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Hamiltonian decompositions of Cayley graphs on abelian groups of even order

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Abstract

Alspach conjectured that any $2k$-regular connected Cayley graph $cay(A,S)$ on a finite abelian group A can be decomposed into k hamiltonian cycles. In 1992, the author proved that the conjecture holds if $S=\{s_1,s_2,\ \dots,\ s_k\}$ is a minimal generating set of an abelian group A of odd order. Here we prove an analogous result for abelian group of even order: If A is a finite abelian group of even order at least 4 and $S=\{s_1,s_2,\ \dots,\ s_k\}$ is a strongly minimal generating set (i.e., $2s_i\notin\langle S-\{s_i\}\rangle$ for each $1\leqslant i\leqslant k$) of A, then $cay(A,S)$ can be decomposed into hamiltonian cycles.

1. Introduction

Let $(A,+)$ be a finite group (we use $+$ for the notation of the operation as our main focus in this paper is on abelian groups) and S be a subset of A with $0\notin S$. The *Cayley graph* $cay(A,S)$ is defined to be the graph G with $V(G)=A$ and $E(G)=\{xy|x,y\in A, x-y\in S \text{ or } y-x\in S\}$. We say the edge xy in $cay(A,S)$ is generated by $s\in S$ if $x-y=s$ or $y-x=s$ and the subgraph Q of $cay(A,S)$ is generated by s if all edges of Q are generated by s.

From the definition, it is clear that any element of S with order 2 generates a 1-factor of $cay(A,S)$ while any element of S with order at least 3 generates a 2-factor of $cay(A,S)$.

Furthermore, $cay(A,S)$ is connected if and only if S generates A.

It is known that any connected Cayley graph on a finite abelian group is hamiltonian [7]. In [1], Alspach conjectured that any $2k$-regular connected Cayley

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graph on a finite abelian group has a hamiltonian decomposition. Clearly, for $k = 1$ the conjecture is trivial. Bermond et al. [2] proved the conjecture for $k = 2$.

Theorem 1.1 (Bermond et al. [2]). *Every* 4*-regular connected Cayley graph* $cay(A, S)$ *on a finite abelian group* A *can be decomposed into two hamiltonian cycles.*

Liu [5] proved that $cay(A, S)$ has a hamiltonian decomposition if $S = \{s_1, s_2, \ldots, s_k\}$ is a generating set of an abelian group A such that $gcd(ord(s_i), ord(s_j)) = 1$ for $i \neq j$. In [6], the author derived a more general result for abelian groups of odd order as follows.

Theorem 1.2 (Liu [6]). *If* A *is an abelian group of odd order and* $S = \{s_1, s_2, \ldots, s_k\}$ *is a minimal generating set of* A, *then* $cay(A, S)$ *has a hamiltonian decomposition.*

We say that a generating set S of a group A is *strongly minimal* if for any $s \in S, 2s$ cannot be generated by the elements in $S - \{s\}$. Clearly, any minimal generating set of a group of odd order is also strongly minimal.

In this paper, we prove the following main result.

Theorem 1.3. *If* A *is a finite abelian group of even order at least* 4 *and* $S = \{s_1, s_2, \ldots, s_k\}$ *is a strongly minimal generating set of* A, *then* $cay(A, S)$ *has a hamiltonian decomposition.*

The next result is a direct consequence to Theorem 1.3 since the condition imposed on the generating set S implies that S is strongly minimal.

Theorem 1.4. *If* A *is a finite abelian group of even order at least* 4 *and* S *is a minimal generating set of* A *such that the quotient group* $A/\langle s \rangle$ *is of odd order for each* $s \in S$, *then* $cay(A, S)$ *has a hamiltonian decomposition.*

2. Direct proof of Theorem 1.4

In this section, we give a direct proof to Theorem 1.4 since we need to use this result in the proof of Theorem 1.3.

Proof of Theorem 1.4. Let A be a finite abelian group of order $2^d(2h + 1) \geqslant 4$ with $d \geqslant 1$ and $S = \{s_1, s_2, \ldots, s_k\}$ be a minimal generating set of A such that $A/\langle s_i \rangle$ is of odd order for each $1 \leqslant i \leqslant k$. Then $ord(s_i) = 2^d(2k_i + 1)$ for each $s_i \in S$. By the Decomposition Theorem of Finite Abelian Groups, any finite abelian group is a direct sum of finitely many cyclic groups with prime-power orders which implies that A can be expressed as a direct sum $A = A_1 \oplus A_2$ with $|A_1| = 2^d$ and $|A_2| = 2h + 1$. For convenience, let $A = \{(x, y) | x \in A_1, y \in A_2\}$. Then for each $1 \leqslant i \leqslant k, s_i = (x_i, y_i)$

with $x_i \in A_1$ and $y_i \in A_2$. By the assumption, $\langle x_i \rangle = A_1$ for each i. Without loss of generality, let $A_1 = Z_{2^d} = \{0, 1, 2, \ldots, 2^d - 1\}$ and let $A_2 = \{u_1, u_2, \ldots, u_{2h+1}\}$. Then each x_i must be an odd integer in A_1. Since S is a minimal generating set of A, $S' = \{y_1, y_2, \ldots, y_k\}$ is a minimal generating set for A_2. By Theorem 1.2, $cay(A_2, S')$ can be decomposed into k hamiltonian cycles $H'_i = u_{\pi_i(1)}u_{\pi_i(2)}\cdots u_{\pi_i(2h+1)}u_{\pi_i(1)}$, $1 \leqslant i \leqslant k$, where each π_i is a permutation on $\{1, 2, \ldots, 2h+1\}$. Note that each edge uv in $cay(A_2, S')$ with $u - v = y_r$ gives rise to $M_{uv} = \{(j, u)(j + x_r, v) | 0 \leqslant j \leqslant 2^d - 1\}$ (generated by $s_r = (x_r, y_r)$) which is a perfect matching between two columns $A_1 \times \{u\}$ and $A_1 \times \{v\}$ (for convenience, we say this matching has a jump x_r). Clearly, different edges in $cay(A_2, S')$ correspond to disjoint matchings between columns in $cay(A, S)$. It follows that the edge-disjoint hamiltonian cycles $H'_i, 1 \leqslant i \leqslant k$, in $cay(A_2, S')$ correspond to edge-disjoint 2-factors $H_i = \bigcup_{uv \in E(H'_i)} M_{uv}$ in $cay(A, S)$. We next show that these $H_i, 1 \leqslant i \leqslant k$, are in fact hamiltonian cycles and thus form a hamiltonian decomposition for $cay(A, S)$. Suppose that, in each H_i with $1 \leqslant i \leqslant k$, the jump for the matching between the columns $A_1 \times \{u_{\pi_i(j)}\}$ and $A_1 \times \{u_{\pi_i(j+1)}\}$ is $x_{r(i,j)} \in \{x_1, x_2, \ldots, x_k\}$, where $1 \leqslant j \leqslant |A_2| = 2h+1$, then H_i is isomorphic to $\bigcup_{0 \leqslant t \leqslant 2^d - 1} (t, u_{\pi_i(1)})(t, u_{\pi_i(2)})\cdots(t, u_{\pi_i(2h+1)})(t + x(i), u_{\pi_i(1)})$, where $x(i) \equiv x_{r(i,1)} + x_{r(i,2)} + \cdots + x_{r(i,2h+1)} \pmod{2^d}$. Since all $x_{r(i,j)} \in \{x_1, x_2, \ldots, x_k\}$ are odd, each $x(i) \pmod{2^d}, 1 \leqslant i \leqslant k$, must be an odd integer in $A_1 = Z_{2^d}$. It follows that each H_i is a hamiltonian cycle in $cay(A, S)$ for $1 \leqslant i \leqslant k$. $\square$

3. Preliminary results

We first recall the well-known concept of cartesian product and a result.

