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Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations

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Abstract

We study the asymptotic behavior of linear evolution equations of the type $\partial_t g = Dg + \mathcal{L}g - \lambda g$, where $\mathcal{L}$ is the fragmentation operator, D is a differential operator, and λ is the largest eigenvalue of the operator $Dg + \mathcal{L}g$. In the case $Dg = -\partial_x g$, this equation is a rescaling of the growth-fragmentation equation, a model for cellular growth; in the case $Dg = -\partial_x(xg)$, it is known that $\lambda = 1$ and the equation is the self-similar fragmentation equation, closely related to the self-similar behavior of solutions of the fragmentation equation $\partial_t f = \mathcal{L}f$.

By means of entropy–entropy dissipation inequalities, we give general conditions for g to converge exponentially fast to the steady state G of the linear evolution equation, suitably normalized. In other words, the linear operator has a spectral gap in the natural L^2 space associated to the steady state. We extend this spectral gap to larger spaces using a recent technique based on a decomposition of the operator in a dissipative part and a regularizing part.

Résumé

Nous étudions le comportement asymptotique d'équations d'évolution linéaires du type $\partial_t g = Dg + \mathcal{L}g - \lambda g$, oú $\mathcal{L}$ est l'opérateur de fragmentation, D est un opérateur differentiel, et λ est la plus grande valeur propre de l'opérateur $Dg + \mathcal{L}g$. Dans le cas $Dg = -\partial_x g$, cette équation est obtenue par changement d'échelle à partir de l'équation de croissance-fragmentation, un modèle pour la croissance cellulaire ; dans le cas $Dg = -\partial_x(xg)$, il est connu que $\lambda = 1$ et cela correspond à l'équation de fragmentation autosimilaire, étroitement reliée au comportement autosimilaire des solutions de l'équation de fragmentation $\partial_t f = \mathcal{L}f$.

Au moyen d'une inégalité d'entropie–dissipation d'entropie, nous donnons des conditions générales pour que g tende exponentiellement vite vers l'état stationnaire G de l'équation d'évolution linéaire, avec la normalisation appropriée. En d'autres termes, l'opérateur linéaire admet un trou spectral dans l'espace L^2 naturel associé à l'état stationnaire. Nous étendons ce résultat de trou spectral à un espace plus grand en utilisant une technique utilisant la décomposition de l'opérateur en une partie dissipative et une partie régularisante.

Keywords: Fragmentation; Growth; Entropy; Exponential convergence; Self-similarity; Long-time behavior

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1. Introduction and main results

In this work we study equations which include a differential term and a fragmentation term. These equations are classical models in biology for the evolution of a population of cells, in polymer physics for the size distribution of polymers, and arise in other contexts where there is an interplay between growth and fragmentation phenomena. The literature on concrete applications is quite large and we refer the reader to [7,12] as general sources on the topic, and to the references cited in [3,8,13] for particular applications. We deal with equations of the following type:

$$\partial_t g_t(x) + \partial_x\big(a(x)g_t(x)\big) + \lambda g_t(x) = \mathcal{L}[g_t](x), \tag{1a}$$

$$g_t(0) = 0 \quad (t \geqslant 0), \tag{1b}$$

$$g_0(x) = g_{in}(x) \quad (x > 0). \tag{1c}$$

The unknown is a function $g_t(x)$ which depends on the time $t \geqslant 0$ and on $x > 0$, and for which an initial condition g_{in} is given at time $t = 0$. The quantity $g_t(x)$ represents the density of the objects under study (cells or polymers) of size x at a given time t. The function $a = a(x) \geqslant 0$ is the *growth rate* of cells of size x. Later we will focus on the growth-fragmentation and the self-similar fragmentation equations, which correspond to $a(x) = 1$ and $a(x) = x$, respectively.

Most importantly, we pick λ to be the largest eigenvalue of the operator $g \mapsto -\partial_x(ag) + \mathcal{L}g$, acting on a function $g = g(x)$ depending only on x; see below for known properties of this eigenvalue and its corresponding eigenvector.

