOGICAL SPACES

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Let T be the forgetful functor from uniform spaces to completely regular topological spaces. We study T-sections, i.e. functors right inverse to T. We develop as tool the notion of spanning a T-section by a class of uniform spaces, and the order-dual notion of cospanning. Coarsest and finest uniform bireflectors and coreflectors associated with a T-section are characterized. Certain effects of the uniform completion reflector on a T-section are expressed in terms of the associated bireflectors.

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uniform space, bireflector, coreflector, completion-stable, spanning, cospanning, completion, completion-true

1. Introduction

Let **Unif** denote the category of uniform spaces and *uniform* (i.e. uniformly continuous) maps, and **Creg** the category of completely regular (thus, uniformizable) topological spaces and continuous maps. Hausdorff separation is not assumed. There is the forgetful functor T: **Unif** → **Creg**. We study functors which equip spaces in **Creg** with compatible uniformities, i.e. functors F: **Creg** → **Unif** with $TF = \mathbf{1}$. Such F is called a *T-section.*

The spanning and cospanning constructions (Section 3) factorize a T-section F as $F = a\varphi = b\mathscr{C}^*$ where a is a bireflector, b is a coreflector, φ is the finest and $\mathscr{C}^*$ the coarsest T-section, and both a and b preserve topology. One main result (5.5) is that the bireflectors thus associated with F occur as a closed interval $[\rho_F, \alpha_F]$ in the partial order 'coarser than' for bireflectors. The dual result (6.3) is restricted to coreflectors that stay above the level of the Čech uniformity. In Section 7 we let the completion reflector γ act on the functors and show, e.g., that F is γ-true (resp., strongly γ-true) iff ρ_F is γ-stable (resp., α_F is γ-stable).

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Our general reference for uniform spaces is [23], for categorical notions [18]. Special terms are defined below.

For $X, Y \in$ **Unif**, $X \leqslant Y$ means that $TX = TY$ and X has coarser uniformity than Y. For functors G, H on the same domain and ranging in **Unif**, $G \leqslant H$ means $GX \leqslant HX$ for each X in the domain. Then $G < H$ means $G \leqslant H$ but $GX \neq HX$ for some X. Join ($\vee$) and meet ($\wedge$), where they exist, refer to $\leqslant$.

$U(X, Y)$ denotes the set of all uniform maps from X to Y. Obvious extensions of this notation are $U(X, \mathscr{A})$ and $U(\mathscr{B}, Y)$ where $\mathscr{A}, \mathscr{B} \subset$ **Unif**.

$\mathbb{R}_u$ shall stand for the real line with its usual uniformity, and $\mathbb{R} = T\mathbb{R}_u$ is the associated topological space. Likewise, $\mathbb{I}_u$ is $[0, 1]$ with its unique uniformity, and $\mathbb{I} = T\mathbb{I}_u$.

All our subcategories will be full and isomorphism-closed, so we do not distinguish between a full subcategory and its class of objects. When **S** is a (co)reflective subcategory of **C**, the (co)reflector $R: \mathbf{C} \to \mathbf{S}$ is sometimes regarded as endofunctor $R: \mathbf{C} \to \mathbf{C}$. A *bireflector* is a reflector R whose reflection maps $i_X: X \to RX$ are bimorphisms; in our setting i_X will be the identity function on the underlying set of X.

2. T-reflectors and T-coreflectors

For $\mathscr{A} \subset$ **Unif**, **init** $\mathscr{A}$ stands for the initial hull (= bireflective hull) of $\mathscr{A}$, i.e. the class of all $X \in$ **Unif** whose uniformity is initial, i.e. weak, for $U(X, \mathscr{A})$. The bireflector $a:$ **Unif** $\to$ **init** $\mathscr{A}$ is given by: aX is initial for $U(X, \mathscr{A})$.

Dually, for $\mathscr{B} \subset$ **Unif**, **fin** $\mathscr{B}$ denotes the final hull (= correflective hull) of $\mathscr{B}$, i.e. the class of objects X final, i.e. strong, for $U(\mathscr{B}, X)$. The coreflector $b:$ **Unif** $\to$ **fin** $\mathscr{B}$ is defined by: bX is final for $U(\mathscr{B}, X)$.

Any bireflector (coreflector) r which preserves topology, i.e. satisfies $Tr = T$, will be called a *T-reflector* (*T-coreflector*).

2.1. Examples of T-reflector. (1) The precompact reflector $p:$ **Unif** $\to$ **Precpt** $=$ **init** $\{\mathbb{I}_u\}$.

(2) Let m be an infinite cardinal. A uniform space is *m-precompact* if it has no uniformly discrete subspace of cardinality m. For fixed m these spaces form a T-reflective subcategory. In case $m = \aleph_0$ we have just **Precpt.** The $\aleph_1$-precompact spaces are also called *separable*, and the corresponding T-reflector is denoted e [23, p. 129; 25].

(3) Another T-reflector with a favored symbol is $c:$ **Unif** $\to$ **init** $\{\mathbb{R}_u\}$ [23, p. 129; 25]. It is clear that $p < c < e$.

(4) Some general ways of creating or changing bireflectors in **Unif** may be found in [10, 21, 22, 24].

2.2. Examples of *T*-coreflectors. (1) Recall that φ denotes the finest section of T. The T-coreflector $\varphi T:$ **Unif** $\to$ **Fine** defines the *fine* uniform spaces.

(2) A uniform space is *subfine* if it admits a uniform embedding into some fine space; equivalently, if it admits an initial map into a (separated) fine space. The subcategory **Subfine** of these spaces is T-coreflective. (The proof in [23, p. 123] is restricted to separated spaces, but can readily be adapted by using the separated reflection.) For a generalization, see [11, p. 100].

(3) The locally fine uniform spaces [23, p. 127].

(4) General methods of constructing T-coreflectors are described in [15, 1.1] and [16, 1.1]; and [17, §5] gives a technique of modifying one to get another (see especially [17, 5.4]). See also [10, 11, 13, 14, 21, 22, 25, 27, 28].

2.3. Proposition. (a) *For a bireflector* a: **Unif** $\to$ **init** $\mathscr{A}$ *these are equivalent*:

(1) $Ta = T$;

(2) $\mathbb{I}_u \in$ **init** $\mathscr{A}$;

(3) $\mathbb{I}_u$ *is uniformly embedded in some* $A \in \mathscr{A}$;

(4) **Precpt** $\subset$ **init** $\mathscr{A}$;

(5) $p \leqslant a$.

