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# CHARACTERIZATION OF THE GRAPHS WITH BOXICITY $\leqslant 2$

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The intersection graph of a family $\mathfrak{M}$ of sets has the sets in $\mathfrak{M}$ as vertices and an edge between two sets iff they have nonempty intersection. Following Roberts [4] the boxicity $b(G)$ of a graph $G$ is defined as the smallest $d$ such that $G$ is the intersection graph of boxes in Euclidean $d$-space, i.e. parallelepipeds with edges parallel to the coordinate axes. In this paper we will give a combinatorial characterization of the graphs with $b(G) \leqslant 2$, called boxicity 2-graphs, by means of the arrangement of zeros and ones in special matrices attached to the graph.

## 1. Introduction and definitions

There are many results about the graphs with $b(G) \leqslant 1$, the so called interval graphs. See Golumbic [2] for a recent survey. The characterization of interval graphs of Fulkerson and Gross in [1] is of particular interest in this connection, because this is in a sense a special case of our characterization of the boxicity 2-graphs. Note, that the class of boxicity 2-graphs contains the class of interval graphs. In the following the set of vertices of a graph $G$ is called $V(G)$, the set of edges $E(G)$. A nonempty subset $M$ of $V(G)$ is called a clique of $G$, if $G$ has only one vertex or the following conditions hold:

$$v_1,\, v_2 \in M,\ v_1 \neq v_2 \Rightarrow v_1 v_2 \in E(G).$$

$$v_3 \in V(G) \setminus M \Rightarrow \exists v_4 \in M : v_3 v_4 \notin E(G)$$

Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and cliques $\{M_1, \ldots, M_m\}$. To $G$ is associated the $n \times n$-adjacency matrix $A(v_1, \ldots, v_n) = (a_{ik})$ with

$$a_{ik} = \begin{cases} 1 & \text{if } v_i v_k \in E(G) \\ 0 & \text{else,} \end{cases}$$

and the $m \times n$-incidence matrix of cliques and vertices $C(M_1, \ldots, M_n; v_1, \ldots, v_n) = (c_{jk})$ with

$$c_{jk} = \begin{cases} 1 & \text{if } v_k \in M_j \\ 0 & \text{else,} \end{cases}$$

called $C$–$V$-matrix of $G$. Both are 0–1-matrices and depend on the numbering of

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vertices and cliques. A 0–1-matrix is said to have the consecutive-ones-property, if there are no zeros between two ones in each column.

**Theorem** [1]. *A graph G is an interval-graph, iff there is a C–V-matrix of G which has the consecutive-ones-property.*

To give a similar characterization of boxicity 2-graphs by means of the consecutive-ones-property is not as easy as for interval-graphs. It is not enough to look at one matrix defined by the graph: Matrices induced by a $C$–$V$-matrix of the graph have to fulfil the condition.

## 2. Characterization of the boxicity 2-graphs

We start with some definitions. Let $G$ be again a graph with vertices $v_1, \ldots, v_n$ and cliques $M_1, \ldots, M_m$. For each index $h$ we consider the set of vertices

$$J_h := \{v_k \mid k = h \text{ or } (k > h \text{ and } a_{ik} = 1 \text{ for some } i \leq h)\}$$

and we define an induced $C$–$V$-matrix $C^{(h)} := (c_{jk}^{(h)})$ by

$$c_{jk}^{(h)} := \begin{cases} c_{jk} & \text{if } v_k \in J_h \text{ and for some } v_r \in J_h \; c_{jr} = 1 \text{ and } a_{kr} = 0 \\ * & \text{else,} \end{cases}$$

where "$*$" is used as an empty symbol filling those places $(j, k)$ of the matrix which are not of interest. Instead of suppressing those entries of the $C$–$V$-matrix $C$ of $G$ we use the asterisk in order to obtain again an $m \times n$-matrix. 0 and 1 appear in $C^{(h)}$ only in columns belonging to vertices of $J_h$ and in those columns we have $c_{jk}^{(h)} = 1$ iff $c_{jk} = 1$. Because of the insignificance of the asterisks we shall consider $C^{(h)}$ again as a 0–1-matrix.