Definition 3.1. The cartesian product $G = G_1 \times G_2$ has vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1 \text{ and } u_2v_2 \in E(G_2) \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)\}$.

Theorem 3.2 (Stong [8]). *Let G_1 and G_2 be graphs that are decomposable into n and m hamiltonian cycles, respectively, with $n \leqslant m$. Then $G_1 \times G_2$ is hamiltonian decomposable if one of the following holds*: (1) $m \leqslant 3n$, (2) $n \geqslant 3$, (3) $|G_1|$ *is even, or* (4) $|G_2| \geqslant 6\lceil m/n \rceil - 3$.

From now on throughout this paper, we let $C_1 = a_1a_2\cdots a_na_1$ and $C_2 = b_1b_2\cdots b_mb_1$ be two cycles. By convention, the subscripts of a are expressed modulo n and the subscripts of b are expressed modulo m.

Definition 3.3. For $0 \leqslant r \leqslant m-1$, the r-pseudo-cartesian product of C_1 and C_2, denoted by $C_1 \times_r C_2$, is the graph which is obtained from the cartesian product $C_1 \times C_2$ by replacing the edge set $\{(a_1, b_i)(a_n, b_i) | 1 \leqslant i \leqslant m\}$ by the edge set $\{(a_1, b_{i+r})(a_n, b_i) | 1 \leqslant i \leqslant m\}$.

From the definition, it is easy to see that $C_1 \times_0 C_2 = C_1 \times C_2 = C_1 \times_m C_2$. For convenience, we call the vertex set $\{(a_i, b_j)|1 \leqslant i \leqslant n\}$ the b_j-row and the vertex set $\{(a_i, b_j)|1 \leqslant j \leqslant m\}$ the a_i-column. Also, we call the edges whose two end-vertices have the same first component vertical edges and the edges with different first components horizontal-type edges in an r-pseudo-cartesian product.

Remark 3.4. If $gcd(r, m) = d$ in an r-pseudo-cartesian product $C_1 \times_r C_2$, then the horizontal-type edges form a 2-factor H which consists of d cycles $B_1, B_2, \ldots, B_d$ of length $(mn)/d$, where each cycle B_i consists of the vertices in the b_{jd+i}-rows for $0 \leqslant j \leqslant (m/d) - 1$, and so any consecutive d rows of $C_1 \times_r C_2$ are on d different cycles of H. Moreover, if we give an orientation to H so that each cycle of H becomes a directed cycle, then for $1 \leqslant i \leqslant d$, all the horizontal-type edges in the rows contained in B_i have the same direction.

The following result [3] extended a result of Kotzig that the cartesian product of any two cycles is hamiltonian decomposable (see [4]).

Theorem 3.5 (Fan et al. [3]). *Any pseudo-cartesian product $C_1 \times_r C_2$ of two cycles C_1 and C_2 can be decomposed into two hamiltonian cycles.*

The next three simple facts are useful in our discussion.

Fact 3.6. *If $u_1u_2 \in E(Q_1)$, and $v_1v_2 \in E(Q_2)$, where Q_1 and Q_2 are two vertex-disjoint cycles, then $C = (Q_1 \cup Q_2 - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ is a cycle.*

Fact 3.7. *Given a cycle C, let u_1u_2 and v_1v_2 be edges of C which are separated by at least two edges. Then $(C - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ is a 2-factor containing at most two cycles.*

Fact 3.8. *Given a cycle C, let u_1u_2 and v_1v_2 be two non-adjacent edges of C. If the order of the four end-vertices appearing on C* is *u_1, u_2, v_1, v_2 along a given direction, then $(C - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ is still a cycle.*

For the following discussions, in $C_1 \times_r C_2$, we color all horizontal-type edges by one color, say blue, and all vertical edges by another color, say red.

Definition 3.9. *An* $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-color switching in $C_1 \times_r C_2$ means that we interchange the colors between two edge sets $\{(a_i, b_j)(a_i, b_{j+1}), (a_{i+1}, b_j)(a_{i+1}, b_{j+1})\}$ and $\{(a_i, b_j)(a_{i+1}, b_j), (a_i, b_{j+1})(a_{i+1}, b_{j+1})\}$.

For convenience, one can simply think a color switching as interchanging the colors between one pair of opposite sides and the other pair of opposite sides in a square formed by two adjacent rows and two adjacent columns. In fact, we will indicate each color switching by a square in the following figures.

Lemma 3.10. *If* $gcd(r,m) = 2t+1 \geqslant 3$, *then, by making the color switchings* $1, 2, \ldots, 2t$ in $C_1 \times_r C_2$ *shown in Fig.* 1 *and the color switching* $x = \{a_3, a_4, b_{2t+1}, b_{2t+2}\}$, *we obtain a blue hamiltonian cycle and, connect four red cycles in the* a_j*-columns for* $j = 1, 2, 3, 4$ *to a single red cycle.*

Proof. By Remark 3.4, all blue edges (namely, horizontal-type edges) form $2t+1$ cycles with each row contained in a single blue cycle and no two of the first $2t+1$ rows in $C_1 \times_r C_2$ are on the same blue cycle. Furthermore, b_1-row and b_{2t+2}-row are on the same blue cycle. It follows from Fact 3.6 that making color switchings $1, 2, \ldots, 2t$ in Fig. 1 will end up with a blue hamiltonian cycle H since each of those color switchings connects two different blue cycles. Suppose that we give orientation to the blue hamiltonian cycle H so that it becomes a directed cycle. Then it is clear that all blue edges in each row have the same direction, the blue edges in b_j-row and the blue edges in b_{j+1}-row have opposite directions for each $1 \leqslant j \leqslant 2t$, and all the blue edges in the b_1-row and the b_{2t+2}-row have the same direction, as they were contained in the same blue cycle originally. We conclude that the blue edges in the b_{2t+1}-row and b_{2t+2}-row have the same direction. It follows from Fact 3.8 that we still have a blue hamiltonian cycle after making the additional color switching $x = \{a_3, a_4, b_{2t+1}, b_{2t+2}\}$ in Fig. 1. On the other hand, it is easy to check that the original four red cycles in the a_j-columns for $j = 1, 2, 3, 4$ are now connected to a single red cycle. $\square$

The following lemma is Lemma 1 in [6]. The variable x is used in Figs. 2 and 3 so that we can start color switchings from any row by choosing a value for x. We need this flexibility when we try to avoid repeated use of an edge later.

Lemma 3.11. *Suppose* $n \geqslant 5$ *and* $gcd(r,m) = 2t+1 \geqslant 3$. *Then, by making the color switchings in* $C_1 \times_r C_2$ *shown in Fig.* 2, *we obtain a blue hamiltonian cycle and connect*

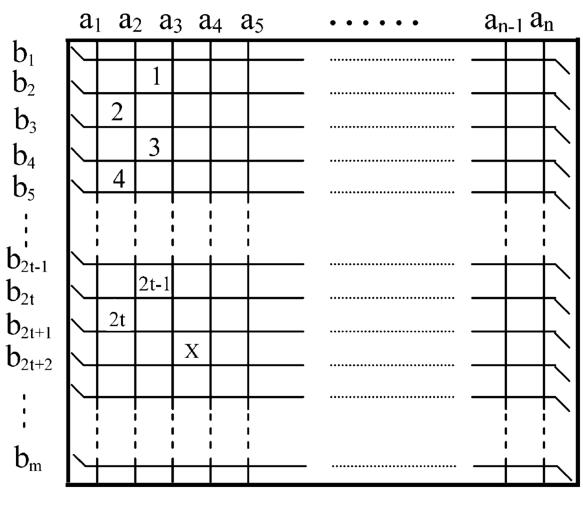


Fig. 1. Color switchings in $C_1 \times_r C_2$ (drawn on a torus).