(b) *For the coreflector* b : **Unif** $\to$ **fin** $\mathscr{B}$ these are equivalent:

(1) $Tb = b$;

(2) **Fine** $\subset$ **fin** $\mathscr{B}$;

(3) $b \leqslant \varphi T$.

Proof. Standard; in (a), (2)$\Rightarrow$(3) by the Hahn–Mazurkiewicz theorem [19, p. 129]. □

3. *T*-sections, span and cospan

Trivially, if the functor F: **Creg** $\to$ **Unif** is defined by $F = a\varphi$ (or $F = b\mathscr{C}^*$) for some T-reflector a (T-coreflector b), then F is a T-section. We show in Proposition 3.2 that every T-section has both these representations.

Let $X \in$ **Creg**. The uniform space $\mathscr{C}^*X$ is defined to have the uniformity initial for the bounded continuous maps from X to $\mathbb{R}_u$. Equivalently, $\mathscr{C}^*X$ is initial for $C(X, \mathbb{I})$ to $\mathbb{I}_u$. However, as set of functions $C(X, \mathbb{I})$ coincides with $U(\varphi X, \mathbb{I}_u)$. Thus $\mathscr{C}^*X$ is initial for $U(\varphi X, \mathbb{I}_u)$. This idea is extended and dualized in Proposition 3.2.

3.1 [3, 4]. *The functors* $\mathscr{C}^*$ *and* φ *are T-sections, and if F is any T-section, then* $\mathscr{C}^* \leqslant F \leqslant \varphi$.

Proof. The claims for φ are clear. Since T preserves initiality, $T\mathscr{C}^*X$ is initial for $C(X, \mathbb{I})$. But the completely regular X is also initial for $C(X, \mathbb{I})$. Thus $T\mathscr{C}^*X = X$. To see that $\mathscr{C}^* \leqslant F$, consider a map g in the initial source $U(\mathscr{C}^*X, \mathbb{I}_u)$. Then $FTg \in U(FX, FT\mathbb{I}_u) = U(FX, \mathbb{I}_u)$, and so $\mathscr{C}^*X \leqslant FX$.

3.2. Definition and Proposition. (a) *Let* $\mathscr{A} \subset \mathbf{Unif}$ *have* $\mathbb{I}_u \in \mathbf{init}\ \mathscr{A}$ *and let* $a : \mathbf{Unif} \to \mathbf{init}\ \mathscr{A}$ *be the associated T-reflector. Define* $\langle \mathscr{A} \rangle : \mathbf{Creg} \to \mathbf{Unif}$ *by*:

$$\langle \mathscr{A} \rangle X \text{ is initial for } U(\varphi X, \mathscr{A}).$$

Then, $\langle \mathscr{A} \rangle = \langle \mathbf{init}\ \mathscr{A} \rangle = a\varphi$, *and this is a T-section, called the T-section spanned by* $\mathscr{A}$.

(b) *Let* $\mathscr{B} \subset \mathbf{Unif}$ *have* $\mathbf{Fine} \subset \mathbf{fin}\ \mathscr{B}$ *and let* $b : \mathbf{Unif} \to \mathbf{fin}\ \mathscr{B}$ *be the associated T-coreflector. Define* $[\mathscr{B}] : \mathbf{Creg} \to \mathbf{Unif}$ *by*:

$$[\mathscr{B}]X \text{ is final for } U(\mathscr{B}, \mathscr{C}^* X).$$

Then, $[\mathscr{B}] = [\mathbf{fin}\ \mathscr{B}] = b\mathscr{C}^*$, *and this is a T-section, called the T-section cospanned by* $\mathscr{B}$.

Proof. Immediate, in light of Proposition 2.3. □

Thus, to say that the T-section F is spanned by $\mathscr{A}$ (resp., cospanned by $\mathscr{B}$) is to say that $F = a\varphi$ (resp., $F = b\mathscr{C}^*$).

3.3. Remarks. The spanning construction was used by Hušek in [20], by the first of the present authors in [2–7], and by us in [8]. (The term 'spanning' comes from [6].)

That the spanning construction yields a T-section in Proposition 3.2(a) follows from the fact that T preserves initiality; this was analyzed in [3, 4]. But T does not preserve finality, and consequently a strict categorical dual of spanning fails to give correspondingly dual results. The notion of cospanning introduced now in Definition 3.2(b) is an order-theoretic dual; it is contrasted with the categorical dual in the following proposition.

3.4. Proposition. *Let* $\mathscr{B} \subset \mathbf{Unif}$ *with* $\mathbf{Fine} \subset \mathbf{fin}\ \mathscr{B}$. *Let* b *be the T-coreflector onto* $\mathbf{fin}\ \mathscr{B}$. *Let the functor* G: $\mathbf{Creg} \to \mathbf{Unif}$ *be defined by*:

$$GX \text{ is final for } C(T\mathscr{B}, X) \text{ from } \mathscr{B}.$$

Then the following are equivalent:

(1) $G = [\mathscr{B}]$;

(2) G *is a T-section*;

(3) $b \geq \mathscr{C}^* T$.

Proof. (1)⇒(2) by Proposition 3.2(b).

(2)⇒(3): Let G be a T-section. For any $Y \in \mathbf{Unif}$ we have:

$$bY \text{ is final for } U(\mathscr{B}, Y);$$

$$GTY \text{ is final for } C(T\mathscr{B}, TY) \text{ from } \mathscr{B}.$$

Since the functions in $U(\mathscr{B}, Y)$ all occur in $C(T\mathscr{B}, TY)$, it follows that bY has finer uniformity than GTY. Moreover $T(bY) = TY = T(GTY)$. Hence $bY \geq GTY$. By 3.1, $GTY \geq \mathscr{C}^* TY$. Thus $b \geq \mathscr{C}^* T$.

(3)⇒(1); Let $b \geqslant \mathscr{C}^*T$. Compare:

GX is final for $C(T\mathscr{B}, X)$ from $\mathscr{B}$;

$[\mathscr{B}]X$ is final for $U(\mathscr{B}, \mathscr{C}^*X)$.

