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Discrete Mathematics 308 (2008) 3574–3585

DISCRETE MATHEMATICS

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Cubes and orientability

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Received 14 February 2007; received in revised form 5 July 2007; accepted 12 July 2007
Available online 13 November 2007

Abstract

We define and study a new class of matroids: cubic matroids. Cubic matroids include, as a particular case, all affine cubes over an arbitrary field. There is only one known orientable cubic matroid: the real affine cube. The main results establish as an invariant of orientable cubic matroids the structure of the subset of acyclic orientations with LV-face lattice isomorphic to the face lattice of the real cube or, equivalently, with the same signed circuits of length 4 as the real cube.

MSC: 52C40; 05B35

Keywords: Cube; Matroid; Orientability of matroids; Oriented matroid

1. Introduction

In this paper we define and study a new class of matroids: cubic matroids. Cubic matroids are matroids over a 2^n element set, standardly over $C^n = \{0, 1\}^n$, that include all affine cubes, i.e. all matroids of affine dependencies of C^n over an arbitrary field F.

The main results concern orientability of matroids in this class. We only know (up to isomorphism) one orientable cubic matroid: the real affine cube. Moreover, we only know one class of orientations of this matroid. The problem we are considering is the following:

Problem 1. *Is the real affine cube the unique orientable cubic matroid*?

A positive answer to Problem 1 leads to a purely combinatorial characterization of the real affine (and linear) dependencies of C^n meaning, in particular, that in the class of cubic matroids orientability implies parallelism and symmetry.

Problem 1 is closely related with the following conjecture of Las Vergnas:

Las Vergnas Cube Conjecture (*Las Vergnas et al. [10]*). The real affine cube has a unique class of orientations.

This conjecture says that somehow the orientation of the real affine cube is encoded in the underlying matroid. If both problems have a positive answer then it seems likely that they will be solved together.

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doi:10.1016/j.disc.2007.07.043

Problem 1 and Las Vergnas Conjecture have a positive answer for $n \leqslant 7$ (in [13] we sketch a proof of both). Las Vergnas Conjecture was first verified computationally by Bokowski et al. [4].

Generalizing those proofs to higher dimensions involves the possibility of an explicit description of the real cube matroid (not necessarily purely combinatorial) which is a difficult theoretical and computational problem. For small dimensions exhaustive enumeration of the hyperplanes of the real cube has been carried out computationally up till dimension 8 by Aischolzer and Aurenhammer [1].

A recursive algorithm to generate hyperplanes of the real affine cube is described in [12].

For relevant information on the asymptotic behaviour of the real linear (and affine) dependencies of random sets of (0, 1) vectors we refer the reader to the papers of Komlós [8], Odlyzco [11], Khan et al. [7]. We mention, in particular, that the probability of a random (0, 1) matrix being singular is asymptotically zero [8,7]. It is conjectured [11,7] that this probability is dominated by the probability that the rows/columns of the matrix contain a circuit of rank 3 (length 4).

Our results are presented in three sections. In the next section we define and establish the main properties of cubic matroids. As a main result of this section we refer to Proposition 6 where we prove that the operation of pulling a vertex onto a hyperplane leaves the class of cubic matroids invariant. We use this operation to construct examples (see Example 1) of cubic matroids which are not representable (over any field).

The operation of pulling a vertex onto a hyperplane does not, in general, preserve orientability. A characterization of pairs hyperplane/element for which orientability is preserved under this operation was given by Fukuda and Tamura [6]. The existence of such a pair in the real affine cube would imply a negative answer to both Problem 1 and Las Vergnas Conjecture as is briefly pointed out in the Final remarks.

The main results of the paper concern orientability of cubic matroids and are obtained in Section 3. Theorems 1 and 2 establish an invariant of orientable cubic matroids. More precisely they say that given an orientable cubic matroid M of rank $(n+1)$, *every orientation class of M has a subset of exactly $n+1$ acyclic orientations with LV-face lattice isomorphic to the face lattice of the real n-dimensional cube and whose structure is independent of the orientation class*; in other terms, *every arrangement of 2^n pseudohyperplanes representing an oriented cubic matroid has exactly $2(n+1)$ topes isomorphic to the n-cross polytope and their relative position within the hyperplane arrangement is independent of the orientation class*.

A direct consequence of Theorems 1 and 2 is Corollary 1 which leads to a reformulation of Problem 1 as a reconstruction problem: *Can the real affine cube be reconstructed from its rank 3 circuits and orientability*?

Note that a positive answer to this question seems implicit in the above mentioned conjecture [7,11] on the probability of a (0, 1) matrix being singular.

The last result of this section, Theorem 3, is a direct application of Theorems 1 and 2 to obtain non-orientability results. It says that affine cubes of rank $\geqslant p+2$ over a field of prime characteristic p are not orientable. This result, implicit in the original paper [3] of Bland and Las Vergnas, is obtained here with a very short proof (see also (1)) of the Final remarks.

We assume the reader is acquainted with matroids and oriented matroids. As general references on matroids we suggest [14,15]. As a general reference on oriented matroids the reader may consult [2].

1.1. Preliminaries and notation

Cubic matroids are matroids over a 2^n element set, standardly defined as matroids over $C^n := \{0, 1\}^n$.

Given a matroid $M = M(E)$ over a set E we denote by $\mathscr{H}(M)$, $\mathscr{C}(M)$, $\mathscr{C}^{\perp}(M)$, or simply $\mathscr{H}, \mathscr{C}, \mathscr{C}^{\perp}$, its families of, respectively: hyperplanes, circuits and cocircuits. The rank and closure of a subset A of E are denoted: $r_M(A)$ and $\mathrm{cl}_M(A)$.

Orientations of a matroid M are described in terms of signatures of the circuits or cocircuits of M. To denote the signed circuits, resp. signed cocircuits of an orientation $\mathscr{M}$ of M (i.e. of an oriented matroid $\mathscr{M}$ whose underlying matroid is M) we use the same letters $\mathscr{C}, \mathscr{C}^{\perp}$. The families of circuits and cocircuits of the underlying matroid are then denoted by $\underline{\mathscr{C}}$ and $\underline{\mathscr{C}}^{\perp}$. Recall that to each circuit, $\underline{X} \in \underline{\mathscr{C}}$, of the underlying matroid corresponds a unique pair of opposite signed circuits $\pm X \in \mathscr{C}$ of the oriented matroid.

In what follows we will work with subsets and signed subsets of C^n as well as with elements of C^n which we identify with "signed" subsets of $[n] := \{1, \ldots, n\}$. We define the terminology and notation.

Notation: Subsets and signed subsets of C^n will always be denoted by capital Latin letters.

A signed subset $X=(X^+,X^-)$ of C^n is an *ordered pair* of disjoint subsets of C^n. The support of the signed set X is the set $\underline{X}:=X^+\cup X^-$.

Given $A\subseteq C^n$ we denote by ${}_{-A}X$ the signed subset obtained from X *reversing signs on* A, i.e. the signed subset ${}_{-A}X=(X^+\backslash A\cup(X^-\cap A), X^-\backslash A\cup(X^+\cap A))$. The signed subset ${}_{-C^n}X=(X^-,X^+)$ is *the opposite of* $X=(X^+,X^-)$ and is denoted $-X$. Given a family of signed subsets $\mathscr{V}$ of C^n, ${}_{-A}\mathscr{V}:=\{{}_{-A}X : X\in\mathscr{X}\}$.

Subsets of $[n]$ will always be denoted by Greek letters, $\alpha,\beta,\ldots$. Given two sets $\alpha,\beta\subseteq[n]$, $\alpha\Delta\beta$ denotes its symmetric difference, i.e. $\alpha\Delta\beta:=\alpha\backslash\beta\cup\beta\backslash\alpha$.

The support of an element $\mathbf{v}\in C^n$ is the set $\underline{\mathbf{v}}:=\{i\in[n]:\mathbf{v}(i)=1\}$. Given a subset $\alpha\subseteq[n]$, $\mathbf{v}(\alpha)$ denotes the restriction of $\mathbf{v}$ to α, and ${}_{\alpha}\mathbf{v}$ denotes the element of C^n obtained from $\mathbf{v}$ interchanging $1\leftrightarrow 0$ on the entries indexed in α, i.e. ${}_{\alpha}\mathbf{v}(i)=1-\mathbf{v}(i)$ *for* $i\in\alpha$ *and* ${}_{\alpha}\mathbf{v}(i)=\mathbf{v}(i)$ *for* $i\notin\alpha$. Remark that the support of ${}_{\alpha}\mathbf{v}$ is $\underline{{}_{\alpha}\mathbf{v}}=\underline{v}\Delta\alpha$.