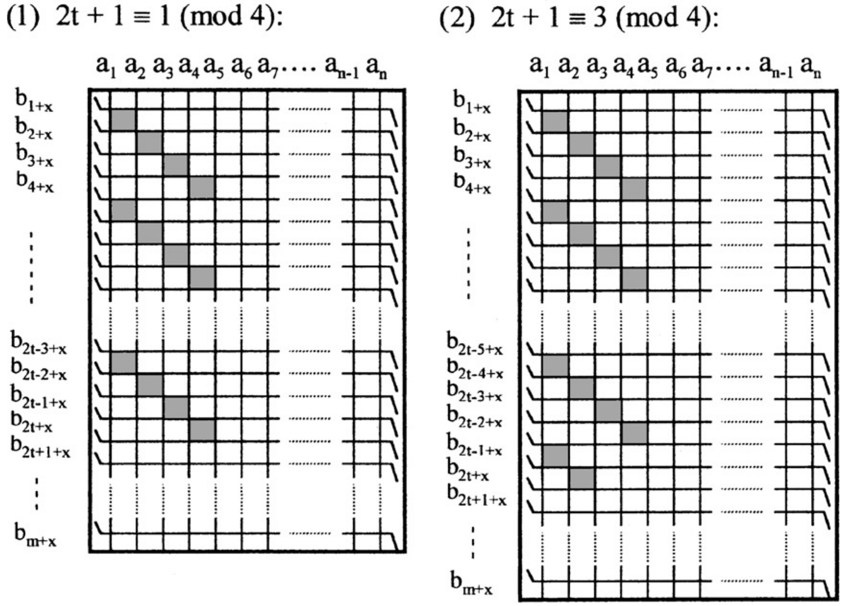


Fig. 2. Color switchings in $C_1 \times_r C_2$ (drawn on a torus), where $0 \leqslant x \leqslant m-1$.

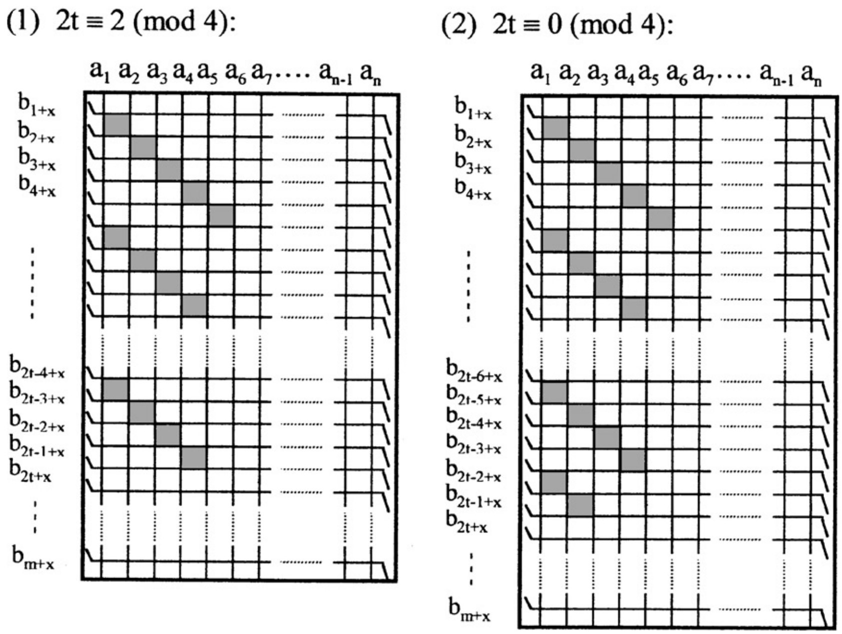


Fig. 3. Color switchings in $C_1 \times_r C_2$ (drawn on a torus), where $0 \leqslant x \leqslant m-1$.

the red cycles in the a_j-columns for $1 \leqslant j \leqslant y$ to a single red cycle, where $y = 3$ if $2t + 1 = 3$, and $y = 5$ otherwise.

Proof. It is clear from Remark 3.4 and Fact 3.6 that the color switchings in Fig. 2 will result in a blue hamiltonian cycle. To see that the a_j-columns for $1 \leqslant j \leqslant y$ are connected to a single red cycle, let us label those color switchings shown in Fig. 2 from top to bottom by $1, 2, \ldots, 2t$. If $2t + 1 = 3$, the result is obvious. Suppose $2t + 1 \geqslant 5$. We will make those color switchings in the increasing order as follows: We first make the color switchings 1,2,3,4 to connect the first five columns to a single red cycle; then make the remaining color switchings a pair $2i - 1$ and $2i$ at each time for $i = 3, 4, \ldots, t$. Each time the color switching $2i - 1$ separates the single red cycle into two red cycles by Fact 3.7, and then the color switching $2i$ connects the two red cycles to form a single red cycle again by Fact 3.6. □

Similar to Lemma 3.11, when $gcd(r, m)$ is even, the next lemma can be seen from Remark 3.4 and Facts 3.6 and 3.7.

Lemma 3.12. *Suppose $n \geqslant 6$. If $gcd(r, m) = 2t$, then, by making the $\{a_i, a_{i+1}, b_i, b_{i+1}\}$-color switchings, where $i = 1$ for $t = 1$ and $i = 1, 2, 3$ for $t = 2$, or the color switchings in $C_1 \times_r C_2$ shown in Fig. 3 for $t \geqslant 3$, we obtain a blue hamiltonian cycle and a red cycle consisting of all the vertices in the a_i-columns for $1 \leqslant i \leqslant y$, where $y = 2$ if $t = 1, 4$ if $t = 2$, and 6 if $t \geqslant 3$.*

At this point, we would like to make a useful remark to Lemmas 3.11 and 3.12.

Remark 3.13. For $m \geqslant 6$, after we apply Lemma 3.11 for $gcd(r, m)$ odd or Lemma 3.12 for $gcd(r, m)$ even to $C_1 \times_r C_2$, each a_i-column, $1 \leqslant i \leqslant n$, has the following property: for any $1 \leqslant f \leqslant m$, at least one of the two vertical edges $e_f = (a_i, b_f)(a_i, b_{f+1})$ and $e_{f+2} = (a_i, b_{f+2})(a_i, b_{f+3})$ is red. Moreover, each a_i-column, $1 \leqslant i \leqslant n$, has either a red P_3 if $gcd(r, m)$ is odd or a red P_4 if $gcd(r, m)$ is even or $m \geqslant 2\, gcd(r, m)$, where P_j is a path on j vertices.

When $gcd(r, m)$ is even, r and m must be even and we have the following simple lemma.

Lemma 3.14. *If $gcd(r, m)$ is even, then, by switching the colors of the two edge sets $E_1 = \{(a_1, b_{2j-1})(a_1, b_{2j}) | 1 \leqslant j \leqslant m/2\} \cup \{(a_n, b_{2j})(a_n, b_{2j+1}) | 1 \leqslant j \leqslant m/2\}$ and $E_2 = \{(a_n, b_i)(a_1, b_{i+r}) | 1 \leqslant i \leqslant m\}$ in $C_1 \times_r C_2$, we obtain a blue hamiltonian cycle as shown with bold edges in Fig. 4 and a red cycle Q consisting of the vertices in the a_1-column and a_n-column. Moreover if we orient Q into a directed cycle, then all the vertical red edges in the a_1-column and a_n-column have the same direction.*

The next special class of graphs (called D_k in [5, Definition 3.8]) plays a key role in our discussion.