The functor T induces a bijection from $U(B, \mathscr{C}^*X)$ to $C(TB, X)$ for each $B \in \mathscr{B}$. To see surjectivity, consider $g \in C(TB, X)$. Let $f = (\mathscr{C}^*g) \circ i_B$ where $i_B : B \to \mathscr{C}^*TB$ is given by $B = bB \geqslant \mathscr{C}^*TB$, with $Ti_B = 1_{TB}$. Then $f \in U(B, \mathscr{C}^*X)$ and $Tf = g$. It follows that $GX = [\mathscr{B}]X$. □

There are important examples (see Examples 6.1) of T-coreflectors which disobey the condition $b \geqslant \mathscr{C}^*T$ of 3.4. Therefore we have to adhere to the notion of cospanning as defined in Proposition 3.2(b). (The condition may as well be called $b > \mathscr{C}^*T$ because $\mathscr{C}^*T$ is not a coreflection.)

3.5. Proposition. *Let F be a T-section. Then, F is both spanned and cospanned by* **range** *F.*

(The proof is now routine.)

We do not know whether $F = \langle \mathscr{A} \rangle = [\mathscr{A}]$ implies $\mathscr{A} =$ **range** F. We doubt this. However, **range** F is 'calculated' from $\mathscr{A}$ in Corollary 5.6 below.

3.6. Definition. Let F be a T-section. We denote by ρ_F the T-reflector onto **init range** F, and by κ_F the T-coreflector onto **fin range** F.

That ρ_F and κ_F indeed preserve topology follows from Propositions 3.5 and 3.2, or directly.

3.7. Proposition. *Let F be a T-section. Then,*

(1) $F = \rho_F \varphi = \kappa_F \mathscr{C}^*$, *and*

(2a) *If* $F = a\varphi$ *for a T-reflector a, then* $\rho_F \leqslant a$;

(2b) *If* $F = b\mathscr{C}^*$ *for a T-coreflector b, then* $b \leqslant \kappa_F$.

Proof. (1) follows from 3.5 and 3.2.

(2a): $F = a\varphi \Rightarrow$ **range** $F \subset$ **range** $a \Rightarrow$ **init range** $F \subset$ **range** a (since **init range** $a =$ **range** a), i.e., **range** $\rho_F \subset$ **range** a, i.e., $\rho_F \leqslant a$. □

3.8. Remarks and Examples. (1) Even for quite 'simple' F, the ρ_F and κ_F can be unfamiliar and difficult to 'compute' (whatever that means): While $\rho_{\mathscr{C}^*} = p$ (clearly), $\kappa_{\mathscr{C}^*}$ is not familiar; and $\kappa_\varphi = \varphi T$ (clearly), while ρ_φ is not familiar.

(2) Not every T-reflector occurs in the form ρ_F. An example is the T-reflector c onto **init**$\{\mathbb{R}_u\}$ (cf. 2.1 above). To see this, consider the T-section $\mathscr{C} = \langle\{\mathbb{R}_u\}\rangle = c\varphi$. (The notation $\mathscr{C}$ is from [12, p. 219].) If there were F with $c = \rho_F$, then $F = \rho_F\varphi = c\varphi = \mathscr{C}$. So it suffices to show that $c \neq \rho_{\mathscr{C}}$. Indeed, $\mathbb{R}_u = c\mathbb{R}_u \neq \rho_{\mathscr{C}}\mathbb{R}_u$, because $\rho_{\mathscr{C}}\mathbb{R}_u$ is precompact, i.e., each uniform $g : \mathbb{R}_u \to c\varphi X$ has precompact range. (If not, there is countably infinite uniformly discrete $\{x_n \mid n \in \mathbb{N}\}$ in $g(\mathbb{R}_u)$, then uniform $h : c\varphi X \to \mathbb{R}_u$ with $h(x_n) = n^2$—use [12, 15.15(b)] to extend $x_n \mapsto n^2$—so that $h \circ g \in U(\mathbb{R}_u, \mathbb{R}_u)$; but $h \circ g$ grows too fast for that.)

(3) So, one wants a characterization of the ρ_F's, or, what is the same thing, a characterization of the T-reflective subcategories of the form **range** $\rho_F =$ **init range** F. We don't think there is a characterization in particularly familiar terms, but Section 5 below is a partial response to the issue. Observe that the conglomerate of all T-reflectors falls into equivalence classes under the relation $a_1 \equiv a_2 \Leftrightarrow a_1\varphi = a_2\varphi$. The equivalence class of any a has a bottom, ρ_F, where $F = a\varphi$, and we shall see that it also has a top, called α_F in Definition 5.2 below. It would seem rather exceptional for such an equivalence class to collapse to a single member, as it does in the case of the precompact reflector p.

(4) Similarly, not every T-coreflector occurs in the form κ_F, but this has two aspects.

First, $\kappa_F \geqslant \mathscr{C}^*T$ always holds (this is easy to see, and also follows immediately from Propositions 3.7 and 5.4), while not every T-coreflector b has $b \geqslant \mathscr{C}^*T$. In fact, both the subfine and the metric-fine coreflectors fail this condition; see Examples 6.1 below.

Second, there are T-coreflectors $b \geqslant \mathscr{C}^*T$ with b not of the form κ_F. Note that κ_F is the top (i.e. finest member) of an equivalence class of T-coreflections under the relation $b_1\mathscr{C}^* = b_2\mathscr{C}^*$. Again, it seems unusual for such an equivalence class to collapse to a single member, as it does for the fine coreflector φT. See Section 6, Remark 7.14(4) and Proposition 7.15.

4. Down-closure and up-closure

We examine a construction, $\downarrow$ (used in this context in [3, 4]), which is much in the nature of things for further analysis of factorizations $F = a\varphi$ (and, roughly dually, $\uparrow$, for $F = b\mathscr{C}^*$).

4.1. Definition. For $\mathscr{A} \subset$ **Unif**, the *down-closure* of $\mathscr{A}$, denoted $\downarrow\mathscr{A}$ is defined by: $X \in \downarrow\mathscr{A}$ means $X \leqslant A$ for some $A \in \mathscr{A}$. If $\downarrow\mathscr{A} = \mathscr{A}$, we say that $\mathscr{A}$ is a $\downarrow$*-class.*

Dually, we define the *up-closure* $\uparrow\mathscr{B}$, and $\uparrow$*-class.*

Clearly $\downarrow(\downarrow) = \downarrow$, and $\uparrow(\uparrow) = \uparrow$ always.

