The $k$-restricted edge-connectivity of a product of graphs

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A B S T R A C T

This work deals with a generalization of the Cartesian product of graphs, the product graph $G_m \ast G_p$ introduced by Bermond et al. [J.C. Bermond, C. Delorme, G. Farhi, Large graphs with given degree and diameter II, J. Combin. Theory, Series B 36 (1984), 32–48]. The connectivity of these product graphs is approached by studying the $k$-restricted edge-connectivity, which is defined as the minimum number of edges of a graph whose deletion yields a disconnected graph with all its components having at least $k$ vertices. To be more precise, we present lower and upper bounds for the $k$-restricted edge-connectivity of $G_m \ast G_p$, and provide sufficient conditions that ensure an optimal value for this parameter. When both $G_m$ and $G_p$ are regular graphs, conditions for guaranteeing that $G_m \ast G_p$ is super-$\lambda(k)$ are also presented, and the particular case where both $G_m$ and $G_p$ are complete graphs is considered.

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1. Introduction

For the graph theoretical terminology and notation not defined here, we refer the reader to [1]. Throughout this paper a graph $G = (V(G), E(G))$ means a finite undirected graph without self-loops or multiple edges, where $V(G)$ and $E(G)$ stand for its vertex set and edge set, respectively. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$. The minimum degree of $G$ is denoted by $\delta(G)$, and the maximum degree of $G$ by $\Delta(G)$.

Extending a given interconnection system to a larger and fault-tolerant one so that the communication delay among nodes of the new network is small enough is a basic objective in network design. One interesting model for this kind of extension consists of considering a number of copies of a given graph $G$, connecting these copies somehow in such a way that certain desirable properties remain and certain useful parameters can be evaluated easily. In this regard the Cartesian product $G \times H$ of graphs is an important tool for obtaining large graphs from smaller ones (hence for designing large-scale interconnection networks), with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. In this work we deal with a generalization of the Cartesian product of graphs, the product graph $G_m \ast G_p$ of two graphs $G_m$ and $G_p$ introduced by Bermond et al. [2].

Definition 1 ([2]). Let $G_m = (V(G_m), E(G_m))$ and $G_p = (V(G_p), E(G_p))$ be two graphs. For each edge $xy \in E(G_m)$ let $\pi_{xy}$ be a permutation of $V(G_p)$ such that $\pi_{xy}^{-1} = \pi_{yx}$. Then the product graph $G_m \ast G_p$ has $V(G_m) \times V(G_p)$ as vertex set, two vertices $(x, x')$, $(y, y')$ being adjacent if either

$x = y$ and $x'y' \in E(G_p)$

or

$xy \in E(G_m)$ and $y' = \pi_{xy}(x')$.

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The product graph $G_m * G_p$ of two given graphs $G_m$, $G_p$ can be viewed as formed by $|V(G_m)|$ disjoint copies of $G_p$, each edge $xy \in E(G_m)$ indicating that some perfect matching between the copies $G^x_p, G^y_p$ (respectively generated by the vertices $x$ and $y$ of $G_m$) is added. The graph $G_m$ is usually called the main graph and $G_p$ is called the pattern graph of the product graph $G_m * G_p$, and every edge of $G_m * G_p$ that belongs to any of the $|E(G_m)|$ perfect matchings between copies of $G_p$ is an intercopy edge of $G_m * G_p$. Some relations between the minimum degree, the maximum degree, and the diameter of a product graph $G_m * G_p$ with the corresponding parameters of its main graph and its pattern graph can be found in [2].

**Lemma 2** ([2]). Let $G_m$ and $G_p$ be two graphs. Then, for every product graph $G_m * G_p$:

1. $\delta(G_m * G_p) = \delta(G_m) + \delta(G_p)$, $\Delta(G_m * G_p) = \Delta(G_m) + \Delta(G_p)$.
2. If both $G_m$ and $G_p$ are connected, then $G_m * G_p$ is also connected and $\text{diam}(G_m) \leq \text{diam}(G_m * G_p) \leq \text{diam}(G_m) + \text{diam}(G_p)$.

Observe that if we choose $\pi_{xy}(x') = x'$ for all $xy \in E(G_m)$ and all $x' \in V(G_p)$, then $G_m * G_p \cong G_m \boxtimes G_p$. Hence the results for the Cartesian product of two graphs will follow directly. On the other hand, if $G_m \cong K_2$ then $G_m * G_p$ results in a permutation graph $(G_p)^p$—as introduced by Chartrand and Harary in [3]. Among a large number of references on Cartesian product graphs or permutation graphs we can outline some particularly interesting papers, as for example [4–12], where the study of the connectivity of these graphs has been addressed.

This work approaches the connectedness of product graphs $G_m * G_p$ by means of studying the $k$-restricted edge-connectivity of these graphs. Given a connected graph $G$ and an integer $k$ such that $1 \leq k \leq |V(G)|/2$, a $k$-restricted edge-cut of $G$ is a set $W \subseteq E(G)$ such that $G - W$ is not connected and all the components of $G - W$ have at least $k$ vertices. Observe that such a $k$-restricted edge-cut may not exist in a graph $G$, then it is said to be $\lambda(k)$-connected. In this case, the minimum cardinality of a $k$-restricted edge-cut of $G$ is denoted by $\lambda(k)(G)$, and called the $k$-restricted edge-connectivity of $G$ (these concepts were introduced by Fàbrega and Fiol [13,14], even though in a slightly different way). Notice that $\lambda(1)(G) = \lambda(G)$ corresponds to the (standard) edge-connectivity of $G$, and $\lambda(2)(G) = \lambda(G)$ is known as the restricted edge-connectivity of $G$, introduced by Esfahanian and Hakimi in [15]. Observe also that $\lambda(i)(G) \leq \lambda(i-1)(G)$ whenever $i < j$.