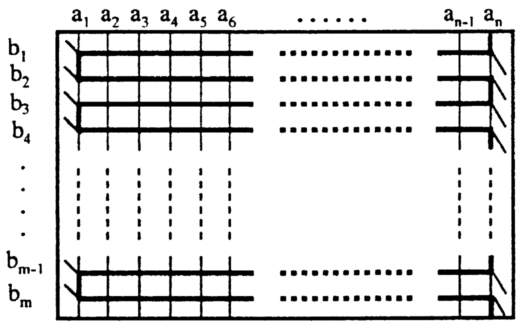


Fig. 4. A hamiltonian cycle in $C_1 \times_r C_2$ (drawn on a torus).

Definition 3.15. For $k \geqslant 2$ and $m, n \geqslant 3$, define $D(k,m,n)$ to be a $2k$-regular graph satisfying:

(1) $V(D(k,m,n)) = \{(a_i, b_j) | 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m\}$,
(2) $E(D(k,m,n))$ can be decomposed into 2-factors $H_1, H_2, \ldots, H_{k-1}$ and F,
(3) $F = \bigcup_{i=1}^{n} F_i$, where each F_i is the cycle $(a_i, b_1)(a_i, b_2) \cdots (a_i, b_m)(a_i, b_1)$, and
(4) for $1 \leqslant j \leqslant k-1$, $H_j \cup F = C_1^j \times_{r_j} C_2$ with the edges of F being vertical, where $0 \leqslant r_j \leqslant m-1$ and $C_1^j = a^{(j)}_{\pi_j(1)} a^{(j)}_{\pi_j(2)} \cdots a^{(j)}_{\pi_j(n)} a^{(j)}_{\pi_j(1)}$ with π_j being a permutation of $\{1, 2, \ldots, n\}$ and $(a_i^{(j)}, b_t) = (a_i, b_{t+h_{i,j}})$ for $1 \leqslant i \leqslant n, 1 \leqslant t \leqslant m$, and $0 \leqslant h_{i,j} \leqslant m-1$.

Clearly, in each $H_j \cup F = C_1^j \times_{r_j} C_2$ for $1 \leqslant j \leqslant k-1$ of a graph $D(k,m,n)$, the vertical edges form F and the horizontal-type edges form H_j. The example shown in Fig. 5 is a graph $D(3,5,6)$ with $r_1 = 1, r_2 = 2, \pi_1 = I$ (the identity), $\pi_2 = (1\,2\,4\,6\,5\,3)$, $h_{i,1} = 0$ and $h_{i,2} \equiv 3i - 1 \bmod 5$ for $1 \leqslant i \leqslant 6$.

For graphs $D(3,m,n)$, Lemma 3.14 in [5] and Lemma 3.9 in [3] together give the next result, where D_3 is replaced by $D(3,m,n)$.

Proposition 3.16. *Suppose that each H_i in a $D(3,m,n)$ consists of $2t_i + 1 \geqslant 3$ cycles for $i = 1$ and 2. If the sets $K_1 = \{\pi_1(1), \pi_1(2), \pi_1(3), \pi_1(4)\}$ and $K_2 = \{\pi_2(1), \pi_2(2), \pi_2(3)\}$ have exactly one common element $\pi_1(4) = \pi_2(1)$, then $D(3,m,n)$ has a hamiltonian decomposition.*

Next, we will show that under certain conditions, graphs $D(k,m,n)$ can be decomposed into hamiltonian cycles. To do so, we first color the edges of $D(k,m,n)$ so that all edges of F are of red color and for $1 \leqslant j \leqslant k-1$, all edges of H_j are of color c_j; we then try to find some edge-disjoint color switchings so that making those color switchings results in k monochromatic hamiltonian cycles.

Lemma 3.17. *Let n be even, $k \geqslant 2$, and $m \geqslant 5$. If each H_i in a $D(k,m,n)$ consists of $2t_i + 1$ cycles for $1 \leqslant i \leqslant k-1$, and the sets K_j for $1 \leqslant j \leqslant m-1$ are mutually disjoint, where $K_1 = \{\pi_1(1), \pi_1(2), \pi_1(3), \pi_1(4)\}$ and $K_j = \{\pi_j(i) | 1 \leqslant i \leqslant 5\}$ for $2 \leqslant j \leqslant k-1$, then $D(k,m,n)$ has a hamiltonian decomposition.*

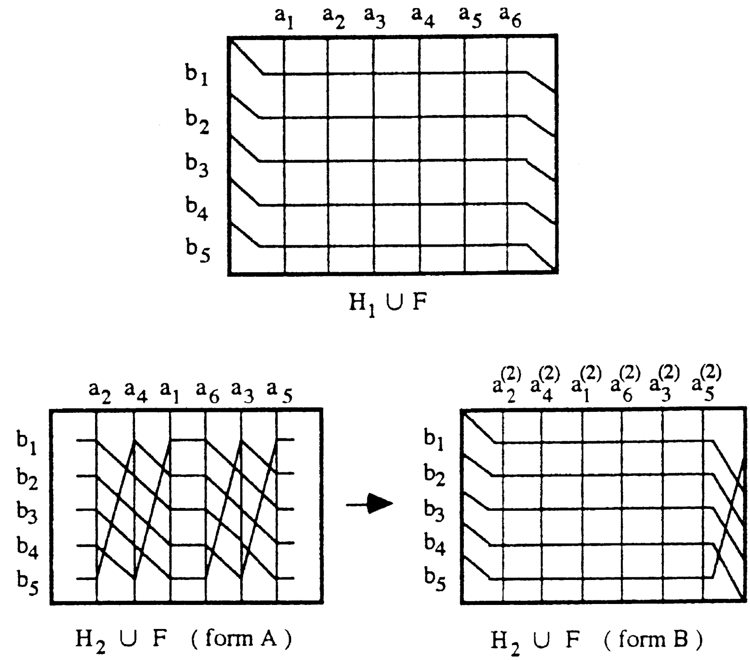


Fig. 5. A graph $D(3,5,6)$ (drawn on a torus).

Proof. Clearly, in each $H_j \cup F = C_1^j \times_{r_j} C_2$ of $D(k,m,n)$ for $1 \leqslant j \leqslant k-1$, the vertical edges form F and thus are red, the horizontal-type edges form H_j and thus are of color c_j. Without loss of generality, we assume π_1 to be the identity permutation and $H_1 \cup F = C_1 \times_{r_1} C_2$ in $D(k,m,n)$, and so $K_1 = \{1,2,3,4\}$. By Remark 3.4, for each $1 \leqslant j \leqslant k-1, 2t_j + 1 = gcd(r_j, m)$ and each row of $H_j \cup F$ (in the sense that we visualize $H_j \cup F$ as in (form B) of Fig. 5) is in the same cycle of H_j. If $t_j = 0$ for some j, then H_j is already a hamiltonian cycle and so we only need to work on the remaining graph $D(k-1,m,n)$ obtained by removing the edges of H_j from $D(k,m,n)$ unless $k=2$, in this case $D(2,m,n) = H_1 \cup F = C_1 \times_{r_1} C_2$ is hamiltonian decomposable by Theorem 3.5. Thus, we may assume that $t_j \geqslant 1$ for each $1 \leqslant j \leqslant k-1$. Since the sets K_j for $1 \leqslant j \leqslant k-1$ are mutually disjoint, we can make edge-disjoint color switchings as follows: First, apply Lemma 3.10 to $H_1 \cup F = C_1 \times_{r_1} C_2$ to obtain a hamiltonian cycle H^* of color c_1 and connect the a_i-columns for $i = 1,2,3,4$ to a single red cycle Q_1. Note that the hamiltonian cycle H^* contains two subpaths $P' = (a_n, b_1)(a_{n-1}, b_1)\dots(a_4, b_1)(a_3, b_1)(a_3, b_2)(a_4, b_2)\dots(a_n, b_2)$ and $P'' = (a_n, b_3)(a_{n-1}, b_3)\dots(a_{h+1}, b_3)(a_h, b_3)(a_h, b_4)(a_{h+1}, b_4)\dots(a_n, b_4)$, where $h = 4$ for $t_1 = 1$ and $h = 3$ for $t_1 \geqslant 2$. Next, for each $2 \leqslant j \leqslant k-1$, we apply Lemma 3.11 to $H_j \cup F = C_1^j \times_{r_j} C_2$ to obtain a hamiltonian cycle of color c_j and connect the $a_{\pi_j(i)}$-columns for $1 \leqslant i \leqslant y$ to a single red cycle Q_j, where $y = 3$ if $gcd(r_j, m) = 2t_j + 1 = 3$, and $y = 5$ otherwise. We remark that when we apply Lemma 3.11 to

$H_j \cup F$ for each $2 \leqslant j \leqslant k-1$, the value of x can be any integer between 0 and $m-1$ which allows us to have a red P_3 (see Remark 3.13) anywhere as we wish in the leftmost a_{l_j}-column of Q_j, where l_j is the smallest index among all a_i-columns contained in Q_j.