4.2. Proposition. *Let* $\mathscr{A}$, $\mathscr{B} \subset$ **Unif.**

(a) *If* $\mathscr{A}$ *is bireflective, so is* $\downarrow\mathscr{A}$.
If $\mathscr{A}$ *is T-reflective, so is* $\downarrow\mathscr{A}$.

(b) *If $\mathscr{B}$ is coreflective, so is $\uparrow\mathscr{B}$.*
If $\mathscr{B}$ is T-coreflective, so is $\uparrow\mathscr{B}$.

Proof. (a) We can suppose $\mathscr{A} = \mathbf{init}\ \mathscr{A}$, and show $\mathbf{init}\downarrow\mathscr{A} \subset \downarrow\mathscr{A}$. So let X be initial for $U(X, \downarrow\mathscr{A})$. Write $U(X, \downarrow\mathscr{A}) = \{f_j : X \to A'_j\}_j$ with $A'_j \leq A_j \in \mathscr{A}$, and let Y be initial for $\{f_j : X \to A_j\}_j$ (with an abuse of language). Since $TA'_j = TA_j$, we have $TX = TY$. Also X is coarser than Y, so that $X \leq Y$. With $Y \in \mathbf{init}\ \mathscr{A} = \mathscr{A}$ we have $X \in \downarrow\mathscr{A}$.

The rest of (a) follows from Proposition 2.3, and (b) is roughly dual. □

4.3. Examples. (1) Let m be an infinite cardinal and $\mathscr{D}(m)$ the T-reflective subcategory of m-precompact spaces from 2.1(2). Clearly $\downarrow\mathscr{D}(m) = \mathscr{D}(m) = \mathbf{init}\ \mathscr{D}(m)$.

(2) Clearly $\downarrow\mathbf{range}\ \mathscr{C}^* = \mathbf{Precpt.}$

(3) The class $\mathbf{init}\{\varphi T\mathbb{R}_u\}$ is not a $\downarrow$-class: $\mathbb{R}_u \in \downarrow\mathbf{init}\{\varphi T\mathbb{R}_u\}$, since $\mathbb{R}_u \leq \varphi T\mathbb{R}_u$, while $\mathbb{R}_u \notin \mathbf{init}\{\varphi T\mathbb{R}_u\}$, since $U(\mathbb{R}_u, \varphi T\mathbb{R}_u)$ consists of bounded functions, as discussed in Remark 3.8(2).

We do not know whether $\mathbf{init}\{\mathbb{R}_u\}$ is down-closed, but doubt it.

While, $\downarrow\mathbf{init}\{\varphi T\mathbb{R}_u\} = \downarrow\mathbf{init}\{\mathbb{R}_u\}$: As noted above, $\mathbb{R}_u \leq \varphi T\mathbb{R}_u$, whence by Proposition 4.2(a) $\downarrow\mathbf{init}\{\mathbb{R}_u\} \subset \downarrow\mathbf{init}\{\varphi T\mathbb{R}_u\}$. Conversely, by [9, Theorem 1] (or in other ways) $\varphi T\mathbb{R}_u \in \mathbf{init}\{\mathbb{R}_u\}$, whence $\downarrow\mathbf{init}\{\varphi T\mathbb{R}_u\} \subset \downarrow\mathbf{init}\{\mathbb{R}_u\}$.

We have, so far, been doing our best to pretend that $\langle\ \rangle$ and $[\ ]$ are dual, and that $\downarrow$ and $\uparrow$ are dual. This is, of course, a fiction, the basic fact being that φ is left-adjoint to T while $\mathscr{C}^*$ is not right-adjoint to T (and T has no right-adjoint). In our considerations, this manifests itself in the facts that T-reflectors a always satisfy $a \leq \varphi T$, while for some T-coreflectors b we have $b \not\geq \mathscr{C}^*T$; the latter confuses the interplay between $[\ ]$ and $\uparrow$. So, for Section 5 below, we shall simply assume $b \geq \mathscr{C}^*T$ when we need to. The issue of what happens without that assumption is discussed in the brief Section 6.

4.4. Proposition. (a) *For any $\mathscr{A} \subset$ **Unif**, we have $\langle\mathscr{A}\rangle = \langle\downarrow\mathscr{A}\rangle$.*

(b) *If $\mathscr{B} \subset$ **Unif** satisfies $B \geq \mathscr{C}^*TB$ for all $B \in \mathscr{B}$, then $[\mathscr{B}] = [\uparrow\mathscr{B}]$.*

The proof is routine.

If $\mathbf{Fine} \subset \mathbf{fin}\ \mathscr{B}$, the condition in Proposition 4.4(b) again amounts to $b \geq \mathscr{C}^*T$, where b is the T-coreflector to $\mathbf{fin}\ \mathscr{B}$.

5. Largest spanning and cospanning classes

Given $\mathscr{A}, \mathscr{B} \subset$ **Unif**, let $\mathbf{max}\ \mathscr{A} = \{A \in \mathscr{A} \mid (\forall A' \in \mathscr{A})\ (A \leq A' \Rightarrow A = A')\}$, and dually, $\mathbf{min}\ \mathscr{B} = \{B \in \mathscr{B} \mid (\forall B' \in \mathscr{B})\ (B \geq B' \Rightarrow B = B')\}$. For general $\mathscr{A}, \mathscr{B}$, these would appear to be uninteresting. Also, for general $\mathscr{A}$, $\mathscr{B}$, it is exceptional when $\downarrow\mathscr{A}$ is bireflective, or when $\uparrow\mathscr{B}$ is coreflective. So the following indicates how special are the classes **range** F, for T-sections F.

5.1. Proposition. *Let F be a T-section. Then,*

(1) $\mathbf{range}\ F = \{X \in \mathbf{Unif} \mid X = FTX\}$.

(2) $\downarrow\mathbf{range}\ F = \{X \in \mathbf{Unif} \mid X \leqslant FTX\}$, *and*
$\uparrow\mathbf{range}\ F = \{X \in \mathbf{Unif} \mid X \geqslant FTX\}$.