For all $B \subseteq V(G)$ nonempty set of vertices of a graph $G$, let $\omega_G(B)$ denote the set of edges of $G$ with one endvertex in $B$ and the other one not in $B$. For any positive integer $k$, the $k$-edge degree of $G$ is defined as $\xi(G)(G) = \min\{|\omega_G(B)| : |B| = k, G[B] \text{ is connected}\}$ (where $G[B]$ stands for the induced subgraph by $B$ in $G$). Clearly, $\xi(1)(G) = \delta(G)$ and $\xi(2)(G) = \delta(G) = \min\{d_G(u) + d_G(v) - 2 : uv \in E(G)\}$, usually known as the minimum edge degree of $G$.

It is well known that $\lambda(G) = \lambda(1)(G) \leq \delta(G)$ and in [15] was proved that $\lambda'(G) = \lambda(2)(G) < \xi(G)$ if $G$ is not a star and its order is at least 4. Apart from the existence of $\lambda(k)(G)$, one important question to be considered concerns its upper bounding. In this regard, a theorem due to Zhang and Yuan [16] is especially useful.

**Theorem 3** ([16]). Let $G$ be a connected graph of minimum degree $\delta$ and order $n \geq 2(\delta + 1)$ that is not isomorphic to any $G_m, s \leq m$, (where $G_m, s \ast m$ consists of $m$ disjoint copies of $K_3$ and a new vertex $u$ adjacent to all the vertices in those copies). For all $k \leq \delta + 1$, $G$ is $\lambda(k)$-connected with $\lambda(k)(G) \leq \xi(k)(G)$.

For other interesting results on the $k$-restricted edge-connectivity of graphs see for example [17–19,5,20–24].

A $\lambda(k)$-connected graph $G$ such that $\lambda(k)(G) = \xi(k)(G)$ is said to be $\lambda(k)$-optimal. Observe that after the deletion from $G$ of a minimum $k$-restricted edge-cut the two resulting components can both have order greater than $k$, even if $G$ is $\lambda(k)$-optimal. Thus, a $\lambda(k)$-optimal graph $G$ is said to be super-$\lambda(k)$ if the deletion of every minimum $k$-restricted edge-cut isolates some component with $k$ vertices. In this regard the following results were obtained in [19].

**Theorem 4** ([19]). Let $G$ be a $\lambda(k+1)$-connected graph such that $\lambda(k)(G) \leq \xi(k)(G)$.

1. $G$ is super-$\lambda(k)$ if and only if $\lambda(k+1)(G) > \xi(k)(G)$.
2. Suppose that $\lambda(k+1)(G) = \xi(k+1)(G)$ and the minimum degree $\delta(G) \geq 2k+1$. Then $G$ is super-$\lambda(k-1)$ for every $t = 0, \ldots, k-1$.

In what follows we give some conditions on $G_m$ and $G_p$ that ensure that $G_m * G_p$ is $\lambda(k)$-connected for $k \geq 2$, and present bounds for $\lambda(k)(G_m * G_p)$. Going one step further, we give sufficient conditions to guarantee an optimal value for $\lambda(k)$, that is, $\lambda(k)(G_m * G_p) = \xi(k)(G_m * G_p)$; moreover, $G_m * G_p$ is ensured to be super-$\lambda(k)$, for the regular case, under some constraints. The main objective of this work is to generalize or extend a previous result obtained in [17] for the restricted edge connectivity $\lambda'(G_m * G_p) = \lambda(2)(G_m * G_p)$ of product graphs, which is recalled in the following theorem.

**Theorem 5** ([17]). Let $G_m$ and $G_p \not\cong K_3$ be two connected graphs. If $\delta(G_p) \geq \Delta(G_m) + 1 \geq 2$, then the product graph $G = G_m * G_p$ is $\lambda'$-connected, and

$$\min [\lambda(G_m) | V(G_p) |, (\delta(G_m) + 1)\lambda'(G_p), \delta(G_m)(\delta(G_p) + 1) + \lambda'(G_p), \xi(G)] \leq \lambda'(G) \leq \xi(G).$$
2. Results

Consider a product graph $G_m \ast G_p$ given by Definition 1, with both $G_m$ and $G_p$ connected, and let $k$ be an integer such that $1 \leq k \leq |V(G_p)|$. Let $B \subseteq V(G_p)$ be a set of $k$ vertices such that $B$ induces a connected subgraph of $G_p$ with $|\omega_{G_p}(B)| = \xi_{(k)}(G_p)$. If $x \in V(G_m)$ is chosen so that $d_{G_m}(x) = \delta(G_m)$, then the set of vertices $xB = \{(x, y) : y \in B\} \subseteq V(G_m \ast G_p)$ satisfies

$$\xi_{(k)}(G_m \ast G_p) \leq |\omega_{G_m}(xB)| = |\omega_{G_p}(B)| + k\delta(G_m) = \xi_{(k)}(G_p) + k\delta(G_m).$$

Hence the following remark holds.