The opposite of an element $\mathbf{v}$ is $\mathbf{v}^*={}_{[n]}\mathbf{v}$. The supports of $\mathbf{v}^*$ and $\mathbf{v}$ are complementary subsets: $\underline{\mathbf{v}^*}=[n]\backslash\underline{\mathbf{v}}$. Given $A\subseteq C^n$, $A^*:=\{\mathbf{v}^*\in C^n:\ \mathbf{v}\in A\}$.

When $\alpha,\beta\subseteq[n]$ are *disjoint* subsets of $[n]$, to simplify the notation, we use $\alpha\beta$ instead of $\alpha\cup\beta$ in variables depending on subsets of $[n]$. For instance, given $\mathbf{v}\in C^n$ we write ${}_{\alpha\beta}\mathbf{v}$ instead of ${}_{(\alpha\cup\beta)}\mathbf{v}$ and if $\beta=\{j\}$ we write ${}_{\alpha j}\mathbf{v}$.

2. Cubic matroids

We recall (see, for instance [14]) that a *partial partition of* $[n]$ is a partition of a subset of $[n]$ into disjoint subsets, denoted $\pi=\{\alpha_1,\ldots,\alpha_k\}$, with $\bigcup_i\alpha_i\subseteq[n]$. The subsets α_i are the blocks of π. The *lattice of partial partitions of* $[n]$, denoted Q_n, is the poset of partial partitions of $[n]$ with the order $\pi_1\leqslant\pi_2$ iff every block of π_2 is a union of blocks of π_1.

To each partial partition $\pi=\{\alpha_1,\ldots,\alpha_k\}$ of $[n]$ corresponds the partition $\tilde{\pi}=\{\alpha_1,\ldots,\alpha_k,\beta\}$ of $[n+1]$, where $\beta:=[n+1]\backslash\bigcup_{i=1}^{k}\alpha_i$. This correspondence defines an isomorphism between the lattice Q_n and the lattice P_{n+1} of partitions of the set $[n+1]$.

Definition 1 (*Subcubes of* C^n). Given $\mathbf{v}\in C^n$ and a partial partition $\pi=\{\alpha_1,\ldots,\alpha_k\}\in Q_n$ *the subcube of* C^n *defined by* $\mathbf{v}$ *and* π is the subset $C(\mathbf{v};\pi):=\{\mathbf{w}\in C^n:\mathbf{w}(\alpha_j)=\mathbf{v}(\alpha_j)$ *or* $\mathbf{w}(\alpha_j)=\mathbf{v}^*(\alpha_j),\ \forall j=1,\ldots,k\}$. If we want to specify the partition π we write $C(\mathbf{v};\alpha_1,\ldots,\alpha_k)$.

A subcube $C(\mathbf{v};\pi)$ of C^n is called a *k-subcube* of C^n if π has k blocks. The family of all the k-subcubes of C^n, $k=1,\ldots,n$, will be denoted $\mathscr{C}_k$.

The poset of all subcubes of C^n, ordered by set inclusion is denoted $\mathrm{Cub}(n)$. $\mathrm{Cub}(n)$ has a maximal element: C^n. The minimal elements of $\mathrm{Cub}(n)$ are the elements of C^n. By adding $\emptyset$ to $\mathrm{Cub}(n)$ we obtain *the lattice* $\widehat{\mathrm{Cub}}(n)$ *of subcubes of* C^n.

The next proposition lists properties of the poset $\mathrm{Cub}(n)$ derived directly from the definition:

Proposition 1. (1) *For every* $\mathbf{v},\mathbf{w}\in C^n$ *and every* $\pi\in Q_n$, $C(\mathbf{v};\pi)=C(\mathbf{w};\pi)$ *iff* $\mathbf{w}\in C(\mathbf{v};\pi)$.

(2) *For every* $\mathbf{v}\in C^n$ *and every* $\pi_1,\pi_2\in Q_n$, $C(\mathbf{v};\pi_1)\subset C(\mathbf{v};\pi_2)$ *iff* $\pi_2<\pi_1$ *in* Q_n.

(3) *For every* $\mathbf{v}\in C^n$ *the interval* $[\mathbf{v},C^n]$ *of* $\widehat{\mathrm{Cub}}(n)$ *is antiisomorphic to the lattice* Q_n, *or equivalently is isomorphic to the dual lattice* Q_n^{op}.

2.1. Rectangles, facets and skew facets of C^n

A 2-subcube or *rectangle* of C^n is a subset $C(\mathbf{v};\alpha,\beta)=\{\mathbf{v},\ {}_{\alpha}\mathbf{v},\ {}_{\alpha\beta}\mathbf{v},\ {}_{\beta}\mathbf{v}\}$, with α,β two disjoint subsets of $[n]$. The pairs $\{\mathbf{v},\ {}_{\alpha}\mathbf{v}\}$, $\{{}_{\alpha}\mathbf{v},\ {}_{\alpha\beta}\mathbf{v}\}$, $\{{}_{\alpha\beta}\mathbf{v},\ {}_{\beta}\mathbf{v}\}$ and $\{\mathbf{v},\ {}_{\beta}\mathbf{v}\}$ are the *edges of the rectangle*, and the pairs $\{\mathbf{v},\ {}_{\alpha\beta}\mathbf{v}\}$, $\{{}_{\alpha}\mathbf{v},\ {}_{\beta}\mathbf{v}\}$ are the *diagonals of the rectangle*.

The $(n-1)$-subcubes of C^n correspond to partial partitions of $[n]$ into $n-1$ blocks. These partitions are of two types, those in which each block is a single element and those in which a block has two elements. The $(n-1)$-subcubes defined by partitions of the first type will be called *facets of* C^n and the $(n-1)$-subcubes defined by partitions of the second kind will be called *skew facets of* C^n.

Thus, the facets of C^n are the subsets of the form: $H_i^0 := \{\mathbf{v} \in C^n : \mathbf{v}(i) = 0\}$ or $H_i^1 := \{\mathbf{v} \in C^n : \mathbf{v}(i) = 1\}$, for $1 \leqslant i \leqslant n$. The skew facets of C^n are the subsets of the form $H_{ij}^0 := \{\mathbf{v} \in C^n : \mathbf{v}(i) - \mathbf{v}(j) = 0\}$ or $H_{ij}^1 := \{\mathbf{v} \in C^n : \mathbf{v}(i) + \mathbf{v}(j) = 1\}$, for $1 \leqslant i < j \leqslant n$.

If $H \subseteq C^n$ is a facet (resp. skew facet) then $H^* = C^n \backslash H$ is also a face (resp. skew facet), the opposite facet (resp. skew facet).

Definition 2 (*Cubic matroids over* C^n. *Cubic matroids*). A cubic matroid over C^n is a matroid $M = M(C^n)$ over the set C^n, whose families of hyperplanes and circuits, $\mathscr{H}, \mathscr{C}$, satisfy the following conditions:

(C1) The facets and skew facets of C^n are hyperplanes of M: $\mathscr{C}_{n-1} \subseteq \mathscr{H}$.
(C2) The rectangles of C^n are circuits of $\mathscr{M}$: $\mathscr{C}_2 \subseteq \mathscr{C}$.

A *Cubic matroid* is a matroid isomorphic to a cubic matroid over C^n.

Clearly, for every field F, the matroid of the affine dependencies of $C^n \subseteq F^n$ over F, denoted $\mathrm{Aff}_F(C^n)$, is a cubic matroid. We refer to these cubic matroids as *affine cube matroids*. In particular $\mathrm{Aff}_{\mathbb{R}}(C^n)$ is the *real affine cube*.

The next two propositions establish general properties of cubic matroids.

Proposition 2. *Let* $M = M(C^n)$ *be a cubic matroid over* C^n.

(1) *For* $n = 1, 2$ $M = \mathit{Aff}_{\mathbb{R}}(C^n)$.
(2) *Every subcube* $C(\mathbf{v}, \pi)$ *of* C^n *is a flat of* M.
(3) *Let* $\pi_1, \pi_2 \in Q_n$ *be such that* π_2 *covers* π_1 *in* Q_n, *then for every* $\mathbf{v} \in C^n$ *the subcube* $C(\mathbf{v}, \pi_1)$ *covers the subcube* $C(\mathbf{v}, \pi_2)$ *in the lattice of flats of* M.
(4) *Every* k*-subcube* $C(\mathbf{v}, \pi)$ *of* C^n *is a flat of* M *of rank* $k + 1$, *in particular* $r(M) = r(C^n) = n + 1$.
(5) *For every* $n \geqslant 2$, *the restriction* $M(H)$, *of* M *to a facet or skew facet* $H \in \mathscr{C}_{n-1}$, *is isomorphic to a cubic matroid over* C^{n-1}.

Proof. (1) Is trivial.

(2) For every $\mathbf{v} \in C^n$ and $\pi \in Q_n$ the subcube $C(\mathbf{v}, \pi)$ is clearly an intersection of $(n-1)$-subcubes of C^n which are hyperplanes of M, therefore $C(\mathbf{v}; \pi)$ is a flat of M.