(3) $\mathbf{range}\ F = \mathbf{max}\downarrow\ \mathbf{range}\ F = \mathbf{min}\uparrow\ \mathbf{range}\ F$.

(4) $\downarrow$**range** *F is T-reflective, and* $\uparrow$**range** *F is T-coreflective.*

Proof. (1) is clear.

(2) $Y \leqslant X = FTX \Rightarrow FTY = FTX = X \geqslant Y$; the reverse is clear. The statement about $\uparrow$ is dual.

(3) If $FTX = X \leqslant X' \leqslant Y = FTY$, then $FTX = FTY$, whence $X = X'$; the reverse holds by definition. The statement about $\uparrow$ is dual.

(4) Let $X \in \mathbf{init}\downarrow\mathbf{range}\ F$. Then there is an initial source $(f_j : X \to Y_j)_{j\in J}$ with $Y_j \in \downarrow\mathbf{range}\ F$, i.e. $Y_j \leqslant FTY_j$. This gives $h_j : FTY_j \to Y_j$ with $Th_j = 1$. Then $(h_j \circ FTf_j : FTX \to Y_j)_{j\in J}$ is a source whose T-image coincides with that of the given initial source. It follows that $X \leqslant FTX$, i.e. $X \in \downarrow\mathbf{range}\ F$. Thus $\downarrow\mathbf{range}\ F$ is initially closed, hence bireflective. It is T-reflective because $\mathbb{I}_u = F\mathbb{I} \in \downarrow\mathbf{range}\ F$. Dually, $\uparrow$ **range** F is finally closed, hence coreflective; it is T-coreflective because it clearly contains **Fine**. □

5.2. Definition. *Let F be a T-section. We shall denote by* α_F *the T-reflector onto* $\downarrow$ **range** *F, and by* ω_F *the T-coreflector onto* $\uparrow$ **range** *F.*

5.3. Proposition. *Let F be a T-section. Then, for each* $X \in \mathbf{Unif}$, *we have*

$$\alpha_F X = X \wedge FTX \quad \text{and} \quad \omega_F X = X \vee FTX.$$

Proof. We show that $X \mapsto X \wedge FTX$ is the reflector for $\downarrow$ **range** F. Clearly, $X \wedge FTX \in \downarrow\mathbf{range}\ F$, and clearly $X = X \wedge FTX \Leftrightarrow X \in \downarrow\mathbf{range}\ F$. Now consider the uniform map $i_x : X \to X \wedge FTX$ with $Ti_X = 1_{TX}$, and any uniform $g : X \to Y$ with $Y = Y \wedge FTY$. Then we have the map $\bar{g} : X \wedge FTX \to Y \wedge FTY = Y$ with $\bar{g} \circ i_X = g$. Uniform continuity of $\bar{g}$ follows from that of g and of FTg.

Dually, $\omega_F X = X \vee FTX$. Note $\omega_F X \geqslant FTX \geqslant \mathscr{C}^* TX$. □

5.4. Proposition. *Let F be a T-section. Then,*

(a) $F = \alpha_F \varphi$; *whenever a is a T-reflector with* $F = a\varphi$, *then* $a \leqslant \alpha_F$.

(b) $F = \omega_F \mathscr{C}$; *whenever b is a T-coreflector with* $F = b\mathscr{C}^*$ *and* $b \geqslant \mathscr{C}^* T$, *then* $\omega_F \leqslant b$. *We have* $\omega_F \geqslant \mathscr{C}^* T$.

Proof. (a) By Proposition 5.3, $\alpha_F\varphi X = \varphi X \wedge FT\varphi X = \varphi X \wedge FX = FX$. We have $\alpha_F Y = Y \wedge FTY = Y \wedge a\varphi TY \geqslant aY \wedge a\varphi TY = aY$, since $\mathbf{1} \geqslant a$ and $\varphi T \geqslant a$.

(b) is dual, with $\mathscr{C}^*$ in place of φ. □

5.5. Corollary. *Let F be a T-section. Then,*

(1) *For any T-reflector a, we have*

$$F = a\varphi \Leftrightarrow \rho_F \leqslant a \leqslant \alpha_F.$$

(2) [3, 4] $\downarrow$ **range** *F is the largest class that spans F.*

In contrast, we can only say that **init range** F is the smallest *bireflective* class that spans F, and that **fin range** F is the smallest *coreflective* class that spans F.

The dual of Corollary 5.5(1) follows in Corollary 6.3 below.

5.6. Corollary. (a) *Let* $\mathscr{A} \subset$ **Unif** *with* $\mathbb{I}_u \in$ **init** $\mathscr{A}$, *and let* $F = \langle \mathscr{A} \rangle$. *Then,* $\downarrow$ **range** $F = \downarrow$ **init** $\mathscr{A}$, *and* **range** $F =$ **max** $\downarrow$ **init** $\mathscr{A}$.

(b) *Let* $\mathscr{B} \subset$ **Unif** *with* **Fine** $\subset \mathscr{B}$, *and let also* $B \geqslant \mathscr{C}^*TB$ *for every* $B \in \mathscr{B}$. *Let* $F = [\mathscr{B}]$. *Then,* $\uparrow$ **range** $F = \uparrow$ **fin** $\mathscr{B}$, *and* **range** $F =$ **min** $\uparrow$ **fin** $\mathscr{B}$.

Proof. (a) Observe **range** $F \subset$ **init** $\mathscr{A}$ by Proposition 3.2. Then use Proposition 5.4 (or Corollary 5.5) and Proposition 5.1(3).

(b) Almost dual. □

5.7. Corollary. *Let F be a T-section, let a be the T-reflector onto* **init** $\mathscr{A}$, *and let b be the T-coreflector onto* **fin** $\mathscr{B}$, *and assume* $b \geqslant \mathscr{C}^*T$. *Then,*

(a) *It is equivalent to say*: $F = \langle \mathscr{A} \rangle$; $F = a\varphi$; $\downarrow$ **range** $F = \downarrow$ **init** $\mathscr{A}$; **range** $F =$ **max** $\downarrow$ **init** $\mathscr{A}$.

(b) *It is equivalent to say*: $F = [\mathscr{B}]$; $F = b\mathscr{C}^*$; $\uparrow$ **range** $F = \uparrow$ **fin** $\mathscr{B}$; **range** $F =$ **min** $\uparrow$ **fin** $\mathscr{A}$.