**Remark 1.** Let $G_m$, $G_p$ be two connected graphs, and let $k$ be an integer, $1 \leq k \leq |V(G_p)|$. Then for every product graph $G_m \ast G_p$ it follows that

$$\xi_{(k)}(G_m \ast G_p) \leq \xi_{(k)}(G_p) + k\delta(G_m).$$

Given two graphs $G_m$, $G_p$, the copy of $G_p$ in $G_m \ast G_p$ corresponding to a vertex $x \in V(G_m)$ will be denoted by $G_m^x$, considered as a subgraph of $G_m \ast G_p$ (i.e., $V(G_m^x) \subseteq V(G_m \ast G_p)$ and $E(G_m^x) \subseteq E(G_m \ast G_p)$).

**Lemma 6.** Let $G_m$ and $G_p$ be two connected graphs with $\delta(G_m), \delta(G_p) \geq 2$. Let $k$ be an integer, $2 \leq k \leq \lfloor |V(G_p)|/2 \rfloor$. If $\delta(G_m) + \delta(G_p) \geq 2\lfloor 5k - 1 \rfloor$, then the graph $G = G_m \ast G_p$ is $\lambda_{(k)}$-connected and $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$. Let $k$ be an integer, $2 \leq k \leq |V(G_p)|/2$. If $\delta(G_m) + \delta(G_p) \geq \max\{5, k - 1\}$, then the graph $G = G_m \ast G_p$ is $\lambda_{(k)}$-connected and $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$. Let $k$ be an integer, $2 \leq k \leq |V(G_p)|/2$. If $\delta(G_m) + \delta(G_p) \geq \max\{5, k - 1\}$, then the graph $G = G_m \ast G_p$ is $\lambda_{(k)}$-connected and $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$.

**Proof.** Observe that $G_m$ and $G_p$ connected implies that $G = G_m \ast G_p$ is connected. Since the inequality $(a + 1)(b + 1) \geq 2(a + b)$ holds for any pair of integers $a \geq 2$ and $b \geq 3$, or $a \geq 3$ and $b \geq 2$, it follows that

$$|V(G_m \ast G_p)| = |V(G_m)| \cdot |V(G_p)| \geq (1 + \delta(G_m))(1 + \delta(G_p)) \geq 2(\delta(G_m) + \delta(G_p) + 1) = 2(\delta(G_m \ast G_p) + 1),$$

as $\delta(G_m) + \delta(G_p) \geq 5$ and $\delta(G_m), \delta(G_p) \geq 2$. Furthermore, from the definition of the product $G = G_m \ast G_p$ it follows that $G$ cannot be isomorphic to any graph $G_m^x, G_p^y$ introduced in Theorem 3 because $|V(G_m)| > 2$ and $|V(G_p)| > 2$. Hence by Theorem 3, $G$ is $\lambda_{(k)}$-connected and $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$ because $\delta(G_m \ast G_p) + 1 = \delta(G_m) + \delta(G_p) + 1 \geq k$. □

The following theorem constitutes the main result of this work.

**Theorem 7.** Let $G_m$ and $G_p$ be two connected graphs. Let $k$ be an integer, $2 \leq k \leq \lfloor |V(G_p)|/2 \rfloor$, and assume that $G_p$ is $\lambda_{(k)}$-connected. If $\delta(G_m) \geq k - 1$ and $\delta(G_p) \geq \Delta(G_m) + 2k - 3$, then the graph $G = G_m \ast G_p$ is $\lambda_{(k)}$-connected and

$$\min\{\lambda(G_m)|V(G_p)|, \lambda(G_m) - k\lambda(G_p), \lambda(G_p) - k\lambda(G_m), \lambda(G_m) + \lambda(G_p), \xi(G_p)\} \geq \lambda(G) \geq \xi(G).$$

**Proof.** Case $k = 2$ follows from Theorem 5 (Theorem 14 of [17]). Hence $k \geq 3$ is assumed for the rest of the proof. Thus, $\delta(G_m) \geq 3 - 1 \geq 2$ and $\delta(G_p) \geq \Delta(G_m) + 2k - 3 \geq 5$ yielding that $G$ is $\lambda_{(k)}$-connected with $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$ by Lemma 6.

We next prove the lower bound for $\lambda_{(k)}(G)$. Let $W \subseteq E(G)$ be a minimum $k$-restricted edge-cut of $G$, $|W| = \lambda_{(k)}(G)$. Thus $G - W$ consists of exactly two connected components, $H$ and $H^*$, such that $|V(H)| \geq k$ and $|V(H^*)| \geq k$. When $|V(H)| = k$, it follows that $\lambda_{(k)}(G) = |W| = |\omega_{G}(V(H))| \geq \xi_{(k)}(G)$, and the lower bound for $\lambda_{(k)}(G)$ holds. Hence, suppose that $|V(H)| \geq k + 1$ and $|V(H^*)| \geq k + 1$.

Let $W = \bigcup_{x \in V(G_m^x)} W_x \cup W_{G_p}$, where $W_x \in E(G_p^x)$ for each $x \in V(G_m)$, and $W_{G_p}$ is only composed by intercopy edges. Clearly, if $W_x = \emptyset$ then $W_x$ is an edge-cut of $G_p^x$ because $W_x$ has minimum cardinality. We need now the following claim.

