Computers and Mathematics with Applications 62 (2011) 3446–3448

Contents lists available at SciVerse ScienceDirect

# Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# On $(C, 1)$ means of sequences

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ARTICLE INFO

*Article history:*
Received 10 June 2011
Accepted 22 August 2011

*Keywords:*
$(C, 1)$ bounded sequences
$(C, 1)$ summable sequences

ABSTRACT

Let $(s_n)$ be a sequence of real numbers such that $\limsup_n \sigma_n = \beta$ and $\liminf_n \sigma_n = \alpha$, where $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k$ and $\beta \neq \alpha$. We prove that $\limsup_n s_n = \beta$ and $\liminf_n s_n = \alpha$ if the following conditions hold:

$$\liminf_n \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k - s_n) \geq (\beta - \alpha)\frac{\lambda}{\lambda - 1} \quad \text{for } \lambda > 1,$$

$$\liminf_n \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (s_n - s_k) \geq (\beta - \alpha)\frac{\lambda}{1 - \lambda} \quad \text{for } 0 < \lambda < 1,$$

where $[\lambda n]$ denotes the integer part of $\lambda n$.



## 1. Introduction

Let $(s_n)$ be a sequence of real numbers, and for each $n$ define the $(C, 1)$ means by

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k \quad (n = 1, 2, \ldots). \tag{1.1}$$

A sequence $(s_n)$ is said to be $(C, 1)$ summable to $s$ if $\lim \sigma_n = s$. A sequence $(s_n)$ is said to be $(C, 1)$ bounded if $(\sigma_n)$ is bounded. It is well-known [1] that

$$\liminf_n s_n \leq \liminf_n \sigma_n \leq \limsup_n \sigma_n \leq \limsup_n s_n \tag{1.2}$$

for any sequence $(s_n)$ of real numbers. If $(s_n)$ is a bounded sequence, then $\liminf_n s_n$ and $\limsup_n s_n$ exist and are finite.

If $(s_n)$ is a bounded sequence, $(s_n)$ is $(C, 1)$ bounded. So $\liminf_n \sigma_n$ and $\limsup_n \sigma_n$ always exist and are finite. That the converse of this implication is not generally true follows from the example of the sequence defined by

$$s_n = \begin{cases} k, & \text{if } n = 2k,\ k = 1, 2, \ldots \\ -k, & \text{if } n = 2k-1,\ k = 1, 2, \ldots. \end{cases}$$

It is clear that $\liminf_n \sigma_n = -\frac{1}{2}$ and $\limsup_n \sigma_n = 0$, but $\liminf_n s_n = -\infty$ and $\limsup_n s_n = \infty$. The main goal of this paper is to show that if $(s_n)$ is a sequence of real numbers such that $\limsup_n \sigma_n = \beta$ and $\liminf_n \sigma_n = \alpha$, where $\beta \neq \alpha$, then

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doi:10.1016/j.camwa.2011.08.060

$\limsup_n s_n = \beta$ and $\liminf_n s_n = \alpha$, provided that

$$\liminf_n \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k - s_n) \geq (\beta - \alpha) \frac{\lambda}{\lambda - 1} \quad \text{for } \lambda > 1,$$

$$\liminf_n \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (s_n - s_k) \geq (\beta - \alpha) \frac{\lambda}{1 - \lambda} \quad \text{for } 0 < \lambda < 1,$$

where $[\lambda n]$ denotes the integer part of $\lambda n$.

## 2. The main result

**Theorem 2.1.** *For a sequence $(s_n)$ of real numbers, let $\limsup \sigma_n = \beta$ and $\liminf \sigma_n = \alpha$, where $\beta \neq \alpha$. If*

$$\liminf_n \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k - s_n) \geq (\beta - \alpha) \frac{\lambda}{\lambda - 1} \quad \text{for } \lambda > 1, \tag{2.1}$$

*and*

$$\liminf_n \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (s_n - s_k) \geq (\beta - \alpha) \frac{\lambda}{1 - \lambda} \quad \text{for } 0 < \lambda < 1, \tag{2.2}$$

*where $[\lambda n]$ denotes the integer part of $\lambda n$, then $\limsup_n s_n = \beta$ and $\liminf_n s_n = \alpha$.*

We note that since $\limsup_n \sigma_n$ and $\liminf_n \sigma_n$ exist for a $(C, 1)$ bounded sequence, then $\limsup_n s_n$ and $\liminf_n s_n$ exist and are equal to $\limsup_n \sigma_n$ and $\liminf_n \sigma_n$, respectively, provided that (2.1) and (2.2) hold.