5.8. Proposition. *Let a be any T-reflector, let b be a T-coreflector with* $b \geqslant \mathscr{C}^*T$, *and assume that*

$$F = a\varphi = b\mathscr{C}^*.$$

Then,

$$ab = ba = FT.$$

Proof. From Propositions 3.7, 5.3, 5.4 we observe:

$$\rho_F \leqslant a \leqslant \alpha_F \leqslant FT \leqslant \omega_F \leqslant b \leqslant \kappa_F.$$

Clearly, the relation $\leqslant$ is preserved under composition with a or b, both on the left and on the right. Thus, from $a \leqslant FT$ one has $ab \leqslant FTb = FT$, and from $b \geqslant FT$ one has $ab \geqslant aFT = a(a\varphi)T = a^2\varphi T = a\varphi T = FT$. The proof of $ba = FT$ is dual, with $\mathscr{C}^*$ in room of φ. □

5.9. Remark. Familiar occurrences of our special bireflectors and coreflectors are: $\rho_{\mathscr{C}^*} = \alpha_{\mathscr{C}^*} = p$, $\kappa_\varphi = \omega_\varphi = \varphi T$, and $\alpha_\varphi = \mathbf{1}_{\mathbf{Unif}}$. We shall see in 7.13 below that $\rho_{\mathscr{C}} < c < \alpha_{\mathscr{C}}$. See also Remark 7.14(2).

6. Coreflectors that crash through the Čech layer

The category **Unif** is rich in T-coreflectors (see Example 2.2 above). However, most of the examples which have been studied fail the condition $b \geqslant \mathscr{C}^*T$, which fact undeniably complicates the analysis of the associated T-section $b\mathscr{C}^*$.

6.1. Examples. (1) The T-coreflector s to the subfine spaces (Example 2.2 above) has [**Subfine**] = [↑**Subfine**] but $s \ngeqslant \mathscr{C}^*T$. To see this, observe that each precompact space is subfine (its completion is compact). Hence $s\mathscr{C}^* = \mathscr{C}^*$, whence by Proposition 3.2(b) [**Subfine**] $= \mathscr{C}^*$. Further, ↑ **Subfine** ⊃ ↑ **Precpt** = **Unif,** so that [↑ **Subfine**] $= \mathscr{C}^*$. However, there exists a precompact X with $sX = X < \mathscr{C}^*TX$.

(2) Let $\mathscr{B}$ be the class of metric-fine spaces and let b be the T-coreflector to $\mathscr{B}$ (see [15]). Then, $[\mathscr{B}] \neq [\uparrow\mathscr{B}]$ and $b \ngeqslant \mathscr{C}^*T$.

Proof: Let D be an uncountable set with the coarsest uniformity admitted by the discrete topology (i.e. the uniformity induced by the one-point compactification). By [15, 3.1], $D \in \mathscr{B}$. But $\mathscr{C}^*TD$ has the uniformity whose base consists of all finite covers. So $\mathscr{C}^*TD > D = bD$, and $b \ngeqslant \mathscr{C}^*T$. Further, by [15, 2.4], $b\mathscr{C}^*TD$ has base of all countable covers. Also, $TX = TD \Rightarrow X \geqslant D \Rightarrow X \in \uparrow\mathscr{B}$. Now, by Proposition 3.2, $[\mathscr{B}] = b\mathscr{C}^*$, and also $[\uparrow\mathscr{B}] = b_1\mathscr{C}^*$, where b_1 is the T-coreflector onto $\uparrow\mathscr{B}$; it exists by Proposition 4.2. Since $\mathscr{C}^*TD \in \uparrow\mathscr{B}$, we have $[\uparrow\mathscr{B}]TD = b_1\mathscr{C}^*TD = \mathscr{C}^*TD$. But $[\mathscr{B}]TD = b\mathscr{C}^*TD$ is not precompact, from above. So $[\mathscr{B}] \neq [\uparrow\mathscr{B}]$.

6.2. Proposition. *Let b be any T-coreflector. Then, $b'X = bX \vee \mathscr{C}^*TX$ defines a T-coreflector b' with $b'\mathscr{C}^* = b\mathscr{C}^*$ and with* **range** b' = (**range** b) ∩ (↑ **range** $\mathscr{C}^*$).

(We omit the easy proof.)

This provides a dual to Corollary 5.5(1) above.

6.3. Corollary. *Let F be a T-section, b any T-coreflector, and $b' = b \vee \mathscr{C}^*T$. Then,*

$$F = b\mathscr{C}^* \Leftrightarrow \omega_F \leqslant b' \leqslant \kappa_F.$$

7. T-reflectors and T-sections versus completion

Many results are known about the interaction of bireflectors in **Unif** with the completion reflector γ; see e.g. [26–29, 24, 25]. The interaction of T-sections with γ has been studied in [6, 7, 8]. Our present purpose is to indicate a connection between the two kinds of interaction. The key to the connection is given by the T-reflectors ρ_F and α_F.

7.1. We take Hausdorff separation as part of the definition of completeness for uniform spaces. We regard the completion reflector as an endofunctor $\gamma : \mathbf{Unif} \to \mathbf{Unif}$ and denote the reflection maps by $\eta_X : X \to \gamma X$. Every η_X is a dense initial map; it is an embedding if and only if TX is Hausdorff. Whenever $f: X \to Y$ is a dense initial map with Y complete, there is a unique uniform isomorphism $h : \gamma X \to Y$ with $h\eta_X = f$. (The reader who trusts this only when TX is Hausdorff, may verify our statement by composing with the uniform separated reflection.) The following lemma is well known for the special cases of products and embeddings.

7.2. Lemma. *If* $(f_j : X \to Y_j)_{j\in J}$ *is an initial source in* **Unif**, *so is* $(\gamma f_j : \gamma X \to \gamma Y_j)_{j\in J}$.