The *fragmentation operator* $\mathcal{L}$ acts on a function $g = g(x)$ as

$$\mathcal{L}g(x) := \mathcal{L}_+ g(x) - B(x)g(x), \tag{2}$$

where the positive part $\mathcal{L}_+$ is given by:

$$\mathcal{L}_+ g(x) := \int_x^\infty b(y,x)g(y)\,dy. \tag{3}$$

The coefficient $b(y,x)$, defined for $y > x > 0$, is the *fragmentation coefficient*, and $B(x)$ is the *total fragmentation rate* of cells of size $x > 0$. It is obtained from b through

$$B(x) := \int_0^x \frac{y}{x} b(x,y)\,dy \quad (x > 0). \tag{4}$$

1.1. Asymptotic behavior

As said above, we pick λ to be the largest eigenvalue of the operator $g \mapsto -\partial_x(ag) + \mathcal{L}g$. Under general conditions on b and a, it is known [3,8] that λ is positive and has a unique associated eigenvector G (up to a factor, of course), which in addition is nonnegative; i.e., there is a unique G solution of,

$$\big(a(x)G(x)\big)' + \lambda G(x) = \mathcal{L}(G)(x), \tag{5a}$$

$$a(x)G(x)|_{x=0} = 0, \tag{5b}$$

$$G \geqslant 0, \quad \int_0^\infty G(x)\,dx = 1. \tag{5c}$$

The associated dual eigenproblem reads:

$$-a(x)\partial_x\phi + \big(B(x) + \lambda\big)\phi(x) = \mathcal{L}_+^*\phi(x), \tag{6a}$$

$$\phi \geqslant 0, \quad \int_0^\infty G(x)\phi(x)\,dx = 1, \tag{6b}$$

where

$$\mathcal{L}_+^*\phi(x) := \int_0^x b(x,y)\phi(y)\,dy, \tag{7}$$

and we have chosen the normalization $\int G\phi = 1$. This dual eigenproblem is interesting because ϕ gives a conservation law for (1):

$$\int_0^\infty \phi(x)g_t(x)\,dx = \int_0^\infty \phi(x)g_{in}(x)\,dx = \mathrm{Cst} \quad (t \geqslant 0). \tag{8}$$

The eigenvector G is an equilibrium of Eq. (1) (a solution which does not depend on time) and one expects that the asymptotic behavior of (1) be described by this particular solution, in the sense that

$$g_t \to G \quad \text{as } t \to \infty, \tag{9}$$

with convergence understood in some sense to be specified, and g normalized so that $\int \phi(x)g(t,x)\,dx = 1$. Furthermore, one expects the above convergence to occur exponentially fast in time; this is,

$$\|g_t - G\| \leqslant C\|g_{in} - G\|e^{-\beta t}, \tag{10}$$

for some $\beta > 0$, in some suitable norm. This latter result has been proved in some particular cases which are essentially limited to the case of B constant [13,6]. In this paper we want to prove this result for more general B, which we do by using entropy methods. In order to describe our results, we need first to describe the entropy functional for this type of equations.

1.2. Entropy

The following *general relative entropy principle* [9,10] applies to solutions of (1):

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \phi(x)G(x)H\big(u(x)\big)\,dx \\
&\quad = \int_0^\infty\int_y^\infty \phi(y)b(x,y)G(x)\big(H\big(u(x)\big) - H\big(u(y)\big) + H'\big(u(x)\big)\big(u(y)-u(x)\big)\big)\,dx\,dy,
\end{aligned} \tag{11}$$

where H is any function and, where here and below,

$$u(x) := \frac{g(x)}{G(x)} \quad (x > 0). \tag{12}$$

When H is a convex function, the right-hand side of Eq. (11) is nonpositive, and in the particular case of $H(x) := (x-1)^2$ we have:

$$\frac{d}{dt}H_2[g|G] = -D^b[g|G] \leqslant 0, \tag{13}$$

where we define:

$$H_2[g|G] := \int_0^\infty \phi G(u-1)^2\,dx, \tag{14}$$

and

$$D^b[g|G] := \int_0^\infty\int_x^\infty \phi(x)G(y)b(y,x)\big(u(x)-u(y)\big)^2\,dy\,dx. \tag{15}$$