*Example.* The graph shown in *Fig. 1* (in which we identify each vertex with its number) has the cliques $M_1 = \{1, 4\}$, $M_2 = \{2, 4\}$, $M_3 = \{4, 6\}$, $M_4 = \{1, 3\}$, $M_5 = \{2, 3\}$, $M_6 = \{2, 5, 7\}$, $M_7 = \{1, 5\}$, $M_8 = \{5, 6, 7\}$. We use this numbering of the cliques in view of the subsequent presentation of our theorem. According

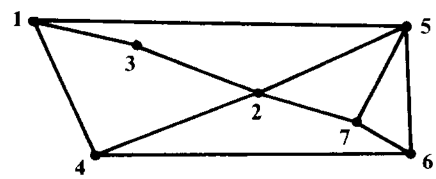


Fig. 1.

to this numbering we have

$$A(G)=\begin{pmatrix}0&0&1&1&1&0&0\\0&0&1&1&1&0&1\\1&1&0&0&0&0&0\\1&1&0&0&0&1&0\\1&1&0&0&0&1&1\\0&0&0&1&1&0&1\\0&1&0&0&1&1&0\end{pmatrix}\qquad C(G)=\begin{pmatrix}1&0&0&1&0&0&0\\0&1&0&1&0&0&0\\0&0&0&1&0&1&0\\1&0&1&0&0&0&0\\0&1&1&0&0&0&0\\0&1&0&0&1&0&1\\1&0&0&0&1&0&0\\0&0&0&0&1&1&1\end{pmatrix}$$

Now we get $J_1=\{1,3,4,5\}$, $J_2=\{2,3,4,5,7\}$, $J_3=\{3,4,5,7\}$, $J_4=\{4,5,6,7\}$, $J_5=\{5,6,7\}$, $J_6=\{6,7\}$, $J_7=\{7\}$, and induced $C$–$V$-matrices are for instance

$$C^{(1)}=\begin{pmatrix}1&*&0&1&0&*&*\\ *&*&0&1&0&*&*\\ *&*&0&1&0&*&*\\ 1&*&1&0&0&*&*\\ *&*&1&0&0&*&*\\ *&*&0&0&1&*&*\\ 1&*&0&0&1&*&*\\ *&*&0&0&1&*&*\end{pmatrix}\qquad C^{(2)}=\begin{pmatrix}*&*&0&1&0&*&0\\ *&1&0&1&0&*&0\\ *&*&0&1&0&*&0\\ *&*&1&0&0&*&0\\ *&1&1&0&0&*&0\\ *&1&0&0&1&*&1\\ *&*&0&0&1&*&*\\ *&*&0&0&1&*&1\end{pmatrix}$$

$$C^{(4)}=\begin{pmatrix}*&*&*&1&0&*&0\\ *&*&*&1&0&*&0\\ *&*&*&1&0&1&0\\ *&*&*&*&*&*&*\\ *&*&*&*&*&*&*\\ *&*&*&0&1&*&1\\ *&*&*&0&1&*&*\\ *&*&*&0&1&1&1\end{pmatrix}$$

Considering $C^{(h)}$ as a 0–1-matrix we say that $C^{(h)}$ has the consecutive-ones-property if there is no zero between ones in the columns (i.e. asterisks are skipped). Then we have:

**Theorem.** *A graph $G$ (with $n$ vertices) is a boxicity 2-graph if and only if there exists a pair of an adjacency matrix $A(G)$ and a $C$–$V$-matrix $C(G)$ such that for each $h\in\{1,\ldots,n\}$ the induced $C$–$V$-matrix $C^{(h)}$ has the consecutive-ones-property.*

First we remark that if the $C$–$V$-matrix of $G$ itself has the consecutive-ones-property, then each induced $C$–$V$-matrix has this property. According to the characterization of interval graphs by Fulkerson–Gross this corresponds to the obvious fact that each interval graph is a boxicity 2-graph.

Secondly we note that one has to examine the consecutive-ones-property only for those induced $C$–$V$-matrices $C^{(h)}$, for which the corresponding set $J_h$ is maximal in view of inclusion. Concerning the previous example this means that one has to look only at $C^{(1)}$, $C^{(2)}$ and $C^{(4)}$ since $J_3 \subseteq J_2$ and $J_7 \subseteq J_6 \subseteq J_5 \subseteq J_4$. As these matrices (which are listed above) have the desired property, the graph of *Fig. 1* is a boxicity 2-graph, according to our theorem.