**Corollary 2.2.** *Let $(s_n)$ be $(C, 1)$ summable to s. If*

$$\liminf_n \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k - s_n) \geq 0 \quad \text{for } \lambda > 1, \tag{2.3}$$

*and*

$$\liminf_n \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (s_n - s_k) \geq 0 \quad \text{for } 0 < \lambda < 1, \tag{2.4}$$

*where $[\lambda n]$ denotes the integer part of $\lambda n$, then $(s_n)$ is convergent to s.*

The conditions (2.3) and (2.4) can be replaced by the weaker conditions

$$\limsup_{\lambda \to 1^+} \liminf_n \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k - s_n) \geq 0 \tag{2.5}$$

and

$$\limsup_{\lambda \to 1^-} \liminf_n \frac{1}{n - [\lambda n]} \sum_{k=[\lambda n]+1}^{n} (s_n - s_k) \geq 0 \tag{2.6}$$

since they are satisfied for all $\lambda > 1$ and all $0 < \lambda < 1$, respectively.

It is shown by Móricz [2] that the conditions (2.5) and (2.6) are Tauberian conditions for $(C, 1)$ summability.

## 3. A lemma

For the proof of Theorem 2.1 we need the following lemma.

**Lemma 3.1** ([3,4]). *Let $(s_n)$ be a sequence of real numbers.*

(i) *For $\lambda > 1$ and sufficiently large n,*

$$s_n - \sigma_n = \frac{[\lambda n] + 1}{[\lambda n] - n} \left(\sigma_{[\lambda n]} - \sigma_n\right) - \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} (s_k - s_n), \tag{3.1}$$

*where $[\lambda n]$ denotes the integer part of $\lambda n$.*

(ii) *For* $0 < \lambda < 1$ *and sufficiently large n,*

$$s_n - \sigma_n = \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n - \sigma_{[\lambda n]}\right) + \frac{1}{n-[\lambda n]}\sum_{k=[\lambda n]+1}^{n}(s_n - s_k), \tag{3.2}$$

*where* $[\lambda n]$ *denotes the integer part of* $\lambda n$*.*

## 4. Proof of Theorem 2.1

Assume that $\limsup_n \sigma_n = \beta$ and $\liminf_n \sigma_n = \alpha$, where $\beta \neq \alpha$. Since

$$\lim_n \frac{[\lambda n]+1}{[\lambda n]-n} = \frac{\lambda}{\lambda - 1} \quad \text{for } \lambda > 1, \tag{4.1}$$

we obtain

$$\limsup_n \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]} - \sigma_n\right) \le (\beta - \alpha)\frac{\lambda}{\lambda - 1} \quad \text{for } \lambda > 1. \tag{4.2}$$

Taking the lim sup of both sides of (3.1) and using (4.2), we have

$$\limsup_n s_n \le \beta + (\beta - \alpha)\frac{\lambda}{\lambda - 1} - \liminf_n \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}(s_k - s_n). \tag{4.3}$$

Taking (2.1) into account, we obtain

$$\limsup_n s_n \le \beta. \tag{4.4}$$

By (1.2),

$$\beta \le \limsup_n s_n. \tag{4.5}$$

Combining (4.4) and (4.5), we have $\limsup_n s_n = \beta$.

Since

$$\lim_n \frac{[\lambda n]+1}{n-[\lambda n]} = \frac{\lambda}{1-\lambda} \quad \text{for } 0 < \lambda < 1, \tag{4.6}$$

we obtain

$$\liminf_n \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma_n - \sigma_{[\lambda n]}\right) \ge (\alpha - \beta)\frac{\lambda}{1-\lambda} \quad \text{for } 0 < \lambda < 1. \tag{4.7}$$

Taking the lim inf of both sides of (3.2) and using (4.7), we have

$$\liminf_n s_n \ge \alpha + (\alpha - \beta)\frac{\lambda}{1-\lambda} + \liminf_n \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}(s_k - s_n). \tag{4.8}$$

Taking (2.2) into account, we obtain

$$\liminf_n s_n \ge \alpha. \tag{4.9}$$

By (1.2),

$$\liminf_n s_n \le \alpha. \tag{4.10}$$

Combining (4.9) and (4.10), we have $\liminf_n s_n = \alpha$. This completes the proof of Theorem 2.1.

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