**Claim.** Each nonempty $W_x \subseteq E(G_p^x)$ is a $k$-restricted edge-cut of $G_p^x$. Then, $|W_x| \geq \lambda_{(k)}(G_p^x)$.

**Proof of the Claim.** We reason by contradiction supposing that some component $I$ of $G_p^x - W_x$ has $|V(I)| \leq k - 1$ (without loss of generality, we assume that $I$ is a subgraph of $H$). Let $u \in V(I)$ be a vector of $I^*$, and let us consider the set of edges $B(u) = \omega_{G_p}(u) \setminus E(I^*)$. Note that $\omega_{G_p}(u) \cap W_x \neq \emptyset$ because $d_{G_p}(u) \geq \delta(G_p) \geq k > |V(I)|$.

First suppose that all components of $H - u$ have at least $k$ vertices. Then the set of edges of $G$

$$W' = (W \setminus (\omega_{G_p}(u) \cap W_x)) \cup (\omega_{G_p}(u) \setminus W_x) \cup B(u)$$

is also a $k$-restricted edge-cut of $G$ because $u$ is not isolated in $G - W'$ (since $\omega_{G_p}(u) \cap W_x$ is not contained in $W'$). As

$$d_{G_p}(u) \leq |\omega_{G_p}(u) \cap W_x| + |V(I)| - 1 \text{ and } \delta(G_p) \geq \Delta(G_m) + 2k - 3 \text{ we obtain that}$$

$$|W'| \leq |W| - |\omega_{G_p}(u) \cap W_x| - |V(I)| - 1 \text{ and } \delta(G_p) \geq \Delta(G_m) + 2k - 3 \text{ we obtain that}$$

$$\leq \lambda_{(k)}(G) - d_{G_p}(u) + |V(I)| - 1 + |V(I)| - 1 + \Delta(G_m)$$

$$\leq \lambda_{(k)}(G) + 2(k - 2) + \Delta(G_m)$$

$$\leq \lambda_{(k)}(G) - 1,$$

a contradiction as $W$ is a minimum $k$-restricted edge-cut of $G$. 

Second suppose that some component $\Omega$ of $H - u$ has at most $k - 1$ vertices (hence $u$ is a cutvertex of $H$). If $\Omega = \{v\}$, the graph $H - v$ is still connected ($v$ is not a cutvertex of $H$), and we can get again a contradiction by reasoning for $v$ as we have just done above for $u$; that is, the set of edges

$$W'' = (W \setminus (\omega_{G_p}(v) \cap W_p)) \cup (\omega_{G_p}(v) \setminus W_p) \cup (\omega_C(v) \setminus E(G_p^\ast))$$

(with $v \in V(G_p^\ast - W_p) \cap V(H)$) is a $k$-restricted edge-cut of $G$ with cardinality $|W''| < |W|$. If $|\Omega| \geq 2$, there exist at least two vertices in $\Omega$ which are not cutvertices of $\Omega$ (just consider two leaves of a spanning tree of $\Omega$). If exactly one of these two vertices is adjacent to $u$, call $t$ to the other vertex; otherwise, let $t$ be whichever of those two vertices that are not cutvertices of $\Omega$. Then the graph $H - t$ is connected, and we can reason for vertex $t$ as we did before for vertices $u$, or $v$, to arrive at a contradiction. Having obtained an absurdity in all possible cases, we deduce that each nonempty $W_x \subset E(G_p^\ast)$ is a $k$-restricted edge-cut of $G_p^\ast$, and the claim holds. □

Let $r$ be the number of copies of $G_p$ in $G$ that are split by $W$ (i.e., copies of $G_p$ having vertices in both $H$ and $H^\ast$), $0 \leq r \leq |V(G_m)|$. Suppose first that $r = 0$. Then all the edges in $W$ are intercopy edges and correspond to $i \geq 1$ perfect matchings between copies of $G_p$ that appear in $G$ as a replacement of $\ell$ edges of $G_m$. Moreover, the set of these $\ell$ edges of $G_m$ must be an edge-cut of $G_m$ (for if not, $G$ - $W$ is still connected), hence $\ell \geq \lambda(G_m)$. Thus $|W| \geq \lambda(G_m)|V(G_p)|$, and the theorem holds. If $r \geq \delta(G_m) - k + 3$, then by Claim $\lambda_{\ell}(G) = |W| \geq (\delta(G_m) - k + 3)\lambda_{\ell}(G_p)$, because at least $r \cdot \lambda_{\ell}(G_p)$ edges must be deleted from $G$ in order to split by $W$ the considered $r$ copies of $G_p$. Then, the theorem also follows in this case.