Proof. There is no loss of generality in assuming that the class J is a set. Let (σ_X, SX) be the uniform T_0-reflection of X. There is an initial map $e: X \to \Pi\gamma Y_j$ with $\pi_j e = \eta_{Y_j} f_j$ (all $j \in J$), and an embedding $\bar{e}$: $SX \to \Pi\gamma Y_j$ with $\bar{e}\sigma_X = e$. Then $\bar{e} = k\hat{e}$ where $\hat{e}$ is an embedding and k is the inclusion map of the closure of $\bar{e}(SX)$ as uniform subspace of $\Pi\gamma Y_j$. Since $\hat{e}\sigma_X$ is a dense initial map of X into a complete space, there is a uniform isomorphism h with $h\eta_X = \hat{e}\sigma_X$. We have $\gamma f_j = \pi_j kh$. Hence $(\gamma f_j)_{j\in J}$ is an initial source. □

We now summarize some results from [8] in 7.3–7.6 below.

7.3. [8]. Let F be any T-section. The subcategory F^{-1}(**Complete**) consisting of all X in **Creg** for which FX is complete, is an epireflective subcategory of **Tych.** (**Tych** = Tychonoff spaces.) We call F *completion-true* if $\gamma F = Fr$ for some endofunctor r of **Creg**. Then $r = T\gamma F$ and $T\gamma F$ is the reflector of **Creg** onto F^{-1}(**Complete**). Examples of completion-true F are;

(1) $F = \mathscr{C}^*$; then F^{-1}(**Complete**) = **Compact** (the term *compact* for us includes Hausdorff) and $T\gamma F = \beta$, the Stone–Čech reflector.

(2) $F = \mathscr{C}$; then F^{-1}(**Complete**) = **Realcompact** and $T\gamma F = \upsilon$, the Hewitt realcompact reflector.

(3) $F = \varphi$; then F^{-1}(**Complete**) = **Topcpl**, the topologically complete spaces, and $T\gamma F = \delta$, the Dieudonné reflector.

For any T-section F, F^{-1}(**Complete**) lies between **Compact** and **Topcpl**. Every epireflective subcategory of **Tych** between **Compact** and **Topcpl** is of the form F^{-1}(**Complete**) for some (in general more than one) completion-true T-section F.

There is an example in [8] of a T-section F for which $T\gamma F$ is not idempotent, hence not the reflector onto F^{-1}(**Complete**). Such F is not completion-true.

Completion-truth of F clearly means $\gamma F = FT\gamma F$. Hence the interest of the following result proved in [8].

7.4. Lemma (*H.-P.A. Künzi*). *For every T-section F,*

$$\gamma F \geqslant FT\gamma F.$$

7.5 [8]. *For a T-section F, the following conditions are equivalent*:
(1) *F is completion-true*;
(2) *F is spanned by some class of complete uniform spaces*;
(3) γ (**range** $F) \subset$ **range** F;
(4) γ(**range** $F) \subset \downarrow$ **range** F.

7.6. [8]. The T-section F is called *strongly completion-true* if $\gamma(\downarrow$ **range** $F) \subset \downarrow$ **range** F. The functor $\mathscr{C}$ is not strongly completion-true. Examples of strongly completion-true T-sections are $\mathscr{C}^*$, φ and, for each infinite cardinal m, the functor $\mathscr{C}^*_m$ spanned by $\mathscr{D}(m)$, the m-precompact spaces (see Examples 4.3: one has $\downarrow$ **range** $\mathscr{C}^*_m = \mathscr{D}(m)$).

7.7. Definition. Let $\mathscr{A}$ be a reflective (or coreflective) subcategory of **Unif**, with reflector (or coreflector) a. We call $\mathscr{A}$ *completion-stable* if $\gamma\mathscr{A} \subset \mathscr{A}$, where $\gamma\mathscr{A} = \{\gamma X \mid X \in \mathscr{A}\}$. The same term is then applied to the functor a.

The following result is essentially folklore.

7.8. Proposition. *Let $\mathscr{A}$ be a bireflective subcategory of* **Unif** *with bireflector a. The following are equivalent*:
(1) *a is completion-stable*;
(2) $\gamma a = a\gamma a$;
(3) $(\gamma a)^2 = \gamma a$;
(4) γa *is a reflector*;
(5) $\gamma\mathscr{A} = \mathscr{A} \cap$ **Complete**;
(6) $\mathscr{A}$ *is the initial hull of some class of complete spaces*;
(7) $\mathscr{A} =$ **init** $\gamma\mathscr{A}$.

Proof. (1)$\Rightarrow$(7) because the unit $\eta_X : X \to \gamma X$ is initial; (7)$\Rightarrow$(6) is trivial; (6)$\Rightarrow$(1) by Lemma 7.2. For (3)$\Rightarrow$(2) it helps if one realizes that the equality sign stands for a canonical natural isomorphism given by composition of units. The other implications are obvious. □

7.9. The T-section F is strongly completion-true if and only if the bireflector α_F is completion-stable. This is immediate from Definitions 7.6 and 7.7 (since α_F corresponds to $\downarrow$ **range** F), but has to be savored with the following result.

7.10. Theorem. *For a T-section F, the following are equivalent*;
(1) *F is completion-true*;
(2) *The T-reflector* ρ_F is completion-stable;
(3) *Some T-reflector in* $[\rho_F, \alpha_F]$ *is completion-stable*;
(4) $\gamma F = a\gamma F$ *for some T-reflector a in* $[\rho_F, \alpha_F]$;
(5) $\gamma F = a\gamma F$ *for every T-reflector a in* $[\rho_F, \alpha_F]$.

Proof. (1)$\Rightarrow$(2): Recall from Definition 3.6 that ρ_F reflects to **init range** F. Let $X \in$ **init range** F. Then there is some initial source $(f_j: X \to FA_j)_{j\in J}$. By Lemma 7.2 the source $(\gamma f_j: \gamma X \to \gamma FA_j)_{j\in J}$ is initial. Assuming F completion-true, we have $\gamma FA_j = FT\gamma FA_j$, so that $\gamma FA_j \in$ **range** F. Thus $\gamma X \in$ **init range** F, and ρ_F is completion-stable.

(2)$\Rightarrow$(3): Trivial.

(3)$\Rightarrow$(4); Let $a \in [\rho_F, \alpha_F]$ be completion-stable. Thus $F = a\varphi$ and $\gamma a = a\gamma a$. Then $\gamma a\varphi = a\gamma a\varphi$, i.e. $\gamma F = a\gamma F$.