**Proof of the theorem.** To prove the necessity let $G$ be a boxicity 2-graph with $V(G) = \{v_1, \ldots, v_n\}$. Then there exists a family $\mathfrak{B} = \{B_1, \ldots, B_n\}$ of boxes $B_i = \{\binom{x}{y} \mid a_i \leq x \leq b_i, c_i \leq y \leq d_i\}$ in the plane, where $B_i$ corresponds to $v_i$, such that $B_i$ and $B_k$ $(i \neq k)$ intersect iff $v_i v_k \in E(G)$. We may assume that all boxes have nonempty interior and that intersecting boxes have interior points in common. Further we may assume that $i \neq k$ implies $c_i \neq c_k$ and that the numbering is chosen such that $c_1 > c_2 > \cdots > c_n$.

The adjacency matrix is set up with respect to this numbering. Now consider the stripes $S_i = \{\binom{x}{y} \mid c_i \leq y < c_{i-1}\}$ for $i = 1, \ldots, n$ (where $c_0 := d_1$) and let $\mathfrak{B}_i$ denote the subfamily of $\mathfrak{B}$ meeting stripe $S_i$. Two boxes of $\mathfrak{B}_i$ intersect, iff they intersect in stripe $S_i$. So the intersection graph of $\mathfrak{B}_i$ is the subgraph $G_i$ of $G$, spanned by the corresponding vertices of $\mathfrak{B}_i$. It is clear, that $G_i$ (if $\mathfrak{B}_i$ nonempty) is an interval graph, the representation by intervals being given by the intersection of the boxes of $\mathfrak{B}_i$ with the line $y = c_i$. What about the sets $J_h$? $J_h$ consists of $v_h$ and all vertices $v_k$ with $k > h$ for which an index $i \leq h$ exists with $v_i v_k \in E(G)$. Because of the special ordering of the vertices this means that the corresponding boxes meet stripe $S_h$, i.e. $J_h \subseteq V(G_h)$.

To set up a $C$–$V$-matrix of $G$ we define an ordering of the cliques of $G$ as follows. First note that $\mathfrak{B}$ has (as any family of boxes in $\mathbb{R}^d$) the Helly-1-property: If $\mathfrak{B}' \subseteq \mathfrak{B}$ and $B_i \cap B_k \neq \emptyset$ for all pairs $B_i, B_k \in \mathfrak{B}'$, then $\bigcap_{B \in \mathfrak{B}'} B \neq \emptyset$.

So every clique of $G$ is represented by a box in $\mathbb{R}^2$, the intersection of the corresponding members of $\mathfrak{B}$. By assumption all these boxes have nonempty interior. So we may choose a point in each of these boxes in such a way that no two points have the same $x$-coordinate. Then we take the linear ordering of the $x$-coordinates of those points from left to right as the ordering of the cliques of $G$ to set up the $C$–$V$-matrix of $G$.

Now look at any column $k$ of a matrix $C^{(h)}$, where $v_k \in J_h$. The ones in the column are just the ones in the same column of $C$, and they indicate the cliques involving $v_k$. By definition $c_{jk}^{(h)} = 0$ if $c_{jk} = 0$ and if there exists a $v_r \in J_h$ such that $v_r \in M_j$ but $v_r v_k \notin E(G)$. In view of the representation this means that the box corresponding to the clique $M_j$ does not meet the box $B_k$, but it is contained in a box $B_r$ with $v_r \in J_h$, which is disjoint to $B_k$. Now remember that boxes corresponding to vertices of $J_h$ belong to the subfamily $\mathfrak{B}_h$ and realize that disjoint members of $\mathfrak{B}_h$ are separated by vertical lines. So $B_r$ is separated from $B_k$ by a vertical line and this says that $c_{jk}^{(h)} = 0$ cannot separate two ones in the $k$th column of $C^{(h)}$.

To prove the sufficiency let a graph $G$ be given with an adjacency-matrix $A(G)$ and a $C$–$V$-matrix $C(G)$ having the properties required in the theorem. We construct a family of boxes in $\mathbb{R}^2$, having $G$ as its intersection graph, so $G$ is a boxicity 2-graph.