Note that we connect four columns to form a single red cycle in $H_1 \cup F$ and connect an odd number of columns to a single red cycle in each $H_j \cup F$ for $j \geqslant 2$. Since n is even and each column is contained in a single red cycle, we now have an odd number $2d+1$ of red cycles, where each red cycle is either a Q_j or a cycle consisting of a single column. Let $0 < z_1 < z_2 < \cdots < z_{2d} < n$ be the integer sequence such that the a_{z_i+1}-columns for $1 \leqslant i \leqslant 2d$ are the leftmost columns of those $2d$ red cycles other than Q_1, where the leftmost column of a red cycle is the column with the smallest index among all columns contained in that red cycle. Then $z_1 = 4$. To connect these red cycles to form a red hamiltonian cycle, we next focus on $H_1 \cup F = C_1 \times_{r_1} C_2$ and find $2d$ additional color switchings $X_1, X_2, \ldots, X_{2d}$ between red edges and edges of color c_1 such that each X_i is between the a_{z_i}-column and the a_{z_i+1}-column. By Remark 3.13, each a_i-column for $4 \leqslant i \leqslant n$ has the following property P: for any $1 \leqslant f \leqslant m$, at least one of the vertical edges $(a_i, b_f)(a_i, b_{f+1})$ and $(a_i, b_{f+2})(a_i, b_{f+3})$ is red. Note that each a_{z_i+1}-column is either the leftmost column of a red cycle consisting of a single column which means all of its vertical edges are red, or the leftmost column of a red cycle Q_j and so we can have a red P_3 in the a_{z_i+1}-column anywhere as we wish according to the earlier remark. Thus, when we define the color switchings $X_1, X_2, \ldots, X_{2d}$ in the following, which only need to use either one red vertical edge or two adjacent red vertical edges from each involved column, we can use red vertical edges freely in each of the a_{z_i+1}-columns. For $1 \leqslant i \leqslant d$, having $X_1, X_2, \ldots, X_{2i-3}, X_{2i-2}$ defined, we then define X_{2i-1} and X_{2i}: Since the $a_{z_{2i}}$-column has property P, one of the two edges $e_1 = (a_{z_{2i}}, b_1)(a_{z_{2i}}, b_2)$ and $e_2 = (a_{z_{2i}}, b_3)(a_{z_{2i}}, b_4)$ is red, we define X_{2i} to be the $\{a_{z_{2i}}, a_{z_{2i}+1}, b_y, b_{y+1}\}$-color switching, where $y = 1$ if e_1 is red, and $y = 3$ otherwise. Then, since one of $e_3 = (a_{z_{2i-1}}, b_{y-1})(a_{z_{2i-1}}, b_y)$ and $e_4 = (a_{z_{2i-1}}, b_{y+1})(a_{z_{2i-1}}, b_{y+2})$ is red by property P, we define X_{2i-1} to be either $\{a_{z_{2i-1}}, a_{z_{2i-1}+1}, b_{y-1}, b_y\}$-color switching or $\{a_{z_{2i-1}}, a_{z_{2i-1}+1}, b_{y+1}, b_{y+2}\}$-color switching depending on whether e_3 or e_4 is red. Clearly, the color switchings $\{X_1, X_2, \ldots, X_{2d}\}$ are edge-disjoint and between red edges and edges of color c_1. Since each color switching X_i connects two different red cycles, by Fact 3.6, making the color switchings $X_1, X_2, \ldots, X_{2d}$ will result in a red hamiltonian cycle. Now, we claim that applying the color switchings $X_1, X_2, \ldots, X_{2d}$ in $H_1 \cup F$ will still end up with a hamiltonian cycle of color c_1. In fact, starting with the hamiltonian cycle H^* of color c_1 which contains two subpaths P' and P'' as noted earlier (P' and P'' are still part of H^* since we did not use any of the edges from H^* before making the color switchings X_j's), let us make those color switchings in the order $X_{2d}, X_{2d-1}, \ldots, X_2, X_1$, a pair X_{2i-1}, X_{2i} at each time: by Fact 3.7, making color switching $X_{2i} = \{a_{z_{2i}}, a_{z_{2i}+1}, b_y, b_{y+1}\}$ with $y = 1$ or 3 will separate the hamiltonian cycle of color c_1 into two cycles with one being $(a_{z_{2i}}, b_1)(a_{z_{2i}-1}, b_1)\ldots(a_4, b_1)(a_3, b_1)(a_3, b_2)(a_4, b_2)\ldots(a_{z_{2i}}, b_2)(a_{z_{2i}}, b_1)$ if $y = 1$ or $(a_{z_{2i}}, b_3)(a_{z_{2i}-1}, b_3)\ldots(a_{h+1}, b_3)(a_h, b_3)(a_h, b_4)(a_{h+1}, b_4)\ldots(a_{z_{2i}}, b_4)(a_{z_{2i}}, b_3)$ if $y = 3$; and then by Fact 3.6, making the color switching X_{2i-1} will connect the two cycles of color c_1 into a hamiltonian cycle of color c_1 again. Thus, the claim follows. Together with the existing monochromatic hamiltonian cycles of colors c_j for $2 \leqslant j \leqslant k-1$, we obtain a hamiltonian decomposition of $D(k,m,n)$. $\square$

In the following proof, we call a path $P_4 = (a_i, b_y)(a_i, b_{y+1})(a_i, b_{y+2})(a_i, b_{y+3})$ in an a_i-column of $C_1 \times_{r_1} C_2$ *odd* if y is odd, and *even* if y is even. Also, we call a vertical edge $(a_i, b_y)(a_i, b_{y+1})$ *odd* if y is odd, and *even* if y is even.