(4)$\Rightarrow$(1): Let $\gamma F = a\gamma F$, $a \in [\rho_F, \alpha_F]$. Since $\alpha_F \leqslant FT$ by Proposition 5.3, we have $\gamma F \leqslant FT\gamma F$. Then, by Künzi's Lemma 7.4, $\gamma F = FT\gamma F$.

(1)$\Rightarrow$(5); Let F be completion-true. Since (1) implies (2), $\gamma\rho_F = \rho_F\gamma\rho_F$. Putting φ on the right gives $\gamma F = \rho_F\gamma F$. Further by 7.5 we have γ(**range** $F) \subset \downarrow$ **range** F, which means $\gamma F = \alpha_F\gamma F$. Thus $\gamma F = \rho_F\gamma F = \alpha_F\gamma F$ and, if $\rho_F \leqslant a \leqslant \alpha_F$, it follows that $\gamma F = a\gamma F$.

(5)$\Rightarrow$(4): Trivial. □

Given any class $\{\mathcal{A}_j \mid j \in J\}$ of completion-stable bireflective subcategories, we have its bireflective supremum **init** $\bigcup_{j\in J} \mathcal{A}_j$ which is again completion-stable, an easy consequence of Proposition 7.8. The following result gives one such supremum more explicitly.

7.11. Theorem. *Let F be a completion-true T-section. There is a finest completion-stable T-reflector in $[\rho_F, \alpha_F]$. We shall denote it by τ_F. We have:*

$$\textbf{range}\ \tau_F = \textbf{init}\ (\textbf{Complete} \cap \downarrow \textbf{range}\ F),$$

and this is the largest completion-stable bireflective subcategory of **Unif** *which spans F.*

Proof. Denote $\mathcal{M}_F := \downarrow$ **range** F. By Proposition 7.8, **init** (**Complete** $\cap\, \mathcal{M}_F$) is completion-stable, and by 7.5 it spans F. Let $\mathcal{A}$ be any completion-stable bireflective subcategory of **Unif** which spans F. Then $\gamma\mathcal{A} \subset \mathcal{A} \subset \mathcal{M}_F$ (see Corollary 5.5). Thus $\gamma\mathcal{A} \subset$ **Complete** $\cap\, \mathcal{M}_F$. But $\mathcal{A} =$ **init** $\gamma\mathcal{A}$ by Proposition 7.8, whence $\mathcal{A} \subset$ **init** (**Complete** $\cap\, \mathcal{M}_F$). □

7.12. Corollary. *The T-section F is strongly completion-true if and only if F is completion-true with $\tau_F = \alpha_F$.*

Recall the T-reflector c to **init**$\{\mathbb{R}_u\}$, with $\mathscr{C} = c\varphi$.

7.13. Example. $\rho_{\mathscr{C}} < c \leqslant \tau_{\mathscr{C}} < \alpha_{\mathscr{C}}$.

Proof. We have $\rho_{\mathscr{C}} < c$ by Remark 3.8, $c \leqslant \tau_{\mathscr{C}}$ since c is completion-stable by Proposition 7.8(6), and $\tau_{\mathscr{C}} < \alpha_{\mathscr{C}}$ by 7.6 and Corollary 7.12.

7.14. Remarks. (1) We have already observed that $\rho_{\mathscr{C}^*} = \alpha_{\mathscr{C}^*}$ is the precompact reflector p. It seems desirable to have conditions for ρ_F and α_F to coincide, and for ρ_F and τ_F to coincide.

(2) The familiar relationships between the compact topological spaces, the functor $\mathscr{C}^*$ and the precompact uniform spaces are imitated to some extent when one changes $\mathscr{C}^*$ to some other completion-true T-section F. Say $F = \mathscr{C}$. Then the compact reflector $\beta = T\gamma\mathscr{C}^*$ becomes the realcompact reflector $\upsilon = T\gamma\mathscr{C}$. Various equivalent characterizations of precompactness have analogues—which need no longer be equivalent among themselves—in the bireflective subcategories corresponding to $\rho_{\mathscr{C}}$, c, $\tau_{\mathscr{C}}$, $\alpha_{\mathscr{C}}$. Any one of these classes has some claim to the term "prerealcompact"; for instance, Alò and Shapiro [1] gave this name to the class **init** $\{\mathbb{R}_u\}$ which corresponds to c. The imitation becomes better when F is strongly completion-true. For instance, the Samuel compactification γp has the two analogues $\gamma\rho_F$ and $\gamma\tau_F$ which are reflectors when F is completion-true; but $\gamma\alpha_F$ is a reflector precisely when F is strongly completion-true (by Proposition 7.8 and 7.9).

(3) We leave aside the question whether there are T-reflectors a other than the identity which satisfy the strong property $\gamma a = a\gamma$ (see [24]). However, if $\gamma a = a\gamma$, then the T-section $F = a\varphi$ is completion-true and $T\gamma F$ is the reflector to the topologically complete spaces.

Proof: We have $\gamma a\varphi = a\gamma\varphi = a\varphi\delta$, from 7.3(3). Thus $\gamma F = F\delta$, and $T\gamma F = \delta$.

(4) Much is known about T-coreflectors versus completion; see e.g. [13–15, 17, 26–29, 25]. A basic result is that every T-coreflector is completion-stable [26, 17]. So one considers T-coreflectors with the stronger property $\gamma b = b\gamma$. Several T-coreflectors are known to have this property, e.g. those that preserve initial maps, in particular the subfine and the locally fine coreflectors [23, p. 127], and those given in [17, 5.4]. One has:

7.15. Proposition. *If the T-coreflector b satisfies* $\gamma b = b\gamma$, *then* $b\mathscr{C}^* = \mathscr{C}^*$.

Proof. $T\gamma(b\mathscr{C}^*X) = Tb\gamma\mathscr{C}^*X = T\gamma\mathscr{C}^*X$, since $Tb = T$. Thus $T\gamma(b\mathscr{C}^*X)$ is compact, and so $b\mathscr{C}^*X$ is precompact. Hence $b\mathscr{C}^*X \leqslant \mathscr{C}^*T(b\mathscr{C}^*X) = \mathscr{C}^*X$, and since $b \geqslant \mathbf{1}$, $b\mathscr{C}^*X = \mathscr{C}^*X$.

Correction added in proof. Example 3.8(2): The argument in parentheses has to be modified by taking $h(x_n) = a_n^2$ with $g(a_n) = x_n$.

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