Consider now the case $1 \leq r \leq \delta(G_m) - k - 2$, and let $S = \{x \in V(G_m) : G_p^x$ is a split (by $W$) copy of $G_p\}$; hence $|S| = r$. For every $x \in S$ let us write $V(G_p^x) = V_x \cup V_x^\ast$, with $V_x \subset V(H), V_x^\ast \subset V(H^\ast)$, and let us denote $k_x = \min(|V_x|, |V_x^\ast|)$ and $s_x = |W_x|$. By Claim $k_x \geq k$ and $s_x \geq \lambda_{\ell}(G_p)$. Taking into account that each $x \in S$ is adjacent in $G_m$ to at least $\delta(G_m) - (r - 1)$ other vertices $y$ of $G_m - S$, then from copy $G_p^x$ we have at least as many edges in $G$ as $k_x(\delta(G_m) - (r - 1)) + s_x$. If $S_1 = \{x \in S : k_x \geq \delta(G_m) - k + 3\} \subset S$ then

$$|W| \geq \sum_{x \in S} (k_x(\delta(G_m) - r + 1) + s_x) \geq |S_1|((\delta(G_p) - k + 3)(\delta(G_m) - r + 1) + \lambda_{\ell}(G_p)) + \sum_{x \in S \setminus S_1} (k_x(\delta(G_m) - r + 1) + s_x).$$

(1)

Take $x \in S \setminus S_1$, that is, $k \leq k_x < \delta(G_m) - k + 2$, and assume without loss of generality that $k_x = |V_x^\ast|$. Let us consider a connected subgraph induced by a set $Q \subset V_x^\ast$ of cardinality $k$, and let us call $e_Q, e_{V_x^\ast \setminus Q}$, respectively, to the number of edges of the subgraphs of $G_p^x$ induced by $Q, V_x^\ast \setminus Q$; similarly, $e_{Q, V_x^\ast \setminus Q}$ is the number of edges of $G_p^x$ with one endvertex in $Q$ and the other one in $V_x^\ast \setminus Q$. Then,

$$\sum_{u \in Q} d_{G_p^x}(u) + \sum_{u \in V_x^\ast \setminus Q} d_{G_p^x}(u) - s_x \geq \xi_{\ell}(G_p) + 2e_Q + (k_x - \delta(G_p)) - s_x,$n

and also

$$\sum_{u \in Q} d_{G_p^x}(u) + \sum_{u \in V_x^\ast \setminus Q} d_{G_p^x}(u) - s_x \geq 2e_Q + 2e_{V_x^\ast \setminus Q} + 2e_{Q, V_x^\ast \setminus Q} \leq 2e_Q + (k_x - k)(k_x - k - 1) + 2k(k_x - k).$$

From these two inequalities we get

$$s_x \geq k_x(\delta(G_p) - k + 1) - k(\delta(G_p) - k + 1) + \xi_{\ell}(G_p),$$

and then

$$k_x(\delta(G_m) - r + 1) + s_x \geq k_x(\delta(G_m) + \delta(G_p) - r - k_x + 2) - k(\delta(G_p) - k + 1) + \xi_{\ell}(G_p) \geq k(\delta(G_m) - r + 1) + \xi_{\ell}(G_p).$$

(2)

having used for the second inequality that the function $f(k_x) = k_x(\delta(G_m) + \delta(G_p) - r - k_x + 2) - k(\delta(G_p) - k + 1) + \xi_{\ell}(G_p)$ takes its minimum for $k_x = k$ when $k_x \in (k, \ldots, \delta(G_p) - k + 2)$ (because it is not difficult to check that the hypothesis $\delta(G_p) \geq 4\lambda(\Delta(G_m)) + 2k - 3$ implies that $f(\delta(G_p) - k + 2) > f(k)$). Therefore, from (1) and (2) and taking into account that $|S \setminus S_1| = r - |S_1|$ it follows that

$$|W| \geq |S_1|((\delta(G_p) - k + 3)(\delta(G_m) - r + 1) + \lambda_{\ell}(G_p)) + |S \setminus S_1|(k(\delta(G_m) - r + 1) + \xi_{\ell}(G_p))$$

$$= (kr + |S_1|)(\delta(G_p) - 2k + 3)(\delta(G_m) - r + 1) + (r - |S_1|)\xi_{\ell}(G_p) + |S_1|\lambda_{\ell}(G_p).$$

(3)
It is not difficult to check that both these lower bounds for $|W|$ take their minimum values for $r = 1$ when $r \in \{1, \ldots, \delta(G_m) - k + 2\}$. So we have

$$\lambda_{(k)}(G) = |W| \geq \begin{cases} r(\delta(G_p) - k + 3)(\delta(G_m) - r + 1) + \lambda_{(k)}(G_p) & \text{if } |S_1| = r, \\ r(k(\delta(G_m) - r + 1) + \xi_{(k)}(G_p)) & \text{if } |S_1| = 0, \end{cases}$$

and the theorem holds when $|S_1| \in \{0, r\}$. Hence suppose that $1 \leq |S_1| \leq r - 1$, so $2 \leq r \leq \delta(G_m) - k + 2$. In this case from (3) we deduce that

$$|W| \geq (\delta(G_p) + (r - 2)k + 3)(\delta(G_m) - r + 1) + \xi_{(k)}(G_p) + \lambda_{(k)}(G_p).$$

The right-hand term of this inequality takes its minimum value when $r = \delta(G_m) - k + 2$, in which case by the hypothesis $\delta(G_m) \geq k - 1$ and $\delta(G_p) \geq \delta(G_m) + 2k - 3$ we obtain

$$\lambda_{(k)}(G) = |W| \geq (\delta(G_p) + (\delta(G_m) - k)k + 3)(k - 1) + \xi_{(k)}(G_p) + \lambda_{(k)}(G_p)$$

$$= (\delta(G_p) - k^2 + 3)(k - 1) + k(\delta(G_m) - k + 3) + \xi_{(k)}(G_p) + \lambda_{(k)}(G_p)$$

$$\geq \delta(G_m)(k(k - 2) + k - 1) - k(k - 1)(k - 2) + k\delta(G_m) + \xi_{(k)}(G_p) + \lambda_{(k)}(G_p)$$

$$\geq k\delta(G_m) + \xi_{(k)}(G_p) + \lambda_{(k)}(G_p)$$

$$\geq \xi_{(k)}(G_m * G_p),$$

the last inequality by Remark 1. Therefore the theorem holds.