Since

$$H_2[g|G] = \int_0^\infty (g-G)^2 \frac{\phi}{G}\,dx = \|g-G\|^2_{L^2(\phi G^{-1}\,dx)} \tag{16}$$

we see that proving $H_2(g(t)|G) \to 0$ implies that the long time trend to equilibrium (9) holds, and proving the entropy–dissipation entropy inequality,

$$H_2[g|G] \leqslant \frac{1}{2\beta} D^b[g|G], \tag{17}$$

implies that the exponentially fast long time trend to equilibrium (10) holds for the same $\beta > 0$ and for the norm $\|\cdot\| = \|\cdot\|_{L^2(\phi G^{-1}\,dx)}$.

The main purpose of our work is precisely to study the functional inequality (17) and establish it under certain conditions on the fragmentation coefficient b. We notice that, while some results on convergence to equilibrium for Eq. (1) are available, no inequalities like (17) were known, so one of the main points of our work is to show that the entropy method is applicable, in certain cases, to give a rate of convergence for this type of equations.

We focus on the two remarkable cases $a(x) = 1$, which corresponds to the so-called growth-fragmentation equation, and $a(x) = x$, which gives the self-similar fragmentation equation. There are several reasons for restricting our attention to them, the main one being that they are the ones most extensively studied in the literature due to their application as models in physics and biology. Also, the inequality (17) depends on the particular properties of the solution ϕ to the dual eigenproblem (6) and the equilibrium G. The existence and properties of these have been studied mainly for the above two particular cases, and are questions that require different techniques and deserve a separate study. One of our main results gives general conditions under which the entropy–entropy dissipation inequality (17) holds, and this may be applied to cases with a more general $a(x)$, once suitable bounds are proved for the corresponding profiles ϕ and G.

In order to understand our two model cases, let us describe them in more detail and give a short review of previously known results for them.

1.3. The growth-fragmentation equation

The growth-fragmentation equation is the following [6]:

$$\partial_t n_t + \partial_x n_t = \mathcal{L} n_t, \tag{18a}$$

$$n_t(0) = 0 \quad (t \geqslant 0), \tag{18b}$$

$$n_0(x) = n_{in}(x) \quad (x > 0). \tag{18c}$$

Here, $n_t(x)$ represents the number density of cells of a certain size $x > 0$ at time $t > 0$. The nonnegative function n_{in} is the initial distribution of cells at time $t = 0$. Eq. (18) models a set of cells which grow at a constant rate given by the drift term $\partial_x n$, and which can break into any number of pieces, as modeled by the right-hand side of the equation. The quantity $b(x, y)$, for $x > y > 0$, represents the mean number of cells of size y obtained from the breakup of a cell of size x. If one looks for solutions of (18a)–(18b) which are of the special form $n_t(x) = G(x)e^{\lambda t}$, for some $\lambda \in \mathbb{R}$, one is led to the eigenvalue problem for the operator $-\partial_x + \mathcal{L}$ and its dual eigenvalue problem, given by Eqs. (5) and (6), respectively. It has been proved [3,8] that quite generally there exists a solution to this eigenvalue problem which furthermore satisfies $\lambda > 0$ and $G > 0$.

For this particular λ, we consider the change:

$$g_t(x) := n_t(x)e^{-\lambda t}. \tag{19}$$

Then, n satisfies the *rescaled growth-fragmentation equation*, which is the particular case of Eq. (1) with $a(x) = 1$. The long time convergence (9) or (10) means here that the generic solutions asymptotically behave like the first eigenfunction $G(x)e^{\lambda t}$ (with same ϕ moment).