Lemma 3.18. *Suppose that n is even, $k \geqslant 2$, and $m \geqslant 6$. If H_1 in $D(k,m,n)$ consists of an even number of cycles, and the sets $K_1 = \{\pi_1(1), \pi_1(n)\}$ and $K_j = \{\pi_j(i) | 1 \leqslant i \leqslant 6\}$ for $2 \leqslant j \leqslant k-1$ are mutually disjoint, then $D(k,m,n)$ has a hamiltonian decomposition.*

Proof. We proceed in a way similar to the proof of Lemma 3.17. Clearly, in each $H_j \cup F = C_1^j \times_{r_j} C_2$ of $D(k,m,n)$ for $1 \leqslant j \leqslant k-1$, the vertical edges form F and thus are red, the horizontal-type edges form H_j and thus are of color c_j. Again, without loss of generality, we assume π_1 to be the identity permutation and $H_1 \cup F = C_1 \times_{r_1} C_2$ in $D(k,m,n)$, and so $K_1 = \{1, n\}$. By Remark 3.4, each H_j consists of $h_j = gcd(r_j, m)$ cycles for $1 \leqslant j \leqslant k-1$. By the assumption, h_1 is even. This implies that m is even and $m \geqslant 2h_j$ for each odd h_j. Since the sets K_j for $1 \leqslant j \leqslant k-1$ are mutually disjoint, we make edge-disjoint color switchings in each $H_j \cup F$ within a_i-columns for $i \in K_j$ as follows: We first apply Lemma 3.14 to $H_1 \cup F = C_1 \times_{r_1} C_2$ to obtain a hamiltonian cycle H^* of color c_1 and connect the a_1-column and a_n-column to a single red cycle Q_1. Note that all even edges in the a_1-column and all odd edges in the a_n-column are still red. Next, for each $2 \leqslant j \leqslant k-1$, we apply Lemma 3.11 when $h_j > 1$ is odd or Lemma 3.12 when h_j is even to $H_j \cup F = C_1^j \times_{r_j} C_2$ to obtain a hamiltonian cycle of color c_j and connect the involved a_i-columns with $i \in K_j$ to a single red cycle Q_j, where the value of x can be any integer between 0 and $m-1$ which allows us to have a red P_4 anywhere as we wish in the leftmost column of Q_j (such a red P_4 exists by Remark 3.13 and the fact that $m \geqslant 2h_j$ when $h_j = gcd(r_j, m)$ is odd). Clearly, the red edges still form a 2-factor and each column is contained in a single red cycle. By Remark 3.13, each a_i-column for $2 \leqslant i \leqslant n-1$ has the following property P: for any $1 \leqslant f \leqslant m$, at least one of the vertical edges $(a_i, b_f)(a_i, b_{f+1})$ and $(a_i, b_{f+2})(a_i, b_{f+3})$ is red and there is a red P_4 in the a_i-column.

Clearly, each column is either a red cycle or contained in a red cycle Q_j. Suppose that the resulting red 2-factor F' has p red cycles. To connect those red cycles to a red hamiltonian cycle, we focus on $H_1 \cup F = C_1 \times_{r_1} C_2$. Similar to Lemma 3.17, we will find an integer sequence $0 < z_1 < z_2 < \cdots < z_{2d} < n$ and additional color switchings $X_1, X_2, \ldots, X_{2d}$ between red edges and edges of color c_1 in $H_1 \cup F$, where each X_i is between the a_{z_i}-column and the a_{z_i+1}-column, such that making these color switchings will end up with a red hamiltonian cycle and a hamiltonian cycle of color c_1, thereby a hamiltonian decomposition of $D(k,m,n)$.

We first find the desired integer sequence $0<z_1<z_2<\cdots<z_{2d}<n$. Let $0<z'_1<z'_2<\cdots<z'_{p-1}$ be the integer sequence such that the $a_{z'_i+1}$-columns for $1\leqslant i\leqslant p-1$ are the leftmost columns of those $p-1$ red cycles other than Q_1. Then $z'_1=1$. If $p-1$ is even, then $p-1=2d$ for some d and we have the desired sequence by setting $z_i=z'_i$ for $1\leqslant i\leqslant 2d$. Assume that $p-1$ is odd, i.e., p is even. Suppose that we have $p-1$ edge-disjoint color switchings Y_i for $1\leqslant i\leqslant p-1$ between red edges and edges of color c_1, with Y_i between $a_{z'_i}$-column and $a_{z'_i+1}$-column. Then, it follows from Fact 3.6 that by making those color switchings $Y_1,\ldots,Y_{p-1}$, we have a red hamiltonian cycle Q. If we give an orientation to Q to obtain a directed cycle, then it is clear that all red edges in the same column have the same direction and for each $1\leqslant i\leqslant p-1$, the red edges in $a_{z'_i}$-column and the red edges in $a_{z'_i+1}$-column have opposite direction. By Lemma 3.14, all red vertical edges in both a_1-column and a_n-column have the same direction. Since n is even, there must exist $1\leqslant z\leqslant n-1$ such that all red vertical edges in both a_z-column and a_{z+1}-column have the same direction. This implies that $z\neq z'_j$ for $1\leqslant j\leqslant p-1$. It follows that a_{z+1}-column is a non-leftmost column of some red cycle Q_w. By Fact 3.8, after making any one additional color switching Y between the a_z-column and a_{z+1}-column using one red edge from each column, we still have a red hamiltonian cycle. Inserting z to the sequence $z'_1<z'_2<\cdots<z'_{p-1}$, we obtain the desired integer sequence $z_1<z_2<\cdots<z_{2d}$, where $2d=p$.

Now, we will find desired edge-disjoint color switchings $X_1,X_2,\ldots,X_{2d}$ between red edges and edges of color c_1 in $H_1\cup F$. Note that for each $z_i\neq z$, the a_{z_i+1}-column is either the leftmost column of a red cycle consisting of a single column which means all of its vertical edges are red, or the leftmost column of a red cycle Q_j and so we can have a red P_4 in that column anywhere as we wish according to the earlier remark. For $1\leqslant i\leqslant d$, having $X_1,X_2,\ldots,X_{2i-3},X_{2i-2}$ defined, we then define X_{2i-1} and X_{2i} in two cases:

Case 1: $z\neq z_{2i-1},z_{2i}$. We define X_{2i-1} and X_{2i} in the same way as in the proof of Lemma 3.17.

Case 2: $z=z_{2i-1}$ or z_{2i}. In this case, the a_{z+1}-column is a non-leftmost column of some red cycle Q_w. For $z+1<n$, the a_{z+1}-column is a non-leftmost column of some red cycle Q_w with $w>1$, and so both the a_{z+1}-column and the leftmost a_{l_w}-column of Q_w have a red P_4 by Property P. Clearly, only the edges from a $D=P_3=(a_{l_w},b_h)(a_{l_w},b_{h+1})(a_{l_w},b_{h+2})$ in the a_{l_w}-column may be used by previously defined X_j's. It follows that, when we apply Lemma 3.11 or Lemma 3.12 to $H_w\cup F$ above, we may choose the value for x appropriately so that the $D=P_3$ is part of a red P_4 in the leftmost a_{l_w}-column of Q_w meanwhile we can make a red P_4 in the a_{z+1}-column either odd or even. Thus, for $z+1<n$, we may assume that the a_{z+1}-column has an even red $P_4=(a_{z+1},b_{2q})(a_{z+1},b_{2q+1})(a_{z+1},b_{2q+2})(a_{z+1},b_{2q+3})$ for $z=z_{2i-1}$, or an odd red $P_4=(a_{z+1},b_{2q-1})(a_{z+1},b_{2q})(a_{z+1},b_{2q+1})(a_{z+1},b_{2q+2})$ for $z=z_{2i}$. Recall that each a_i-column has property P for $2\leqslant i\leqslant n-1$, all even vertical edges in the a_1-column and all odd vertical edges in the a_n-column are red. For $z=z_{2i}$, we define color switchings X_{2i-1} and X_{2i} in the same way as in the proof of Lemma 3.17 except that we take $y=2q-1$ or $2q+1$ depending on whether $(a_{z_{2i}},b_{2q-1})(a_{z_{2i}},b_{2q})$ or

$(a_{z_{2i}}, b_{2q+1})(a_{z_{2i}}, b_{2q+2})$ is red. For $z = z_{2i-1}$, we first define X_{2i-1} to be $\{a_{z_{2i-1}}, a_{z_{2i-1}+1}, b_y, b_{y+1}\}$-color switching for $y = 2q$ or $2q+2$ depending on whether $(a_{z_{2i-1}}, b_{2q})(a_{z_{2i-1}}, b_{2q+1})$ or $(a_{z_{2i-1}}, b_{2q+2})(a_{z_{2i-1}}, b_{2q+3})$ is red, then define X_{2i} to be $\{a_{z_{2i}}, a_{z_{2i}+1}, b_{y-1}, b_y\}$-color switching or $\{a_{z_{2i}}, a_{z_{2i}+1}, b_{y+1}, b_{y+2}\}$-color switching depending on whether $(a_{z_{2i}}, b_{y-1})(a_{z_{2i}}, b_y)$ or $(a_{z_{2i}}, b_{y+1})(a_{z_{2i}}, b_{y+2})$ is red. Clearly, each X_{2j-1} uses only even red vertical edges while each X_{2j} uses only odd red vertical edges, and $X_1, \ldots, X_{2d}$ are edge-disjoint.