(3) Consider a partial partition $\pi_1 = \{\alpha_1, \ldots, \alpha_k\} \in Q_n$. A partial partition $\pi_2 \in Q_n$ covers π_1 iff π_2 is obtained from π_1 in one of the following ways: (i) deleting a block form π_1 or (ii) replacing two blocks of π_1 by their union. We may consider w.l.o.g. that in case (i) $\pi_2 = \{\alpha_1, \ldots, \alpha_{k-1}\}$ and that in case (ii) $\pi_2 = \{\alpha_1, \ldots, \alpha_{k-1} \cup \alpha_k\}$. To simplify, let $\mathbf{v} \in C^n$ and set $C_1 := C(\mathbf{v}; \pi_1)$, $C_2 := C(\mathbf{v}; \pi_2)$. Consider $C_2' := C({}_{\alpha_k}\mathbf{v}; \pi_2)$. Observe that, in both cases, $C_1 = C_2 \cup C_2'$ and so C_1 covers C_2 iff $C_2' \subseteq \mathrm{cl}_M(C_2 \cup {}_{\alpha_k}\mathbf{v})$.

To prove this inclusion consider $\mathbf{w}' \in C_2' \backslash (C_2 \cup {}_{\alpha_k}\mathbf{v})$ then ${}_{\alpha_k}\mathbf{w} \in C_2$ and $R := \{\mathbf{v}, {}_{\alpha_k}\mathbf{v}, \mathbf{w}', {}_{\alpha_k}\mathbf{w}'\}$ is a circuit of M with $R \backslash \mathbf{w}' \subseteq C_2 \cup {}_{\alpha_k}\mathbf{v}$, implying that $\mathbf{w}' \in \mathrm{cl}_M(C_2 \cup {}_{\alpha_k}\mathbf{v})$.

(4) Consider a k-subcube $C(\mathbf{v}, \pi)$ of C^n (i.e. π has k blocks). Let $\gamma : \pi < \pi_{k-1} < \cdots \pi_2 < \hat{1}$ be a maximal chain of Q_n between π and the maximal element $\hat{1} = \{[n]\}$. By definition of Q_n, γ has length k. By (2) and (3) the chain: $\emptyset < \mathbf{v} = C(\mathbf{v}, [n]) < \cdots < C(\mathbf{v}, \pi_{k-1}) < C(\mathbf{v}, \pi)$ is a maximal chain of length $k+1$ in the lattice of flats of M, therefore $r_M(C(\mathbf{v}, \pi)) = k + 1$.

(5) Left to the reader. $\square$

Proposition 3. *Let* $M = M(C^n)$ *be a cubic matroid over* C^n. *Then every subset* $A \subseteq C^n$ *such that* $|A| \geqslant 2^{n-1} + 1$ *satisfies* $r_M(A) = n + 1$.

Proof. The proof is by induction on n. The cases $n = 1, 2$ are trivial. For the induction step observe that given a subset $A \subseteq C^n$ such that $|A| \geqslant 2^{n-1} + 1$ and a pair, H, H^*, of opposite facets of C^n the intersections $A_0 := A \cap H$ and $A_1 := A \cap H^*$ must be both nonempty and one of them, say A_0, must verify $|A_0| \geqslant 2^{n-2} + 1$. By Proposition 2(5) $M(H)$ is isomorphic to a cubic matroid over C^{n-1}, the induction assumption implies that $\mathrm{cl}_M(A_0) = H$. Since $A_1 \neq \emptyset$, $\mathrm{cl}_M(A) = \mathrm{cl}_M(A_0 \cup A_1) = C^n$, i.e. $r_M(A) = r_M(C^n) = n + 1$. $\square$

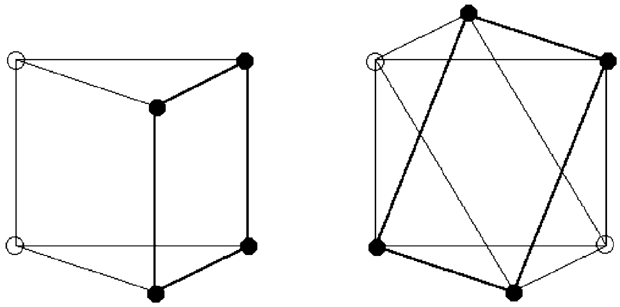

Fig. 1.

Proposition 4 (*Elimination properties for rectangles*). *Let $M = M(C^n)$ be a cubic matroid over C^n. Consider two rectangles, R and R', of M, such that $R \cap R' = \{\mathbf{v},\ {}_{\alpha}\mathbf{v}\}$.*

(1) If $\{\mathbf{v},\ {}_{\alpha}\mathbf{v}\}$ is an edge of both R and R' then we have $R = \{\mathbf{v},\ {}_{\alpha}\mathbf{v},\ {}_{\alpha\beta}\mathbf{v},\ {}_{\beta}\mathbf{v}\}$ and $R' = \{\mathbf{v},\ {}_{\alpha}\mathbf{v},\ {}_{\alpha\gamma}\mathbf{v},\ {}_{\gamma}\mathbf{v}\}$, for some subsets $\beta, \gamma \subset [n]$ such that $\alpha \cap (\beta \cup \gamma) = \emptyset$. In this case the rectangle $C(\mathbf{v}; \alpha, \beta \Delta \gamma) = R \Delta R' = \{{}_{\beta}\mathbf{v},\ {}_{\alpha\beta}\mathbf{v},\ {}_{\alpha\gamma}\mathbf{v},\ {}_{\gamma}\mathbf{v}\}$ is the unique circuit of M contained in $(R \cup R')\backslash \mathbf{v}$.

(2) If $\{\mathbf{v},\ {}_{\alpha}\mathbf{v}\}$ is a diagonal of both R and R' then we have $R = \{\mathbf{v},\ {}_{\beta}\mathbf{v}, {}_{\alpha}\mathbf{v},\ {}_{\alpha\backslash\beta}\mathbf{v}\}$ and $R' = \{\mathbf{v},\ {}_{\gamma}\mathbf{v},\ {}_{\alpha}\mathbf{v},\ {}_{\alpha\backslash\gamma}\mathbf{v}\}$, for some subsets $\beta, \gamma \subset \alpha$. In this case the rectangle $C({}_{\beta}\mathbf{v}; \beta\Delta\gamma, (\alpha\backslash\beta)\Delta\gamma) = R\Delta R' = \{{}_{\alpha\backslash\beta}\mathbf{v},\ {}_{\gamma}\mathbf{v},\ {}_{\beta}\mathbf{v},\ {}_{\alpha\backslash\gamma}\mathbf{v}\}$ is the unique circuit of M contained in $(R \cup R')\backslash\{\mathbf{v}\}$.

Proof. The two situations are pictured in Fig. 1A and B, where the rectangles R and R' are marked with a thin line and the rectangle $R\Delta R'$ with a thick line. The proof is straightforward since by Proposition 2(4) a rectangle is a flat of rank 3 of M. □

We conclude this section describing a "local" operation on matroids: *pulling an element onto a hyperplane*. This operation is a non-oriented version of the operation with the same name introduced by Fukuda and Tamura in [6] for oriented matroids (see e.g. [2]).

Although our operation does not in general preserve orientability it transforms a cubic matroid into a new cubic matroid. We exemplify its use by constructing examples of non-representable cubic matroids.

Definition 3. Let $M = M(E)$ be a matroid of rank r over a set E. A pair $(H, e) \in \mathscr{H} \times E$ of a hyperplane H and an element e of M is said to be *in general position in M* if for every hyperline L contained in H the subset $H_L := L \cup e$ is a hyperplane of M.

Proposition 5. *Let $M = M(E)$ be a matroid of rank r over a set E, with no loops or coloops.*

Assume that there is a hyperplane G and an element $e \notin G$ such that the pair $(G, e) \in \mathscr{H} \times E$ is a pair in general position in M. Denote by $\mathscr{G}$ (resp. $\mathscr{X}$) the collections of hyperplanes (resp. circuits) of M defined by

$$\mathscr{G} := \{H \in \mathscr{H} : H \subseteq G \cup e\},$$

$$\mathscr{X} := \{C \in \mathscr{C} : |C| = r + 1 \text{ and } |C \cap (G \cup e)| = r\}.$$

Then

(1) *$\mathscr{H}_{G,e} := \mathscr{H}\backslash\mathscr{G} \cup \{G \cup e\}$ is the collection of hyperplanes of a new matroid of rank r over E, the matroid obtained from M pulling the element e onto the hyperplane G, denoted $M_{G,e} = M_{G,e}(E)$.*
(2) *The family of circuits, $\mathscr{C}_{G,e}$, of the matroid $M_{G,e}$ is given by $\mathscr{C}_{G,e} := \mathscr{C}\backslash\mathscr{X} \cup \{C \cap (G \cup e) : C \in \mathscr{X}\}$.*
(3) *For every $n \geqslant 3$ if $M = M(C^n)$ is a cubic matroid then $M_{G,e} = M_{G,e}(C^n)$ is a cubic matroid.*

Proof. (1) Verifying that $\mathscr{H}_{G,e}$ is the family of hyperplanes of a matroid is a routine checking of the axioms for hyperplanes of a matroid:

(H1) $H_1, H_2 \in \mathscr{H}_{G,e}$ and $H_1 \subseteq H_2 \Longrightarrow H_1 = H_2$.
(H2) If $H_1, H_2 \in \mathscr{H}_{G,e}$ are two distinct hyperplanes of $\mathscr{H}_{G,e}$ then for every $x \notin H_1 \cup H_2$ there is $H_3 \in \mathscr{H}_{G,e}$ such that $(H_1 \cap H_2) \cup x \subseteq H_3$.