**Remark 2.** Let $k \geq 1$ be an integer and $G$ a graph with minimum and maximum degree $\delta(G)$ and $\Delta(G)$ respectively, and $k$-edge degree $\xi_{(k)}(G)$. Then

$$k\delta(G) - k(k - 1) \leq \xi_{(k)}(G) \leq k\Delta(G) - 2(k - 1).$$

The following results states, roughly speaking, that if $\lambda_{(k)}(G_p) = \xi_{(k)}(G_p)$, then this optimality is inherited by $G_m * G_p$ provided that the number of vertices of $G_p$ is large enough.

**Corollary 8.** Let $G_m$ and $G_p$ be two connected graphs, and let $k$ be an integer, $2 \leq k \leq \lfloor |V(G_p)|/2 \rfloor$. Suppose that $\delta(G_m) \geq k - 1$, $\delta(G_p) \geq \Delta(G_m) + 2k - 3$, $\lambda(G_m)|V(G_p)| \geq k(\Delta(G_p) + \delta(G_p) - 4k + 5)$, and also that $G_p$ is $\lambda_{(k)}$-optimal, that is, $\lambda_{(k)}$-connected with $\lambda_{(k)}(G_p) = \xi_{(k)}(G_p)$. Then the graph $G_m * G_p$ is also $\lambda_{(k)}$-optimal, that is, $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$.

**Proof.** From Remarks 1 and 2, it follows that:

$$\lambda(G_m)|V(G_p)| \geq k(\Delta(G_p) + \delta(G_p) - 4k + 5)$$

$$= k(\Delta(G_p) - 2(k - 1) + k(\delta(G_p) - 2k + 3)$$

$$\geq \xi_{(k)}(G_p) + k\Delta(G_m)$$

$$\geq \xi_{(k)}(G_m * G_p);$$

$$\delta(G_m) - k + 3) \lambda_{(k)}(G_p) \geq 2\xi_{(k)}(G_p) \geq \xi_{(k)}(G_p) + k\delta(G_p) - k(k - 1)$$

$$\geq \xi_{(k)}(G_p) + k\delta(G_m) + k(k - 2)$$

$$\geq \xi_{(k)}(G_m * G_p);$$

$$\delta(G_m)(\delta(G_p) - k + 3) + \lambda_{(k)}(G_p) \geq k\delta(G_p) + \xi_{(k)}(G_p) \geq \xi_{(k)}(G_m * G_p).$$

Hence $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$ follows from Theorem 7.

With the following result we still guarantee $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$ even though $G_p$ need not be $\lambda_{(k)}$-optimal. To achieve such a goal, constraints on minimum and maximum degrees of $G_m$ are required. Note that the upper bound on $\Delta(G_m)$ is ensured to be larger than $\delta(G_m)$ because $\delta(G_m) \geq k + 2$.

**Corollary 9.** Let $G_m$ and $G_p$ be two connected graphs, and let $k$ be an integer, $2 \leq k \leq \lfloor |V(G_p)|/2 \rfloor$. Suppose that $\delta(G_m) \geq k + 2$, $\Delta(G_m) \leq \delta(G_m)(\delta(G_m) - k) - k - 4$, $\delta(G_p) \geq \Delta(G_m) + 2k - 3$, and $\lambda(G_m)|V(G_p)| \geq k(\Delta(G_p) + \delta(G_p) - 4k + 5)$. Suppose also that $G_p$ is $\lambda_{(k)}$-connected with $\lambda_{(k)}(G_p) \geq \xi_{(k)}(G_p) - k(\Delta(G_m) - \delta(G_m) + k)$. Then the graph $G_m * G_p$ is also $\lambda_{(k)}$-optimal, that is, $\lambda_{(k)}(G_m * G_p) = \xi_{(k)}(G_m * G_p)$. 
Proof. We again compute the first three contributions in the lower bound for \( \lambda_{(k)}(G_m \ast G_p) \) given by Theorem 7. The inequality \( \lambda(G_m)|V(G_p)| \geq \xi_{(k)}(G_m \ast G_p) \) is proved as in Corollary 8. For the other two terms:

\[
\delta(G_m)(\delta(G_p) - k + 3) + \lambda_{(k)}(G_p) \geq (k + 2)(\Delta(G_m) + k) + \xi_{(k)}(G_p) - k(\Delta(G_m) - \delta(G_m) + k)
\]

\[
= \xi_{(k)}(G_p) + k\delta(G_m) + 2(\Delta(G_m) + k)
\]

\[
\geq \xi_{(k)}(G_p) + k\delta(G_m)
\]

\[
\geq \xi_{(k)}(G_m \ast G_p),
\]

as a consequence of the hypothesis on \( \lambda_{(k)}(G_p) \) and by Remark 1. Also by Remark 2, \( \xi_{(k)}(G_p) \geq k\delta(G_p) - k(k - 1) \geq k(\Delta(G_p) + k - 2) \) and using the hypothesis on the upper bound for \( \Delta(G_m) \) we can write:

\[
(\delta(G_m) - k + 3)\lambda_{(k)}(G_p) \geq (\delta(G_m) - k + 3)(\xi_{(k)}(G_p) - k(\Delta(G_m) - \delta(G_m) + k))
\]

\[
\geq \xi_{(k)}(G_p) + (\delta(G_m) - k + 2)k(\Delta(G_m) + k - 2) - (\delta(G_m) - k + 3)k(\Delta(G_m) - \delta(G_m) + k)
\]

\[
= \xi_{(k)}(G_p) + k\left(\delta(G_m)(\delta(G_m) - k + 1) - (\Delta(G_m) - k + 4)\right)
\]

\[
\geq \xi_{(k)}(G_p) + k\delta(G_m) \geq \xi_{(k)}(G_m \ast G_p).
\]

Therefore, from Theorem 7 it follows that \( \lambda_{(k)}(G_m \ast G_p) = \xi_{(k)}(G_m \ast G_p). \) \( \square \)

When both \( G_m \) and \( G_p \) are regular graphs, we can ensure that \( G_m \ast G_p \) is super-\( \lambda_{(k)} \) provided that \( \lambda_{(k+1)}(G_p) \) is large enough.

Corollary 10. Let \( k \geq 1 \) be an integer. Let \( G_m \) be an \( r \)-regular connected graph and let \( G_p \) be an \( s \)-regular connected graph, with \( r \geq 2k - 1, s \geq r + 2k - 1. \) Suppose that \( \lambda(G_m)|V(G_p)| \geq 2ks(k - 2), \) and also that \( G_p \) is \( \lambda_{(k+1)} \)-connected with \( \lambda_{(k+1)}(G_p) \geq 2(s - k). \) Then the \((r+s)\)-regular graph \( G_m \ast G_p \) is super-\( \lambda_{(k)} \).

Proof. By Remark 2 and Lemma 2

\[
\xi_{(k)}(G_m \ast G_p) \leq k(r + s) - 2(k - 1) \leq 2k(s - k) + 1,
\]

having used the facts that \( r \leq s - 2k + 1 \) and \( k \geq 1. \) Then

\[
\lambda(G_m)|V(G_p)| \geq 2ks(k - 2) + 2 > \xi_{(k)}(G_m \ast G_p).
\]

Let us show next three other inequalities involving \( \xi_{(k)}(G_m \ast G_p) \) (note that \( s - k \geq 1 \)):

\[
(\delta(G_m) - (k + 1) + 3)\lambda_{(k+1)}(G_p) \geq (k + 1)\lambda_{(k+1)}(G_p) \geq (k + 1)2(s - k)
\]

\[
\geq 2ks(k - s) + 1 \geq \xi_{(k)}(G_m \ast G_p);
\]

\[
\delta(G_p)(\delta(G_m) - (k + 1) + 3) + \lambda_{(k+1)}(G_p) = r(s - k + 2) + \lambda_{(k+1)}(G_p)
\]

\[
\geq (2k - 1)(s - k + 2) + 2(s - k) \geq 2ks - k + 1 \geq \xi_{(k)}(G_m \ast G_p);
\]

\[
\xi_{(k+1)}(G_m \ast G_p) \geq \xi_{(k)}(G_m \ast G_p) + \delta(G_m \ast G_p) - 2k
\]

\[
= \xi_{(k)}(G_m \ast G_p) + r + s - 2k
\]

\[
\geq \xi_{(k)}(G_m \ast G_p) + 4k - 3
\]

\[
\geq \xi_{(k)}(G_m \ast G_p).
\]

Therefore, from Theorem 7 it turns out that

\[
\lambda_{(k+1)}(G_m \ast G_p) > \xi_{(k)}(G_m \ast G_p),
\]

and thus \( G_m \ast G_p \) is super-\( \lambda_{(k)} \) by means of Theorem 4. \( \square \)

Consider next the complete graph \( K_n \) on \( n \geq 2 \) vertices. For all integer \( k \) such that \( 1 \leq k \leq n \) and for each set \( X \subseteq V(K_n) \) of cardinality \( k \), the number of edges connecting \( X \) to \( V(K_n) \setminus X \) is \( |\omega(X)| = k(n - k) \), hence \( \xi_{(k)}(K_n) = k(n - k) \). Moreover, \( k(n - k) \) is a strictly increasing function of \( k \) when \( 1 \leq k \leq \lfloor n/2 \rfloor \), and then

\[
\xi_{(1)}(K_n) < \xi_{(2)}(K_n) < \cdots < \xi_{(\lfloor n/2 \rfloor)}(K_n).
\]

As a consequence, the following remark holds.

Remark 3. Let \( n \geq 2 \) and \( k \) be integers, with \( 1 \leq k \leq n \). Then \( \xi_{(k)}(K_n) = k(n - k) \) holds for the complete graph \( K_n \). Furthermore, for all \( k \) such that \( 1 \leq k \leq \lfloor n/2 \rfloor \), \( K_n \) is \( \lambda_{(k)} \)-connected and \( \lambda_{(k)} \)-optimal; that is,

\[
\lambda_{(k)}(K_n) = \xi_{(k)}(K_n) = k(n - k).
\]
Corollary 11. For all integers $k \geq 1$, $\ell \geq 2k + 1$ and $n \geq \ell + 2k - 1$ the graph $K_\ell \ast K_n$ is super-$\lambda_{(j)}$ for every $1 \leq j \leq k$.