From the choice of the integer sequence $0 < z_1 < z_2 < \cdots < z_{2d} < n$, it follows that making the color switchings $X_1, X_2, \ldots, X_{2d}$ results in a red hamiltonian cycle. Note that the hamiltonian cycle H^* of color c_1 contains subpaths $P(j) = (a_n, b_{2j-1})(a_{n-1}, b_{2j-1})\ldots(a_2, b_{2j-1})(a_1, b_{2j-1})(a_1, b_{2j})(a_2, b_{2j})\ldots(a_n, b_{2j})$ for $1 \leqslant j \leqslant m/2$. Similar to the proof of Lemma 3.17, we conclude that we still end up with a hamiltonian cycle of color c_1 after those color switchings X_i's. Together with the existing hamiltonian cycles of colors c_j for $2 \leqslant j \leqslant k-1$, we obtain a hamiltonian decomposition for $D(k, m, n)$. $\square$

4. Proof of Theorem 1.3

Let A be a finite abelian group, $S = \{s_1, s_2, \ldots, s_k\}$ be a generating set of A such that $0 \notin S$ and $s \in S$ implies $-s \notin S$. For $J = \langle s_k \rangle$, let $A_1 = A/J$ and $\bar{S} = \{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{k-1}\}$, where we use $\bar{x}$ to represent the coset $x + J$. Let $m = ord(s_k) \geqslant 3$. Then all edges in $cay(A, S)$ which are generated by s_k form a 2-factor F. Furthermore, F consists of $n = |A_1| = |A|/m$ cycles of length m.

Definition 4.1. For any edge $\bar{x}\,\bar{y}$ of $cay(A_1, \bar{S})$, where $\bar{x} - \bar{y} = \bar{s}_i \in \bar{S}$, we call the edge set $\{u_1u_2 | \bar{u}_1 = \bar{x}, \bar{u}_2 = \bar{y}$ and $u_1 - u_2 = s_i\}$ of $cay(A, S)$ the *lifting edge set* of the edge $\bar{x}\ \bar{y}$.

Definition 4.2. For any subgraph $\bar{Q}$ of $cay(A_1, \bar{S})$, the subgraph Q of $cay(A, S)$ with the edge set being the union of the lifting edge sets of the edges of $\bar{Q}$ is called the subgraph lifted by $\bar{Q}$ and we say $\bar{Q}$ lifts to Q.

It is easy to see from the above definitions that edge-disjoint subgraphs of $cay(A_1, \bar{S})$ lift to edge-disjoint subgraphs of $cay(A, S)$.

The following connection between Cayley graphs $cay(A, S)$ on abelian group A and graphs $D(k, m, n)$ is Lemma 5 in [6] (where D_k is changed to $D(k, m, n)$ here).

Proposition 4.3. *If the Cayley $cay(A_1, \bar{S})$ can be decomposed into $k-1$ hamiltonian cycles $\bar{H}_j = \bar{g}_{\pi_j(1)}\bar{g}_{\pi_j(2)}\cdots\bar{g}_{\pi_j(n)}\bar{g}_{\pi_j(1)}$ for $1 \leqslant j \leqslant k-1$, where each π_j is a permutation of $\{1, 2, \ldots, n\}$, then $cay(A, S) \cong D(k, m, n)$ with each H_j being the 2-factor lifted by $\bar{H}_j$ and F being the 2-factor generated by s_k.*

The above proposition allows us to obtain a hamiltonian decomposition of Cayley graph $cay(A, S)$ by applying Lemmas 3.17 and 3.18. To guarantee the conditions imposed on π_j in Lemmas 3.17 and 3.18 are satisfied, we give the following simple results.

Lemma 4.4. *If A is a finite abelian group of even order at least 4 and $S = \{s_1, s_2, \dots, s_k\}$ is a strongly minimal generating set of A, then $|A| \geqslant 4 \cdot 3^{k-1}$.*

Proof. We proceed by induction on k. For $k = 1$, the result is clear. Assume the result for $k < h$ (with $h \geqslant 2$). Now, we consider $k = h$. Since A is of even order and S is a generating set of A, S has an element of even order, say s_1. Let $S' = \{s_1, s_2, \dots, s_{k-1}\}$ and $A' = \langle S' \rangle$. Then A' has even order and S' is a strongly minimal generating set of A' as S is strongly minimal. By the induction hypothesis, $|A'| \geqslant 4 \cdot 3^{k-2}$. Since S is a strongly minimal generating set of A, there are at least three cosets of A' in A. Thus, $|A| \geqslant 3|A'| \geqslant 4 \cdot 3^{k-1}$. $\square$

The following two propositions are Lemmas 2.5 and 3.3 in [5].

Proposition 4.5. *Let A be a finite abelian group which is generated by $S = \{s_1, s_2, \dots, s_k\}, A_1$ be the subgroup of A which is generated by $S' = \{s_1, s_2, \dots, s_{k-1}\}$ and $J = \langle s_k \rangle$. If $A_1 \cap J = \{0\}$, then $cay(A, S) = cay(A_1, S') \times cay(J, \{s_k\})$.*

Proposition 4.6. *Let G be hamiltonian decomposable 4-regular graph of order $n \geqslant 9$. Then, for each hamiltonian decomposition H_1 and H_2 of G, there exists a path $P = u_1u_2u_3u_4u_5u_6$ such that $P_1 = u_1u_2u_3u_4$ is on H_1 while $P_2 = u_4u_5u_6$ is on H_2.*

Lemma 4.7. *For $d \geqslant 2$, if G is a hamiltonian decomposable $2d$-regular multigraph of order $n > \max\{12(d-1), 36(d-2)\}$, then, for any hamiltonian decomposition $H_1, H_2, \dots, H_d$ of G, there are d vertex-disjoint paths $P(1) = u_1v_1$ and $P(j) = u_jv_jw_jx_jy_jz_j$ for $2 \leqslant j \leqslant d$ such that $P(i)$ is on H_i for $1 \leqslant i \leqslant d$.*

Proof. For $t \leqslant d-1$, suppose t subpaths $P(j), d-t+1 \leqslant j \leqslant d$, have been chosen on the cycles H_j. The $6t$ vertices of those $P(j)$'s divide the remaining $n-6t$ vertices of H_{d-t} into $6t$ subpaths and let P be a longest one of those subpaths. Then P has at least 6 vertices if $t \leqslant d-2$ and at least two vertices if $t = d-1$ as $n > \max\{12(d-1), 36(d-2)\}$. Thus, we may choose a desired subpath of P to be $P(d-t)$. $\square$

Similarly, we can derive the following lemma.