(H1) Let $H_1, H_2 \in \mathscr{H}_{G,e}$, with $H_1 \subseteq H_2$. Since (G, e) is a pair in general position, no hyperplane $H \in \mathscr{H} \backslash \mathscr{G}$ is contained in $G \cup e$, therefore either $H_1 = H_2 = G \cup e$ or $H_1, H_2 \in \mathscr{H} \backslash \mathscr{G}$. In this case they are both hyperplanes of M implying that $H_1 = H_2$.

(H2) Consider two distinct hyperplanes $H_1, H_2 \in \mathscr{H}_{G,e}$ and $x \notin H_1 \cup H_2$. We consider separately the two cases: (1) $H_1, H_2 \in \mathscr{H} \backslash \mathscr{G}$ and (2) $H_1 \in \mathscr{H} \backslash \mathscr{G}$ and $H_2 = G \cup e$.

Case 1: If $H_1, H_2 \in \mathscr{H} \backslash \mathscr{G}$ there is a hyperplane $H_3 \in \mathscr{H}$ satisfying (H2). If $H_3 \in \mathscr{H} \backslash G$ then $H_3 \in \mathscr{H}_{G,e}$ satisfies (H2), if $H_3 \in \mathscr{G}$ then $H_3 \subset G \cup e$ and $H_3' = G \cup e \in \mathscr{H}_{G,e}$ satisfies (H2).

Case 2: Consider $H_1 \in \mathscr{H} \backslash \mathscr{G}$ and $H_2 = G \cup e$. Note that in this case $x \notin G \cup e$. Observe that if $e \notin H_1$ then $H_1 \cap (G \cup e) = H_1 \cap G$ is a subset of some hyperline L of M contained in G. Since the pair (G, e) is in general position and $x \neq e$ we conclude that $H_3 := \mathrm{cl}_M(L \cup x)$ is a hyperplane of $\mathscr{H} \backslash G$ satisfying (H2). In the case $e \in H_1$, $H_1 \cap G$ is a subset of some flat F of rank $r-3$ of M and the fact that (G, e) is in general position and $x \notin G \cup e$ implies that $H_3 := \mathrm{cl}_M(F \cup \{e, x\})$ is a hyperplane of $\mathscr{H} \backslash \mathscr{G}$ satisfying (H2).

(2) We leave the proof to the reader.

(3) Let $M = M(C^n)$ be a cubic matroid with rank greater or equal to 4 ($n \geqslant 3$). Assume that $(G, \mathbf{v}) \in \mathscr{H} \times C^n$ is a pair in general position in M. Observe that from (2) we conclude that every circuit of M with rank smaller or equal to $r-1$ is a circuit of $M_{G,e}$, therefore for$r \geqslant 4$ every rectangle of C^n is a rectangle of $M_{G,e}$. We are left with proving that every facet and skew facet of C^n is a hyperplane of $M_{G,e}$.

Note that given a facet or a skew facet H of C^n for every element $\mathbf{w} \notin H$, i.e. $\mathbf{w} \in H^*$ there is a rectangle R such that $\mathbf{w} \in R$ and $|R \cap H| = |R \cap H^*| = 2$. This implies that for $r \geqslant 4$ the pair $(H, \mathbf{w})$ is not a pair in general position in M. Note also that for $r \geqslant 4$, $M(H)$ has no coloops, therefore H is not contained in $G \cup e$ and consequently it must be a hyperplane of $M_{G,e}$. $\square$

Example 1 (*Non-representable cubic matroids*). Consider $M = \mathrm{Aff}_{\mathbb{R}}(C^{p+1})$ where p is a prime number. Let $G = \{\mathbf{e_1}, \ldots, \mathbf{e_{p+1}}\}$ be the hyperplane of M whose elements are the vectors of the canonical basis of $\mathbb{R}^{p+1}$ and let $\mathbf{u} = \sum_{i=1}^{p+1} \mathbf{e_i}$. The pair $(G, \mathbf{u})$ is in general position in M since the hyperplanes contained in $G \cup \mathbf{u}$ are the following: $G := \{\mathbf{v} \in C^n : \mathbf{v}.(\sum_{i=1}^{p+1} \mathbf{e_i}) = 1\}$ and $G \backslash \mathbf{e_i} \cup \mathbf{u} := \{\mathbf{v} \in C^{p+1} : \mathbf{v}.\mathbf{h_i} = 1\}$ where $\mathbf{h_i} = \sum_{j=1, j \neq i}^{p+1} \mathbf{e_j} - (p-1)\mathbf{e_i}$, all having exactly n elements. By Proposition 6 the matroid $M' := M_{G,\mathbf{u}}$, obtained from M pulling $\mathbf{u}$ onto G, is a cubic matroid. Next we prove that M' contains two minors one of them M_1', representable only over fields of characteristic p, the other M_2', not representable over fields of characteristic p. Since representability is hereditary for minors we conclude that M' is not representable.

Consider $M_1' := M'(G \cup G^* \cup \mathbf{u})$ and $M_2' := M' \backslash \mathbf{u}$. M_1' is a Lazarson matroid (see [14,3, p. 141]) which is representable only over fields of characteristic p. M_2', is representable over $\mathbb{R}$ (observe that $M_2' = \mathrm{Aff}_{\mathbb{R}}(C^{p+1} \backslash \mathbf{u})$) but not over a field of characteristic p since it contains as a minor the uniform matroid $U_{p+2,2}$. Note that $U_{p+2,2} = M_2'/L$ where L is the hyperline of M_2' defined as the intersection of the facet H_{p+1}^0 of C^{p+1} with the affine hyperplane of $\mathbb{R}^{p+1}$ defined by $H : \mathbf{x}.\mathbf{h} = 0$, with $\mathbf{h} = -\mathbf{e_1} + \sum_{i=2}^{p+1} \mathbf{e_i}$. The hyperline L is contained in exactly $p+2$ hyperplanes of M_2', namely the hyperplanes $H_{p+1}^0 : \mathbf{x}.\mathbf{e_{p+1}} = 0$ and $H_j : \mathbf{x}.\mathbf{h_j} = 0$, where $\mathbf{h}_j = -\mathbf{e_1} + \sum_{i=2}^{p} \mathbf{e_i} - j\mathbf{e_{p+1}}$, for $j = -1, 0, \ldots, p-1$. This implies that $M_2'/L = U_{p+2,2}$.

3. Orientability of cubic matroids

We recall that a matroid M is orientable if there is an oriented matroid $\mathscr{M}$ whose underlying matroid is M.

Two orientations $\mathscr{M}$ and $\mathscr{M}'$ of the same matroid $M = M(E)$ are in the same class iff one is obtained from the other *reversing signs on a subset* $A \subseteq E$, i.e. if $\mathscr{C}(\mathscr{M}') {=}_{-A} \mathscr{C}(\mathscr{M})$ and/or $\mathscr{C}^{\perp}(\mathscr{M}') {=}_{-A} \mathscr{C}^{\perp}(\mathscr{M})$. In this case we write $\mathscr{M}' {=}_{-A} \mathscr{M}$.

The orientation class of an oriented matroid $\mathscr{M} = \mathscr{M}(E)$, denoted $\mathcal{O}(\mathscr{M})$, is the set of all orientations obtained from $\mathscr{M}$ reversing signs on a subset of E: $\mathcal{O}(\mathscr{M}) = \{{}_{-A}\mathscr{M} : A \subseteq E\}$.

Acyclic orientations of a matroid play a central role in oriented matroid theory. They are abstract convex polytopes having a well behaved face lattice. This lattice, introduced in [9], is known ([2]) as the Las Vergnas face lattice, or simply the LV-face lattice, of the (acyclic) orientation.

We recall that given a subset E of $\mathbb{R}^n$ the order of $\mathbb{R}$ induces a canonical orientation of the matroid $\mathrm{Aff}_{\mathbb{R}}(E)$, denoted $\mathscr{A\!f\!f}(E)$. This orientation is always acyclic and the LV-face lattice of this acyclic orientation is isomorphic to the face lattice of the polytope $\mathrm{conv}(E)$ of $\mathbb{R}^n$.