Let us give a short review of existing results on the asymptotic behavior of Eq. (1) for constant $a(x)$. In [9,10] the general entropy structure of Eq. (1) and related models was studied, and used to show a result of convergence to the equilibrium G without rate for quite general coefficients b, but under the condition that the initial condition be bounded by a constant multiple of G [10, Theorem 4.3]. Some results were obtained later for the case of *mitosis* with a constant total fragmentation rate $B(x) \equiv B$, which corresponds to the coefficient $b(x, y) = 2\delta_{y=x/2}$: this was studied in [13], where exponential convergence of solutions to the equilibrium G was proved. Similar results were obtained for the mitosis case when $B(x)$ is bounded above and below between two positive constants (i.e., for $b(x, y) = 2B(x)\delta_{y=x/2}$). An exponential speed of convergence has also been proved in [6] allowing for quite general fragmentation coefficients b, provided that the total fragmentation rate $B(x)$ is a constant. Of course, in the above results existence of the equilibrium G and the eigenfunction ϕ were also proved as a necessary step to study the asymptotic behavior of Eq. (1). The problem of existence of these profiles was studied in its own right in [3,8], where results are obtained for general fragmentation coefficients b, without the restriction that the total fragmentation rate should be bounded.

1.4. The self-similar fragmentation equation

The fragmentation equation is

$$\partial_t f_t = \mathcal{L} f_t, \tag{20a}$$

$$f_0(x) = f_{in}(x) \quad (x > 0), \tag{20b}$$

which models a set of clusters undergoing fragmentation reactions at a rate $b(x, y)$. In the study of the asymptotic behavior of f_t, one usually restricts attention to fragmentation kernels which are homogeneous of some degree $\gamma - 1$, with $\gamma > 0$:

$$b(rx, ry) = r^{\gamma-1} b(x, y) \quad (r > 0,\ x > y > 0), \tag{21}$$

so $B(x) = B_0 x^\gamma$ for some $B_0 > 0$. Then one looks for *self-similar solutions*, this is, solutions of the form,

$$f_t(x) = (t+1)^{2/\gamma} G\big((t+1)^{1/\gamma} x\big), \tag{22}$$

for some nonnegative function G. Observe that $f_0 = G$ in this case. We omit the case $\gamma = 0$, as self-similar solutions have a different expression in this case, and the asymptotic behavior of the fragmentation equation is also different, and must be treated separately. Such a remarkable function G is called a self-similar profile and is solution to the eigenvalue problem (5) with $a(x) = x$ and $\lambda = 1$. In this case one can easily show that in fact $\lambda = 1$ is the largest eigenvalue of the operator $g \mapsto -\partial_x(xg) + \mathcal{L}g$. Existence of solutions G of Eq. (5) in this setting has been studied in [4] and also in [3]. The corresponding dual equation (6) is explicitly solvable in this case, with $\phi(x) = Cx$ for some normalization constant C.

The above suggests to define g through the following change of variables:

$$f_t(x) = (t+1)^{2/\gamma} g\left(\frac{1}{\gamma}\log(t+1), (t+1)^{1/\gamma} x\right) \quad (t, x > 0), \tag{23}$$

or, writing g_t in terms of f,

$$g_t(x) := e^{-2t} f\big(e^{\gamma t} - 1, e^{-t} x\big) \quad (t, x > 0). \tag{24}$$

Then, g_t satisfies the *self-similar fragmentation equation*:

$$\partial_t g_t + x\partial_x g_t + 2g_t = \gamma \mathcal{L} g_t, \tag{25a}$$

$$g_0(x) = f_{in}(x) \quad (x > 0). \tag{25b}$$

We may redefine $b(x, y)$ to include the factor γ in front of $\mathcal{L}g_y$, and omit γ in the equation. Then, this equation is of the form (1) with $a(x) = x$ and $\lambda = 1$. The long time convergence (9) or (10) means here that generic solutions asymptotically behave like the self-similar solution $(t+1)^{2/\gamma} G((t+1)^{1/\gamma} x)$.

The problem of convergence to self-similarity for the fragmentation equation (20) was studied in [4], and then in [10]. Results on existence of self-similar profiles and convergence of solutions to them, without a rate, are available in [4, Theorems 3.1 and 3.2] and [10, Theorems 3.1 and 3.2], and are obtained through the use of entropy methods. To our knowledge, no results on the rate of this convergence were previously known.

1.5. Assumptions on the fragmentation coefficient

We turn to the precise description of our results, for which we will need the following hypotheses.