*Construction.* In an $x$–$y$-coordinate-system in $\mathbb{R}^2$ we relate the $j$th clique $M_j$ to the line $x = j$ $(j \in \{1, \ldots, m\})$. Also we relate $J_h$ $(h \in \{1, \ldots, n\})$ to the line $y = n - h$. We call such lines now "levels". If $v_k \in M_j$ and $v_k \in J_h$ we label the point $(j, n - h)$ with $v_k$. Note that some points may have more than one label. Let $P_k$ be the set of points with label $v_k$ and denote by $B_k$ the convex hull conv $P_k$ of the set $P_k$. We claim that $M = \{B_k \mid k = 1, \ldots, n\}$ is a representation of $G$ by boxes, i.e. $B_k$ $(k \in \{1, \ldots, n\}$ corresponds to $v_k$ and for each pair $i \neq k$ the equivalence $B_i \cap B_k \neq \emptyset \Leftrightarrow a_{ik} = 1$ holds. In order to prove this consider the sets $J_h$ containing a given vertex $v_k$. By definition of $J_h$ we have $v_k \in J_k$, but $v_k \notin J_h$ for $h > k$, and if the first 1 in the $k$th row of the adjacency matrix occurs in the column $r \leq k$, then $v_k$ belongs exactly to $J_r, J_{r+1}, \ldots, J_k$.

This implies: Whenever a line $x = j$ contains points labelled $v_k$ at all, then exactly the points of level $n - k$, $n - k + 1, \ldots, n - r$ are labelled $v_k$, with $r$ depending only on $k$. From this it is obvious, that the sets $B_k$ indeed are boxes, and that $B_i \cap B_k \neq \emptyset$ requires that $P_i$ and $P_k$ have elements at the same level.

Now consider any element $a_{ik}$ $(i < k)$ of the adjacency matrix. If $a_{ik} = 1$, then there is (at least one) clique, say $M_j$, which contains $v_i$ and $v_k$. Since $v_i \in J_i$ and $v_k \in J_i$ (because of $a_{ik} = 1$), the point $(j, n - i)$ is labelled $v_i$ and $v_k$, i.e. the boxes $B_i$ and $B_k$ have nonvoid intersection. Now let $a_{ik} = 0$. If $v_i$ and $v_k$ do not appear together in some $J_h$, i.e. no pair of points of $P_i$ and $P_k$ has the same level, then $B_i \cap B_k = \emptyset$, as noted above. So we assume $\{v_i, v_k\} \subseteq J_h$ for some $h$. We have necessarily $h \leq i$, which implies $\{v_i, v_k\} \subseteq J_i$. We claim that $P_i$ and $P_k$ and so $B_i$ and $B_k$ are strictly separated by a vertical line. Otherwise we would have at level $n - i$ a point labelled $v_k$ between two points labelled $v_i$ or a point labelled $v_i$ between two points labelled $v_k$. The first would give a contradiction to the consecutive-ones-property of $C^{(i)}$ in column $i$, the latter a contradiction to the consecutive-ones-property of $C^{(i)}$ in column $k$. □

For illustration we apply the construction given in the second part of the proof to our example. In *Fig. 2* the intersections of the levels and the lines $y = j$ are replaced by small rectangles, and the numbers in the rectangles are their labels.

**Remarks.**

1. It seems to be rather hopeless to look for a fast algorithm for deciding whether a given graph possesses a pair of matrices $(A(G), C(G))$ with the desired

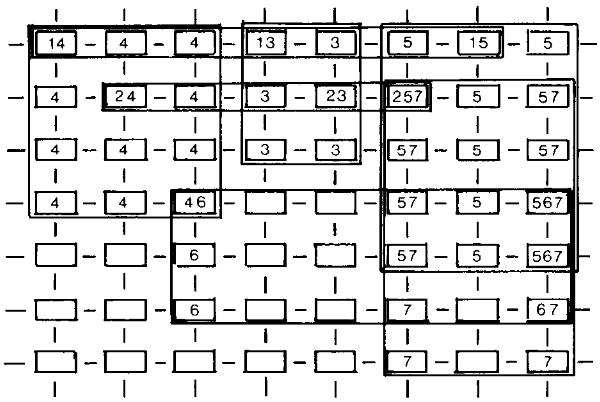


Fig. 2.

properties or not. So it would be of interest to have some simple necessary conditions for 2-boxicity of graphs working with any numbering of vertices and cliques for ruling out most of the graphs.

2. Another open problem is whether there exists a characterization of boxicity 2-graphs by means of forbidden subgraphs corresponding to the characterization of interval graphs given by Lekkerkerker–Boland [3].

## References

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