On the other hand, via Folkman–Lawrence Topological Representation Theorem for oriented matroids [5], each acyclic orientation in a orientation class $\mathcal{O}(\mathscr{M})$ is represented by a pair of opposite maximal regions (topes) of the cell decomposition of the unit sphere $S^{r(\mathscr{M})-1}$ of $\mathbb{R}^{r(\mathscr{M})}$ determined by a signed arrangement of pseudospheres, representing an oriented matroid in this class. Topes are PL-balls whose face lattice is dual to the LV-face lattice of the corresponding acyclic orientation (see e.g. [2]).

Throughout this section we will make extensive use of orthogonality between signed circuits and signed cocircuits of an oriented matroid. We recall that two signed subsets $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ of a set E are *orthogonal* if either $\underline{X} \cap \underline{Y} = \emptyset$ or $X(\underline{X} \cap \underline{Y}) \neq \pm Y(\underline{X} \cap \underline{Y})$. The signatures $\mathscr{C}$ and $\mathscr{C}^\perp$ of the circuits and of the cocircuits of an oriented matroid are orthogonal and this property is enough to recover one of them given the other.

A consequence of our main results, Theorems 1 and 2, is that we can restate Problem 1 and Las Vergnas Conjecture (see Remark 2) as problems about whether or not the oriented matroid $\mathscr{A\!f\!f}(C^n)$ can be reconstructed from its signed circuits of length 4 or, equivalently, from its signed cocircuits of length 2^{n-1}. These partial lists of signed circuits, resp. signed cocircuits, of the oriented real cube matroid $\mathscr{A\!f\!f}(C^n)$ are denoted $\mathscr{R}$, resp. $\tilde{\mathscr{F}}$ and play a central role in what follows. We precise the definition and notation.

3.1. The families $\mathscr{F}$, $\tilde{\mathscr{F}}$ and $\mathscr{R}$

Let $\mathscr{A\!f\!f}(C^n)$ be the oriented matroid of affine dependencies of C^n over $\mathbb{R}$ and denote by $\mathscr{F}$ (resp. $\tilde{\mathscr{F}}$) its family of signed cocircuits complementary of the facets (resp. facets and skew facets) of C^n and by $\mathscr{R}$ its signed rectangles (circuits of rank 3). We have:

$$\mathscr{F} := \{\pm X_i^0, \pm X_i^1 : \ X_i^0 := (H_i^1, \emptyset), \ X_i^1 := (H_i^0, \emptyset)\}_{i \in [n]},$$

$$\tilde{\mathscr{F}} := \mathscr{F} \ \cup \ \{\pm X_{ij}^0, \pm X_{ij}^1 : X_{ij}^0 := (H_i^0 \cap H_j^1, H_i^1 \cap H_j^0), X_{ij}^1 := (H_i^0 \cap H_j^0, H_i^1 \cap H_j^1)\}_{1 \leqslant i < j \leqslant n},$$

$$\mathscr{R} = \{R(\mathbf{v}; \alpha, \beta) = (\{\mathbf{v}, \ {}_{\alpha\beta}\mathbf{v}\}, \{{}_{\alpha}\mathbf{v} \ {}_{\beta}\mathbf{v}\}) = \mathbf{v}^+ \ {}_{\alpha}\mathbf{v}^- \ {}_{\alpha\beta}\mathbf{v}^+ \ {}_{\beta}\mathbf{v}^- : \mathbf{v} \in C^n, \alpha \uplus \beta \subseteq [n]\}.$$

Our first theorem says that every orientation class of an orientable cubic matroid of rank $(n+1)$ contains at least $(n+1)$ acyclic orientations whose LV-face lattice is isomorphic to the face lattice of the n-dimensional real cube.

Theorem 1. *Let $M = M(C^n)$ be an orientable cubic matroid and $\mathscr{M}$ an orientation of M. Then*:

(1) *There is a unique orientation $\mathscr{M}_0 \in \mathcal{O}(\mathscr{M})$ such that $\tilde{\mathscr{F}} \subset \mathscr{C}^\perp(\mathscr{M}_0)$ and $\mathscr{R} \subseteq \mathscr{C}(\mathscr{M}_0)$.*
(2) *Let $\mathscr{M}_0$ be the orientation of $\mathscr{M}$ defined in* (1). *$\mathscr{M}_0$ is acyclic and its LV-face lattice is isomorphic to the face lattice of the n-dimensional cube. Moreover*, *for every $i \in [n]$ the orientation ${}_{-H_i}\mathscr{M}_0(={}_{-H_{-i}}\mathscr{M}_0)$ is also an acyclic reorientation of $\mathscr{M}$ with LV-face lattice isomorphic to the face lattice of the n-dimensional cube.*

We divide the proof of Theorem 1 in several lemmas.

Lemma 1. *Let $M = M(C^n)$ be an orientable cubic matroid. Let $\mathscr{M}$ be an orientation of M then the following three conditions are equivalent*:

(1) $\mathscr{F} \subseteq \mathscr{C}^\perp$.
(2) $\tilde{\mathscr{F}} \subseteq \mathscr{C}^\perp$.
(3) $\mathscr{R} \subseteq \mathscr{C}$.

Proof. (1) $\Longrightarrow$ (2) Given a skew facet H there are two facets G, G' such that the intersection $L = H \cap G \cap G'$ is an LV-face of corank 2 of the orientation $\mathscr{M}$ and $C^n = H \cup G \cup G'$. The signed cocircuit complementary of H is then directly obtained by elimination between the modular pair of positive signed cocircuits complementary of the facets G and G'.

(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are direct consequences of the orthogonality between signed circuits and signed cocircuits of an oriented matroid. $\square$

Lemma 2. *Let $\mathscr{M}$ be an orientation of a (orientable) cubic matroid $M(C^n)$. Assume that $\mathscr{M}$ satisfies the following two conditions*:

(i) *For some $i \in [n]$, the signed sets $X_i^0 = (H_i^1, \emptyset)$, $X_i^1 = (H_i^0, \emptyset)$ are signed cocircuits of $\mathscr{M}$.*
(ii) *There exists a signed rectangle $R(\mathbf{v}; \alpha, i) = \mathbf{v}^+ \; {}_{\alpha}\mathbf{v}^- \; {}_{\alpha i}\mathbf{v}^+ \; {}_{i}\mathbf{v}^-$ which is a signed circuit of $\mathscr{M}$.*

Then the following conditions are satisfied:

1. *For every $\mathbf{w} \in C^n$ and every pair of disjoint subsets $\beta, \gamma \subseteq [n]\backslash i$ the signed rectangle $R(\mathbf{w}; \beta, \gamma i) = \mathbf{w}^+ \, {}_{\beta}\mathbf{w}^- \; {}_{\beta\gamma i}\mathbf{w}^+ \; {}_{\gamma i}\mathbf{w}^-$ is a signed circuit of $\mathscr{M}$.*
2. *$\tilde{\mathscr{F}} \subset \mathscr{C}^{\perp}(\mathscr{M})$ and $\mathscr{R} \subseteq \mathscr{C}(\mathscr{M})$.*

Remark 1. First observe that by orthogonality with the signed cocircuits X_i^0 and X_i^1, every rectangle $C(\mathbf{w}; \beta, \gamma i)$, $\beta \uplus \gamma \subseteq [n]\backslash i$, must be signed in $\mathscr{M}$ in one of the following two ways: either as $\pm R(\mathbf{w}; \beta, \gamma i)$ where $R(\mathbf{w}; \beta, \gamma i) = \mathbf{w}^+ \; {}_{\beta}\mathbf{w}^- \; {}_{\beta\gamma i}\mathbf{w}^+ \; {}_{\gamma i}\mathbf{w}^-$ or as $\pm R'(\mathbf{w}; \beta, \gamma i)$ where $R'(\mathbf{w}; \beta, \gamma i) = \mathbf{w}^+ \; {}_{\beta}\mathbf{w}^- \; {}_{\beta\gamma i}\mathbf{w}^- \; {}_{\gamma i}\mathbf{w}^+$.

Proof. (1) First we consider the case $\mathbf{w} = \mathbf{v}$, where $\mathbf{v}$ is an element of a rectangle $R(\mathbf{v}; \alpha, i)$ in condition (ii) of the lemma, and then the general case.

Case 1: $\mathbf{w} = \mathbf{v}$. This case is proven by contradiction. Assume that for some $\beta \uplus \gamma \subseteq [n]\backslash i$, the signed rectangle $R(\mathbf{v}; \beta, \gamma i) = \mathbf{v}^+ \; {}_{\beta}\mathbf{v}^- \; {}_{\beta\gamma i}\mathbf{v}^+ \; {}_{\gamma i}\mathbf{v}^-$ is not a signed circuit of $\mathscr{M}$. Then, by Remark 2, $R' = R'(\mathbf{v}; \beta, \gamma i) = \mathbf{v}^+ \; {}_{\beta}\mathbf{v}^- \; {}_{\beta\gamma i}\mathbf{v}^- \; {}_{\gamma i}\mathbf{v}^+$ is a signed circuit of $\mathscr{M}$.