Hypothesis 1.1. For all $x > 0$, $b(x, \cdot)$ is a nonnegative measure on the interval $[0, x]$. Also, for all $\psi \in \mathcal{C}_0([0, +\infty))$, the function $x \mapsto \int_{[0,x]} b(x, y)\psi(y)\,dy$ is measurable.

Hypothesis 1.2. There exists $\kappa > 1$ such that

$$\int_0^x b(x, y)\,dy = \kappa B(x) \quad (x > 0). \tag{26}$$

Hypothesis 1.3. There exist $0 < B_m \leqslant B_M$ and $\gamma > -1$ (for growth-fragmentation) or $\gamma > 0$ (for self-similar fragmentation) such that

$$B_m x^\gamma \leqslant B(x) \leqslant B_M x^\gamma \quad (x > 0). \tag{27}$$

Hypothesis 1.4. There exist $\mu, C > 0$ such that for every $\epsilon > 0$,

$$\int_0^{\epsilon x} b(x, y)\,dy \leqslant C\epsilon^\mu B(x) \quad (x > 0).$$

Hypothesis 1.5. For every $\delta > 0$ there exists $\epsilon > 0$ such that

$$\int_{(1-\epsilon)x}^x b(x, y)\,dy \leqslant \delta B(x) \quad (x > 0). \tag{28}$$

Both Hypotheses 1.4 and 1.5 are "uniform integrability" hypotheses: they say that the measure $b(x, y)\,dy$ is not concentrated at $y = 0$ or $y = x$, in some quantitative uniform way for all $x > 0$. For some parts of our results we will also need the following hypothesis:

Hypothesis 1.6. The measure $b(x, \cdot)$ is (identified with) a function on $[0, x]$, and there exists $P_M > 0$ (actually, it must be $P_M \geqslant 2$ due to the definition of B) such that

$$b(x, y) \leqslant P_M \frac{B(x)}{x} \quad (x > y > 0). \tag{29}$$

Hypothesis 1.7. There exists $P_m > 0$ such that

$$b(x, y) \geqslant P_m \frac{B(x)}{x} \quad (x > y > 0). \tag{30}$$

We list the above hypothesis in a rough order of least to most restrictive. When taken all together they can be summarized in an easier way: Hypotheses 1.1–1.7 are equivalent to assuming Hypotheses 1.1, 1.2 and that there exists $0 < B_m < B_M$ satisfying,

$$2B_m x^{\gamma-1} \leqslant b(x, y) \leqslant 2B_M x^{\gamma-1} \quad (0 < y < x), \tag{31}$$

for some $\gamma > -1$, or $\gamma > 0$ in the case of growth-fragmentation. We give Hypotheses 1.1–1.7 instead of this simpler statement for later reference: some of the results to follow use only a subset of these hypotheses, and it is preferable not to use the more restrictive conditions where they are not needed.

As an example of coefficients which satisfy all Hypotheses 1.1–1.7 we may take b of *self-similar* form, i.e.,

$$b(x, y) := x^{\gamma-1} p\left(\frac{y}{x}\right) \quad (x > y > 0), \tag{32}$$

where $p:(0,1)\to(P_m,P_M)$ is a function bounded above and below, and γ within the specified range (for the main theorem below one must have $\gamma\in(0,2]$ for self-similar fragmentation, and $\gamma\in(0,2)$ for growth-fragmentation). A bit more generally, one can take

$$b(x,y):=h(x)x^{\gamma-1}p\left(\frac{y}{x}\right)\quad(x>y>0),\tag{33}$$

with $h(x):(0,+\infty)\to[B_m,B_M]$ a function bounded above and below.

1.6. Previous results

We recall the following result, readily deduced from [3, Theorem 1 and Lemma 1] and [4, Theorem 3.1]:

Theorem 1.8. *(See [3].) Assume Hypotheses* 1.1–1.4. *There exists a unique triple* (λ,G,ϕ) *with* $\lambda>0$, $G\in L^1$ *and* $\phi\in W^{1,\infty}_{\mathrm{loc}}$ *which satisfy* (5) *and* (6) (*in the sense of distributional solutions for* (5a)*, and of a.e. equality for* (6a)).
In addition,

$$G(x)>0,\qquad \phi(x)>0\quad(x>0),\tag{34}$$

$$\text{for all }\alpha>0,\quad x\mapsto x^{\alpha}a(x)G(x)\in W^{1,1}(0,+\infty).\tag{35}$$