Proof. Let us first show that the requirements of Corollary 10 are satisfied for $G_m \simeq K_\ell$ and $G_p \simeq K_n$:

$$
\lambda(G_m) | V(G_p)| = (\ell - 1)n \geq 2k(n - 1) + (\ell - 1) > 2k(n - 1 - k) + 1;
$$

$$
\lambda_{(k+1)}(G_p) = (k + 1)(n - (k + 1)) \geq 2(n - k - 1),
$$

the last equality by Remark 3. Therefore $K_\ell \ast K_n$ is super-$\lambda_{(k)}$. Further, as this implies that $\lambda_{(k)}(K_\ell \ast K_n) = \xi_{(k)}(K_\ell \ast K_n)$, from Theorem 4 it follows that $K_\ell \ast K_n$ is also super-$\lambda_{(j)}$ for every $1 \leq j \leq k$ when $k \geq 2$, ending the proof. \hfill $\Box$

We just have studied the graph $K_\ell \ast K_n$ when $\ell \geq 3$. With the following result we can still approach the $k$-restricted edge connectivity of the graph $K_\ell \ast K_n$ when $\ell = 2$.

Theorem 12. Let $n$ and $k$ be integers such that $2 \leq k < n$. Then $K_2 \square K_n$ is $\lambda_{(k)}$-connected with

$$
\lambda_{(k)}(K_2 \square K_n) = n < k(n - k + 1) = \xi_{(k)}(K_2 \square K_n);
$$

hence $K_2 \square K_n$ is not $\lambda_{(k)}$-optimal.

Proof. Let us denote by $G_1$ and $G_2$ the two disjoint copies of $K_n$ in $G = K_2 \square K_n$, and call $M$ to the set of edges connecting $G_1$ to $G_2$. Observe that $M$ is a $k$-restricted edge-cut of $G$, hence $G$ is $\lambda_{(k)}$-connected.

Let us first compute $\xi_{(k)}(K_2 \square K_n)$. To this end, for each $2 \leq k < n$ we consider two sets $X_1 \subseteq V(G_1)$, $X_2 \subseteq V(G_2)$, such that $|X_1| = j$, $|X_2| = k - j$ ($0 \leq j < k$), and $G[X_1 \cup X_2]$ is connected. Without loss of generality assume $j \leq k - j$. Then, taking into account Remark 3,

$$
|\omega_C(X_1 \cup X_2)| = |\omega_C(X_1)| + |\omega_C(X_2)| + |X_1| + |X_2| - 2||X_1, X_2||
$$

$$
\geq \xi_{(j)}(G_1) + \xi_{(k-j)}(G_2) + j + (k - j) - 2j
$$

$$
= j(n - j - 1) + (k - j)(n + j - k + 1)
$$

$$
\geq k(n - k + 1),
$$

because we can check that the minimum value of $j(n - j - 1) + (k - j)(n + j - k + 1)$ is attained when $j = 0(X_1 = \emptyset)$; hence

$$
\xi_{(k)}(G) = \xi_{(k)}(G_2) + |X_2| = k(n - k) + k = k(n - k + 1).
$$

With respect to the value of $\lambda_{(k)}(G)$ note first that $\lambda_{(k)}(G) \leq |M| = n < k(n - k + 1) = \xi_{(k)}(G)$, the strict inequality because $2 \leq k < n$. To prove the equality $\lambda_{(k)}(G) = n$ it suffices to see that $\lambda_{(k)}(G) \geq n$. Consider a minimum $k$-restricted edge-cut of $G$, $W \subseteq E(G)$, and suppose $W \neq M$. In this case $G - W$ consists of two connected components, $H, H^*$, at least one of them sharing with one of $V(G_1)$ or $V(G_2)$ a number of $1 \leq j \leq n - 1$ vertices; let us assume without loss of generality that $1 \leq j = |V(H) \cap V(G_1)| \leq n - 1$. If $2 \leq j \leq n - 2$ then

$$
\lambda_{(k)}(G) = |W| \geq |V(H) \cap V(G_1)| \cdot |V(H^*) \cap V(G_1)| = j(n - j) \geq j + (n - j) = n,
$$

and we are done. When $j = 1(|V(H^*) \cap V(G_1)| = n - 1)$ necessarily $|V(H) \cap V(G_2)| \geq k - 1 \geq 1$, because $|V(H)| \geq k \geq 2$. If $|V(H) \cap V(G_2)| \geq 2$ at least one edge in $M$ connects $H$ and $H^*$, then

$$
\lambda_{(k)}(G) = |W| \geq (n - 1) + 1 = n;
$$

and if $|V(H) \cap V(G_2)| = 1$ (hence $k = 2$),

$$
\lambda_{(k)}(G) = |W| \geq |V(H) \cap V(G_1)| \cdot |V(H^*) \cap V(G_1)| + |V(H) \cap V(G_2)| \cdot |V(H^*) \cap V(G_2)|
$$

$$
= 2(n - 1) \geq n.
$$

The case $j = n - 1$ is treated similarly, by interchanging the roles of $H$ and $H^*$. \hfill $\Box$

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