If $\gamma = \emptyset$, i.e. if $R' = \mathbf{v}^+ \; {}_{\beta}\mathbf{v}^- \; {}_{\beta i}\mathbf{v}^- \; {}_{i}\mathbf{v}^+$, consider the signed circuit $R = \mathbf{v}^+ \; {}_{\alpha}\mathbf{v}^- \; {}_{\alpha i}\mathbf{v}^+ \; {}_{i}\mathbf{v}^-$ of $\mathscr{M}$ satisfying condition (ii) of the lemma. Since ${}_{i}\mathbf{v}$ has different signs in R and R', elimination for signed circuits implies that there is a signed circuit S of $\mathscr{M}$ such that $S^+ \subseteq (R^+ \cup R'^+)\backslash \; {}_{i}\mathbf{v}$ and $S^- \subseteq (R^- \cup R'^-)\backslash \; {}_{i}\mathbf{v}$. On the other hand, the rectangles R and R' have the edge $\{\mathbf{v}, \; {}_{i}\mathbf{v}\}$ in commun therefore by Proposition 4(1) S must be the signed rectangle: $S = {}_{-\alpha}\mathbf{v}^- \; {}_{\alpha i}\mathbf{v}^+ \; {}_{\beta i}\mathbf{v}^- \; {}_{\beta}\mathbf{v}^-$. This signed set is not orthogonal to one of the positive cocircuit X_i^0 or X_i^1, a contradiction.

If $\gamma \neq \emptyset$ and $R' = R'(\mathbf{v}; \beta, \gamma i) = \mathbf{v}^+ \; {}_{\beta}\mathbf{v}^- \; {}_{\beta\gamma i}\mathbf{v}^-_{\gamma i}\mathbf{v}^+$ is a signed circuit of $\mathscr{M}$ then, by the case $\gamma = \emptyset$ we know that $R = R(\mathbf{v}; \beta\gamma, i) = \mathbf{v}^+ \; {}_{\beta\gamma}\mathbf{v}^- \; {}_{\beta\gamma i}\mathbf{v}^+ \; {}_{i}\mathbf{v}^-$ is a signed circuit of $\mathscr{M}$. R and R' have the diagonal $\{\mathbf{v}, \; {}_{\beta\gamma i}\mathbf{v}\}$ in commun. Eliminating ${}_{\beta\gamma i}\mathbf{v}$ between R and R' we conclude, by Proposition 4(2) that the signed rectangle $S = {}_{\beta}\mathbf{v}^- \; {}_{\beta\gamma}\mathbf{v}^- \; {}_{\gamma i}\mathbf{v}^- \; {}_{i}\mathbf{v}^+$ is a signed circuit of $\mathscr{M}$. This circuit is not orthogonal to one of the positive cocircuits X_i^0, X_1^1, a contradiction.

Case 2: The general case is now obtained as a consequence of Case 1. Consider $\mathbf{w} = {}_{\beta}\mathbf{v} \in C^n$, $\beta \neq \emptyset$. If $i \notin \beta$ then, by Case 1, $R(\mathbf{v}; \beta, i) = R(\mathbf{w}; \beta, i)$ is a signed circuit of $\mathscr{M}$ and (2) holds in this case. If $i \in \beta$ then, by Case 1,$R(\mathbf{v}; \beta\backslash i, i) = R(\mathbf{w}; \beta\backslash i, i)$ is a signed circuit of $\mathscr{M}$ and condition (ii) of the lemma is verified replacing the given circuit by $R(\mathbf{w}; \beta, i)$ implying that in this case the result also follows from Case 1.

(2) By Lemma 1 it is enough to prove that $\mathscr{F} \subseteq \mathscr{C}^{\perp}(\mathscr{M})$. We prove that for every $j \in [n]\backslash i$, $X_j^0 = (H_j^1, \emptyset)$ and $X_j^1 = (H_j^0, \emptyset)$ are signed cocircuits of $\mathscr{M}$. We consider the case X_j^0, the other being similar. Consider $j \in [n]\backslash i$ and $\mathbf{v} \in H_j^1$. Let $Y_j = (Y_j^+, Y_j^-)$ be the signed cocircuit of $\mathscr{M}$ with support $\underline{Y_j} = H_j^1$ such that $\mathbf{v} \in Y_j^+$. Consider another element $\mathbf{w} = {}_{\alpha}\mathbf{v}$ of H_j^1 (note that certainly $j \notin \alpha$). If $i \notin \alpha$ then, by (1), we know that $R = R(\mathbf{v}; \alpha, ij)$ is a signed circuit of $\mathscr{M}$, moreover $R \cap Y_j = \{\mathbf{v}, \mathbf{w}\}$. Since $\mathbf{v}$ and $\mathbf{w}$ have different signs in R, we conclude that $\mathbf{w} \in Y_j^+$. If $i \in \alpha$, using the same argument with $R = R(\mathbf{v}; \alpha, j)$, we also conclude that $\mathbf{w} \in Y_j^+$. Therefore Y_j is the positive cocircuit X_j^0. $\square$

Lemma 3. *Let $\mathscr{M}$ be an orientation of a (orientable) cubic matroid $M(C^n)$, satisfying the condition*:

(i) $X_i^0 = (H_i^1, \emptyset)$, $X_i^1 = (H_i^0, \emptyset) \in \mathscr{C}^\perp$.

Then one of the following conditions is satisfied:

(1) $\tilde{\mathscr{F}} \subset \mathscr{C}^\perp$ *and* $\mathscr{R} \subseteq \mathscr{C}$,
(2) ${}_{-H_i^0}\tilde{\mathscr{F}} \subset \mathscr{C}^\perp$ *and* ${}_{-H_i^0}\mathscr{R} \subseteq \mathscr{C}$.

Proof. Let $C = C(\mathbf{v}; \alpha, i)$ be a rectangle of $\mathscr{M}$. Condition (i) of the lemma (see Remark 2) implies that either $R = R(\mathbf{v}; \alpha, i)$ or $R' = R'(\mathbf{v}; \alpha, i)$ is a signed rectangle of $\mathscr{M}$. In the first case Lemma 2 implies that $\tilde{\mathscr{F}} \subset \mathscr{C}^\perp$ and $\mathscr{R} \subseteq \mathscr{C}$. In the second case Lemma 2 implies that $R' = R'(\mathbf{v}; \alpha, i)$ is a signed rectangle of $\mathscr{M}$ for every $\mathbf{v} \in C^n$ and $\alpha \subseteq [n]\backslash i$. Arguing by contradiction, as in Lemma 2 we conclude that for every $\mathbf{w} \in C^n$ and every disjoint subsets $\beta, \gamma \subseteq [n]\backslash i$ the signed rectangle $R'(\mathbf{w}; \beta, \gamma i)$ is a signed rank 3 circuit of $\mathscr{M}$, therefore ${}_{-H_i}\mathscr{R} \subseteq \mathscr{C}$. By orthogonality with the signed rectangles of ${}_{-H_i}\mathscr{R}$ we conclude that ${}_{-H_i}\tilde{\mathscr{F}} \subset \mathscr{C}^\perp$. $\square$

Proof of Theorem 1. Consider and oriented cubic matroid $\mathscr{M}$.

(1) Let $Y = (Y^+, Y^-)$ and $Z = (Z^+, Z^-)$ be the signed cocircuits of $\mathscr{M}$ complementary of the facets H_n^0 and H_n^1, respectively. Since $C^n = H_n^0 \uplus H_n^1$ there are exactly two acyclic reorientations of $\mathscr{M}$ containing the positive signed cocircuits $X_n^0 = (H_n^1, \emptyset)$ and $X_n^1 = (H_n^0, \emptyset)$ namely the orientations $\mathscr{M}' := {}_{-B_1}\mathscr{M}$ and $\mathscr{M}'' := {}_{-B_2}\mathscr{M}$ where $B_1 := Y^- \cup Z^-$ and $B_2 := Y^+ \cup Z^-$ (observe that $B_2 = B_1 \Delta H_n^0$). By Lemma 3 exactly one of these two orientations, say $\mathscr{M}'$, satisfies the condition $\tilde{\mathscr{F}} \subset \mathscr{C}^\perp(\mathscr{M}')$ and $\mathscr{R} \subseteq \mathscr{C}(\mathscr{M}')$ and the other the condition ${}_{-H_n}\tilde{\mathscr{F}} \subset \mathscr{C}^\perp(\mathscr{M}'')$ and ${}_{-H_n}\mathscr{R} \subseteq \mathscr{C}(\mathscr{M}'')$.