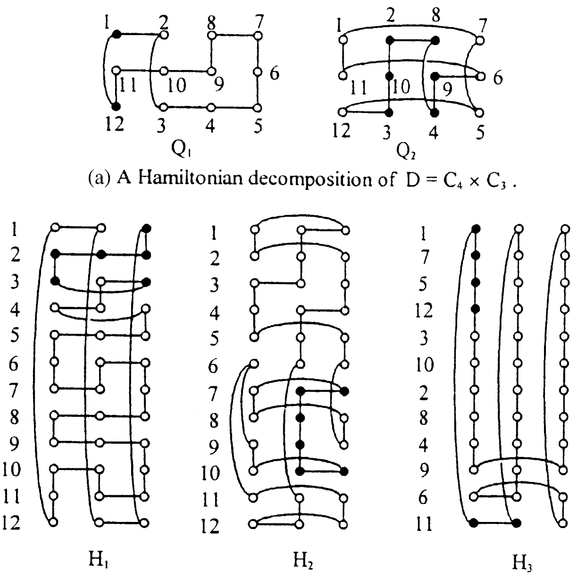
Lemma 4.8. *Let $d \geqslant 2$. If G is a hamiltonian decomposable $2d$-regular multigraph of order $n > \max\{20(d-1), 25(d-2)\}$, then, for any hamiltonian decomposition $H_1, H_2, \dots, H_d$ of G, there are d vertex-disjoint paths $P(1) = u_1v_1w_1x_1$ and $P(j) = u_jv_jw_jx_jy_j$ for $2 \leqslant j \leqslant d$ such that $P(i)$, is on H_i for $1 \leqslant i \leqslant d$.*

In the following discussions, we use the standard notation C_i to denote a cycle on i vertices for $i \geqslant 3$.

Remark 4.9. Fig. 6(a) shows a hamiltonian decomposition Q_1, Q_2 of $C_4 \times C_3$ with two disjoint paths of 2 and 6 vertices (shown by black dots) on Q_1 and Q_2, respectively, and Fig. 6(b) shows a hamiltonian decomposition H_1, H_2, H_3 of the graph $G = C_4 \times C_3 \times C_3$ with three vertex-disjoint paths $P(j) = u_j v_j w_j x_j y_j z_j$ (shown by black dots) such that each $P(j)$ is on H_j for $1 \leqslant j \leqslant 3$.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let A be a finite abelian group of even order at least 4 and $S = \{s_1, s_2, \ldots, s_k\}$ be a strongly minimal generating set of A. Then each s_i has order at least 3. We now prove $cay(A, S)$ has a hamiltonian decomposition by induction on k. For $k = 1$, the result is trivial. For $k = 2$, the result follows from Theorem 1.1. Assume the result for $k < k'$. Now, we consider $k = k' \geqslant 3$. We first assume that S has an element s_h of order less than 6. Then $\langle s_h \rangle \cap \langle S - \{s_h\} \rangle = \{0\}$ as S is strongly minimal. It follows from Proposition 4.5 that $cay(A, S) \cong cay(\langle s_h \rangle, \{s_h\}) \times cay(A', S')$, where $S' = S - \{s_h\}$ and $A' = \langle S' \rangle$. Since S is a strongly minimal generating set of A, S' is a strongly minimal generating set of A'. By Theorem 1.2 for $|A'|$ odd or the induction hypothesis for $|A'|$ even, $cay(A', S')$ can be decomposed into $k-1$ hamiltonian cycles. Clearly, $cay(\langle s_h \rangle, \{s_h\})$ is a cycle of length $ord(s_h)$. Since A is of even order, either $ord(s_h)$ is even or A' is of even order and so $|A'| \geqslant 4 \cdot 3^{k-2} \geqslant 6(k-1) - 3$ by Lemma 4.4. It follows from Theorem 3.2(3) or (4) that $cay(A, S) \cong cay(\langle s_h \rangle, \{s_h\}) \times cay(A', S')$ is hamiltonian decomposable. Now, we assume that every element of S has order at least 6. By Theorem 1.4, we may assume



(a) A Hamiltonian decomposition of $D = C_4 \times C_3$.

(b) A Hamiltonian decomposition of $G = C_4 \times C_3 \times C_3 = D \times C_3$.

Fig. 6. Hamiltonian decomposition of $C_4 \times C_3$ and $C_4 \times C_3 \times C_3$.

that there is an element in S, say s_k, such that the quotient group $A_1 = A/J$ is of even order n, where $J = \langle s_k \rangle$. Let $m = |J|$ and $A_1 = \{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n\}$. Then $m \geqslant 6$ and $n \geqslant 4$ is even. Since S is a strongly minimal generating set of A, $\bar{S} = \{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_{k-1}\}$ is a strongly minimal generating set of A_1. By the induction hypothesis, $cay(A_1, \bar{S})$ can be decomposed into $k-1$ hamiltonian cycles $\bar{H}_j = \bar{g}_{\pi_j(1)}\bar{g}_{\pi_j(2)} \cdots \bar{g}_{\pi_j(n)}\bar{g}_{\pi_j(1)}$ for $1 \leqslant j \leqslant k-1$, where each π_j is a permutation of $\{1, 2, \dots, n\}$. By Proposition 4.3, $cay(A, S) \cong D(k, m, n)$ with each H_j being the 2-factor lifted by $\bar{H}_j$. In each $H_j \cup F = C_1^j \times_{r_j} C_2$, let $t_j = gcd(m, r_j)$. By symmetry, we may assume that t_i is even for $1 \leqslant i \leqslant \beta$ and t_i is odd for $i \geqslant \beta + 1$. By Lemma 4.4, $|A_1| \geqslant 4 \cdot 3^{k-2}$. We consider the following two cases.

Case 1: $d = k - 1 \geqslant 3$. Then $|A_1| \geqslant 4 \cdot 3^{k-2} = 4 \cdot 3^{d-1} > \max\{20(d-1), 36(d-2)\}$ unless $d = k - 1 = 3$ and $|A_1| \leqslant 40$. For $d = k - 1 = 3$, since A_1 is of even order $n \geqslant 4 \cdot 3^2$ and $\bar{S} = \{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ is a strongly minimal generating set of A_1, we must have either $|A_1| > 40$ or $A_1 = 36$ and exactly one element in $\bar{S}$ is of order 4 and the other two are of order 3. Furthermore, for the later case, we must have $\langle \overline{s_i} \rangle \cap \langle \overline{s_j} \rangle = \{0\}$ for $i \neq j$ which implies that $cay(A_1, \bar{S}) \cong C_3 \times C_3 \times C_4$ by Proposition 4.5. Thus, we have either $|A_1| > \max\{20(d-1), 36(d-2)\}$ or $cay(A_1, \bar{S}) \cong C_3 \times C_3 \times C_4$. It follows from Lemmas 4.7 and 4.8 and Remark 4.9 that we may assume the permutations π_j for $1 \leqslant j \leqslant k-1$ in the above hamiltonian decomposition of $cay(A_1\bar{S})$ satisfy the conditions in Lemmas 3.17 and 3.18. Thus, by Lemmas 3.17 and 3.18, $cay(A, S) \cong D(k, m, n)$ has a hamiltonian decomposition.

Case 2: $d = k - 1 = 2$. In this case, we have $n = |A_1| \geqslant 12$. For $n = 12$, since $\bar{S} = \{\bar{s}_1, \bar{s}_2\}$ is a strongly minimal generating set of A_1, we must have $\langle \bar{s}_1 \rangle \cap \langle \bar{s}_2 \rangle = \{0\}$ and it follows from Proposition 4.5 that $cay(A_1, \bar{S}) \cong C_3 \times C_4$. If one of t_1 and t_2 is even, say t_1, then it follows from Lemmas 3.18 and 4.7, and Remark 4.9 that $cay(A, S) \cong D(3, m, n)$ has a hamiltonian decomposition. Now, suppose that both t_1 and t_2 are odd. By deleting one H_i for $t_i = 1$ and applying Theorem 3.5, we may assume $t_1 > 1$ and $t_2 > 1$. Then the result follows from Propositions 3.16 and 4.6. $\square$

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