(2) Let $\mathscr{M}_0$ be the orientation in $\mathscr{O}(\mathscr{M})$ satisfying the condition $\tilde{\mathscr{F}} \subset \mathscr{C}^\perp(\mathscr{M}_0)$ and $\mathscr{R} \subseteq \mathscr{C}(\mathscr{M}_0)$. It is clear that $\mathscr{M}$ has LV-face lattice isomorphic to the face lattice of the n-dimensional cube. For every $i \in [n]$ the reorientation $\mathscr{M}_i := {}_{-H_i^0}\mathscr{M}_0$ is an orientation in the class $\mathscr{O}(\mathscr{M})$ and clearly ${}_{-H_i^0}\tilde{\mathscr{F}} \subset \mathscr{C}^\perp(\mathscr{M}_i)$ and ${}_{-H_i^0}\mathscr{R} \subseteq \mathscr{C}(\mathscr{M}_i)$, implying that $\mathscr{M}_i$ is an acyclic orientation of M with LV-facets: H_i^0, H_i^1 and H_{ij}^0, H_{ij}^1 for every $j \in [n]\backslash i$. Therefore the LV-face lattice of $\mathscr{M}_i$ is isomorphic to the face lattice of the n-dimensional cube. $\square$

Theorem 1 says that every orientation class of an orientable cubic matroid contains, at least, $(n+1)$ acyclic orientations whose LV-face lattice is isomorphic to the face lattice of the n-dimensional cube. Topologically, via Folkman–Lawrence Topological Representation Theorem, each orientation $\mathscr{M} = \mathscr{M}(C^n)$ of a cubic matroid is represented by a signed arrangement of 2^n pseudohyperplanes of the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. This arrangement of pseudospheres determines a cell decomposition $\Delta(\mathscr{M})$ of the unit sphere S^n whose maximal cells correspond to the acyclic reorientations in the same orientation class.

From this point of view Theorem 1 says that for every orientation $\mathscr{M}$ of a cubic matroid the corresponding cell decomposition $\Delta(\mathscr{M})$ contains at least $2(n+1)$ maximal topes, bounded by the 2^n hyperplanes and whose face lattice is isomorphic to the face-lattice of the n-cross polytope.

The next theorem, Theorem 2, is more precise. It says that no other tope of $\Delta(\mathscr{M})$ has this property, moreover it specifies the relative position of these $2(n+1)$ topes within the cell complex.

Theorem 2. *Let $M = M(C^n)$ be an orientable cubic matroid. Every orientation class, $\mathscr{O}(\mathscr{M})$ of M satisfies the following two, equivalent, conditions*:

(1) *$\mathscr{O}(\mathscr{M})$ contains exactly $(n+1)$ acyclic orientations, $\mathscr{M}_0, \mathscr{M}_1, \ldots, \mathscr{M}_n$ whose LV-face lattice is isomorphic to the face lattice of the n-dimensional cube. Moreover, given two orientations $\mathscr{M}, \mathscr{M}' \in \{\mathscr{M}_0, \mathscr{M}_1, \ldots, \mathscr{M}_n\}$ there is a unique pair of opposite LV-facets, $\{H, H^*\}$, of both $\mathscr{M}$* and *$\mathscr{M}'$ such that $\mathscr{M}' = {}_{-H}\mathscr{M}$ $(= {}_{-H^*}\mathscr{M})$.*
(2) *Let $\Delta(\mathscr{M})$ denote the cell decomposition of the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ determined by the signed arrangement of pseudohyperplanes representing an orientation $\mathscr{M} \in \mathscr{O}(M)$. $\Delta(\mathscr{M})$ has exactly $2(n+1)$ topes, $\pm T_0, \pm T_1 \ldots \pm T_n$, whose face lattice is isomorphic to the face lattice of the n-cross-polytope. Moreover, given two topes $T, T' \in \{\pm T_0, \pm T_1 \ldots \pm T_n\}$ such that $T' \neq -T$, T and T' have exactlyone vertex (signed cocircuit) in commun.*

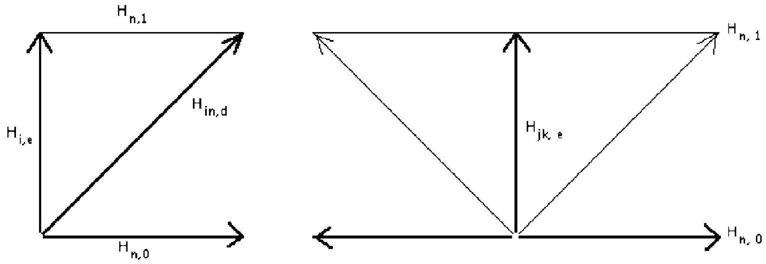

Fig. 2.

Proof. The equivalence between (1) and (2) is standard (see [5,2]). We just prove (1). In order to do so we must prove that, given an orientation $\mathscr{M}$ of M, apart from the $(n+1)$ acyclic reorientations described in Theorem 1 as having LV-face lattice isomorphic to the face lattice of the n-dimensional cube no other acyclic reorientation in the same class has this property.

This is a consequence of the next two lemmas. Lemma 4 saying that the LV-facets of such an acyclic orientation must be facets or skew facets of C^n and Lemma 5 saying that such an acyclic orientation always contains as LV-facets a pair $\{H_i^0, H_i^1\}$ of opposite facets of C^n.

Lemma 3 and Theorem 1 then imply that, being $\mathscr{M}_0$ the unique acyclic orientation of $\mathcal{O}(M)$ satisfying the condition $\tilde{\mathscr{F}} \subseteq \mathscr{C}^{\perp}(\mathscr{M}_0)$, the other n acyclic orientations with LV-face lattice isomorphic to the face lattice of the n-dimensional cube must be $\mathscr{M}_i = {}_{-H_i^0}\mathscr{M}$. Observe that $\mathscr{M}_0$ and $\mathscr{M}_i$ have in commun the pair of LV-facets $\{H_i^0, H_i^1\}$ and $\mathscr{M}_i$ and $\mathscr{M}_j$ have in commun the pair of LV-facets $\{H_{ij}^0, H_{ij}^1\}$, for every $1 \leqslant i < j \leqslant n$. $\square$

Lemma 4. *If $M = M(C^n)$ is an orientable cubic matroid over C^n then every subset A of C^n such that $|A| = 2^{n-1}$ and $r(A) \leqslant n$ is a facet or a skew facet of C^n.*

Proof. Note that, by Theorem 1, we may consider w.l.o.g. that M is oriented with an orientation $\mathscr{M}$ satisfying the condition $\tilde{\mathscr{F}} \subseteq \mathscr{C}^{\perp}$. The proof then follows by induction on n. For $n = 1, 2$ the result is trivial since in these cases $\mathrm{Aff}_{\mathbb{R}}(C^n)$ is the unique cubic matroid over C^n. For the induction step, let A be a subset of C^n in the conditions of the lemma. If A is one of the facets of C^n there is nothing to prove, otherwise consider the intersections $A_0 := A \cap H_n^0$ and $A_1 := A \cap H_n^1$. By Propositions 3(5) and 4, we must have $|A_0| = |A_1| = 2^{n-2}$ and induction implies that A_0, resp. A_1, must be a facet or a skew facet of the $(n-1)$-cube H_n^0, resp. H_n^1. If A_0 is a facet of H_n^0, i.e. $A_0 = H_n^0 \cap H_i^{\varepsilon} \cap H_{in}^{\delta}$ for some $i \in [n-1]$, $\varepsilon, \delta \in \{0, 1\}$ then A_0 is a LV-face of corank 2 of $\mathscr{M}$ and the contraction $\mathscr{M}/A_0$ must be the real acyclic geometry of rank 2 with three points depicted in Fig. 2A. The hypothesis $r(A) \leqslant n$ implies that in this case A must be either the facet H_i^{ε} or the skew facet H_{in}^{δ}. If A_0 is a skew facet of H_n^0, then $A_0 = H_n^0 \cap H_{jk}^{\varepsilon}$ for some $j, k \in [n-1]$, $\varepsilon \in \{0, 1\}$. In this case $\mathscr{M}/A_0$ must be the real geometry of rank 2 with 4 points. Fig. 2B depicts this case, where the thin vectors corresponding to the partition of $H_n^1 \cap H_{jk}^{1-\varepsilon}$ into positive and negative elements of the signed cocircuit complementary of H_{jk}^{ε}. Since $|A| = 2^{n-1}$ and $r(A) \leqslant n$ we conclude that, in this case, A must be the skew facet H_{jk}^{ε} of C^n. $\square$

Lemma 5. *Let $\mathscr{M}$ be an acyclic orientation of a cubic matroid $M = M(C^n)$ whose LV-face lattice is isomorphic to the face lattice of the n-dimensional cube. If a skew facet H_{ij}^{ε} of C^n is a LV-facet of $\mathscr{M}$ then one and only one of the following conditions is satisfied*:

(1) *H_i^0, H_i^1 is a pair of LV-facets of M.*
(2) *H_j^0, H_j^1 is a pair of LV-facets of M.*

Proof. Assume that H_{ij}^0 is a LV-facet of $\mathscr{M}$. Let L and L' be the flats of corank 2 defined by $L := H_{ij}^0 \cap H_i^0 \cap H_j^0$ and $L' := H_{ij}^0 \cap H_i^1 \cap H_j^1$. The contractions $\mathscr{M}/L$ and $\mathscr{M}/L'$ are real rank 2 geometries with three points, containing the positive cocircuit $X_{ij} = (H^1_{ij}, \emptyset)$ of $\mathscr{M}$ and therefore acyclic. One of the extreme points of $\mathscr{M}/L$ is $H^0_{ij}\backslash L$, the other one must be, either $H^0_i\backslash L$ or $H^0_j\backslash L$, implying, respectively, that H^0_i or H^0_j is a LV-facet of $\mathscr{M}$. Similarly, one of the extreme points of $\mathscr{M}/L'$ is $H^0_{ij}\backslash L'$, the other is one of the other two points: $H^1_i\backslash L'$ or $H^1_j\backslash L'$ implying that either H^1_i or H^1_j is a LV-facet of $\mathscr{M}$

To conclude the proof of the lemma we prove, by contradiction, that H^0_i and H^1_j cannot be both LV-facets of $\mathscr{M}$ (this is enough since, by hypothesis, the face lattice of $\mathscr{M}$ is isomorphic to the face lattice of the n-dimensional cube). Assume that H^0_i and H^1_j are both facets of $\mathscr{M}$. Let $\mathbf{v}$ be an element of H^0_{ij} and consider the rectangle $C = C(\mathbf{v}; i, j)$ of M. By the hypothesis of the lemma H^0_{ij} and H^1_{ij} are LV-facets of M. Orthogonality with the corresponding positive signed cocircuits of $\mathscr{M}$ implies that either $R = \mathbf{v}^+ \ {}_i\mathbf{v}^- \ {}_{ij}\mathbf{v}^- \ {}_j\mathbf{v}^+$ or $R' = \mathbf{v}^+ \ {}_i\mathbf{v}^+ \ {}_{ij}\mathbf{v}^- \ {}_j\mathbf{v}^-$ is a signed circuit of $\mathscr{M}$. But R is not orthogonal to the positive cocircuit $X^0_i = (H^1_i, \emptyset)$ of $\mathscr{M}$ and R' is not orthogonal to the positive cocircuit $X^1_j = (H^0_j, \emptyset)$ of $\mathscr{M}$, leading the contradiction. □

Note that from Lemma 4 we conclude that every *orientable* cubic matroid must have the same family of cocircuits with 2^{n-1} elements. The next corollary extends this result to the family of circuits with four elements.

Corollary 1. *Every orientable cubic matroid $M = M(C^n)$ satisfies the following conditions*:

(1) *The hyperplanes of M with 2^{n-1} elements are the facets and skew facets of C^n.*
(2) *The circuits of length 4 of M are the rectangles of C^n.*
(3) *Every class of orientations of M contains a unique acyclic orientation whose signed circuits of length 4 and/or signed cocircuits of length 2^{n-1} are signed as in $\mathscr{A}\!f\!f(C^n)$.*

Proof. (1) was proved in Lemma 4.

(2) The proof is by induction on n. Assume that $C = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is a circuit of rank 3 of M. By Lemma 4 either C is contained in a facet or skew facet, and in this case induction implies that either C is a rectangle, or $|C \cap H| = 2$ for every facet or skew facet H of C^n. In this case assume w.l.o.g. that $\mathbf{v} =_\alpha \mathbf{u}$ for some α strictly contained in $[n]$. If $|\alpha| < n - 1$ then for every $i, j \in [n]\backslash\alpha$ C is contained in a skew facet H_{ij}. If $|\alpha| = n - 1$ then, for $n \geqslant 4$, there is $i, j \in \alpha$ such that C is contained in a skew facet H_{ij}. We are left with the case $n = 3$ in this case, by Lemma 4, C must be a rectangle of C^3.

(3) is an immediate consequence of (1), (2) and Theorem 1. □

Remark 2. Corollary 1 together with Theorems 1 and 2 leads to the following reformulation of Problem 1 and Las Vergnas Conjecture as reconstruction problems:

Problem 1. Can the *real affine cube* $\mathrm{Aff}_{\mathbb{R}}(C^n)$ be reconstructed from *orientability* and *its circuits with* 4 *elements or/and hyperplanes with* 2^{n-1} *elements*?

Las Vergnas Conjecture. The *oriented matroid* $\mathscr{A}\!f\!f(C^n)$ can be reconstructed from its *underlying matroid*, $\mathrm{Aff}_{\mathbb{R}}(C^n)$, and its *signed rectangles or/and its positive cocircuits*.

The next theorem is a direct application of Theorems 1 and 2 to obtain non-orientability results about cubic matroids.

Theorem 3 (*Implicit in Bland and Las Vergnas [3]*). *Let F be a field of prime characteristic p. Then for every $n \geqslant p+1$ the affine cube over F, $\mathrm{Aff}_F(C^n)$, is not orientable.*

Proof. Since orientability is hereditary for minors it is enough to prove the theorem for $n = p+1$. Set $M := \mathrm{Aff}_F(C^{p+1})$ and assume that M is orientable. Let $\mathscr{M}$ be an orientation of M satisfying the conditions of Theorem 1. The subset $\underline{C} = \{\mathbf{e_1} \ldots \mathbf{e_{p+1}}, \mathbf{u}\}$ where $\mathbf{e_i}$ is the point represented by the ith vector of the canonical basis and $\mathbf{u} = \sum_{i=1}^{p+1} \mathbf{e_i}$ must be a circuit of M because F has characteristic p.

For every positive cocircuit $X^0_i = (H^1_i, \emptyset)$ of $\mathscr{M}$ we have $\underline{C} \cap H^1_i = \{\mathbf{e_i}, \mathbf{u}\}$ therefore the signature of the circuit $\underline{C}$ in $\mathscr{M}$ must be $\pm C$ where $C = (\{\mathbf{e_1}, \ldots, \mathbf{e_{p+1}}\}, \ \mathbf{u})$. This implies that $\mathbf{u}$ is not an extremal point of $\mathscr{M}$ in contradiction with the assumption that $\mathscr{M}$ satisfies theconditions of Theorem 1. □

4. Final remarks

(1) The short proof of Theorem 3 extends to the following generalized version of Theorem 3 which also applies to the cubic matroids of Example 1:

Theorem 3′. *Let $M = M(C^n)$ be a cubic matroid, $n \geqslant 3$. If M contains a pair hyperplane/element $(G, \mathbf{u})$ in general position in M satisfying the following condition: G has a basis $B = \{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ such that for every facet $H_i(\mathbf{u})$ of C^n containing $\mathbf{u}$, $B \cap H_i(\mathbf{u}) = \{\mathbf{e_i}\}$, then the matroid $M_{G,e}$, obtained from M by pulling $\mathbf{u}$ onto G, is not orientable.*

An alternative proof of Theorem 3′ consists in observing that the matroid $M_{G,e}$ contains, as a minor, the minor minimal non-orientable matroid M_n of Bland and Las Vergnas [3].

(2) The operation of pulling an element onto a hyperplane defined in Section 2 when applied to an oriented matroid does not, in general, preserve orientability. Fukuda and Tamura proved [6] that orientability is preserved when the element and the hyperplane are not only in general position in the matroid but also *near* each other in the *oriented* matroid.

Roughly speaking a pair (G, e) of a hyperplane and an element are near each other in an oriented matroid $\mathscr{M}(E)$ if the pair (G, e) is in general position in the underlying matroid and all the hyperplanes of M contained in $G \cup e$ determine the same partition of the complement, $E \backslash (G \cup e)$, in the oriented matroid.

If an oriented matroid $\mathscr{M}$ contains a pair (G, e) of a hyperplane and an element near each other then $\mathscr{M}$ induces not only a canonical orientation $\mathscr{M}_{G,e}$ of the matroid $M_{G,e}$ obtained by pulling *e onto G*, but also a new orientation $\mathscr{M}_{eGe}$ of the matroid M, obtained *by pulling the element e across the hyperplane G* (see [6]).

From this point of view an answer to the next question is relevant for a complete understanding of orientable cubic matroids:

Question 1. Does the oriented real affine cube $\mathscr{A}\!f\!f(C^n)$ contain a pair (G, e) of a hyperplane and an element *near* each other?

Observe that if such a pair (G, e) does exist then we, immediately, conclude that Problem 1 and Las Vergnas Conjecture have a negative answer. Since Las Vergnas Conjecture is true for $n \leqslant 7$ [4,13] the existence of such a pair requires $n > 7$.

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