In the case of the growth-fragmentation equation $(a(x)\equiv1)$ *it holds that* $\phi(0)>0$.
In the case of the self-similar fragmentation equation $(a(x)=x)$*, assume in addition that, for some* $k<0$ *and* $C>1$,

$$\int_0^x y^k b(x,y)\,dy\leqslant Cx^kB(x)\quad(x>0).\tag{36}$$

Then,

$$\int_0^\infty x^kG(x)\,dx<+\infty.\tag{37}$$

The above result can actually be proved under weaker hypotheses on the fragmentation coefficient b (see [3]) but we will only use the given version. We note that (37) is derived in [4] under more restrictive conditions on b (in particular, it is asked there that b is of the self-similar form (32)); however, the same proof can be followed just by using (36), and we omit the details here.

1.7. Main results

The following theorem gathers our main results in this work.

Theorem 1.9. *Consider Eq.* (1) *in the self-similar fragmentation case* $(a(x)=x)$ *or the growth-fragmentation case* $(a(x)=1)$*, and assume Hypotheses* 1.1–1.7. *Also*:

- *For self-similar fragmentation, assume* $\gamma\in(0,2]$.
- *For growth-fragmentation, assume* $\gamma\in(0,2)$ *or alternatively that for some* $B_b>0$,

$$b(x,y)=\frac{2B_b}{x}\quad(x>y>0).\tag{38}$$

Denote by $G>0$ *the asymptotic profile* (*self-similar profile in the first case, first eigenfunction in the second case*) *as well as by* ϕ *the first dual eigenfunction* ($\phi(x)=x$ *in the first case*). *These equations have a spectral gap in the space* $L^2(\phi G^{-1}\,dx)$.

1. *More precisely, there exists $\beta > 0$ such that*

$$\forall g \in X, \quad H_2(g|G) \leqslant \frac{1}{2\beta} D^b(g|G)$$

(*the right-hand term being finite or not*), *where we have defined*:

$$H := L^2\big(G^{-1}\phi\big), \qquad X := \left\{ g \in H;\ \int_0^\infty g\phi = \int_0^\infty G\phi = 1 \right\}.$$

Consequently for any $g_{in} \in X$ the solution $g \in C([0,\infty); L^1_\phi)$ to Eq. (1) *satisfies,*

$$\forall t \geqslant 0 \quad \big\| g(t,.) - G \big\|_H \leqslant e^{-2\beta t} \| g_{in} - G \|_H.$$

2. *For this second part, in the case of growth-fragmentation, we need to assume additionally that $\gamma \in (0,1)$ (no additional assumption is needed for the self-similar fragmentation case). There exists $\bar{k} = \bar{k}(\gamma, P_M) \geqslant 3$ and for any $a \in (0,\beta)$ and any $k > \bar{k}$ there exists $C_{a,k} \geqslant 1$ such that for any $g_{in} \in \mathcal{H} := L^2(\theta)$, $\theta(x) = \phi(x) + x^k$, $\int \phi g_{in} = 1$, there holds*:

$$\forall t \geqslant 0 \quad \big\| g(t) - G \big\|_{\mathcal{H}} \leqslant C_{a,k} e^{-at} \| g_{in} - G \|_{\mathcal{H}}.$$

The main improvement of this result is that it allows us to treat total fragmentation rates B which are not bounded, for which no results were available. Also, the result is obtained through an entropy–entropy dissipation inequality, which is not strongly tied to the particular form of the equation, and may be useful in related equations. We also observe that this is to our knowledge the first result on rate of convergence to self-similarity for the fragmentation equation.

In our result we include just one case where B is a constant, with b given by the particular form (38) in the case of growth-fragmentation. The difficulty in allowing for more general b (with B still constant) is not in the entropy–entropy dissipation inequality, but in the estimates for the solution ϕ of the dual eigenproblem (6). The estimates in Section 4.2 are not valid in this case, and in fact one expects that ϕ should be bounded above and below between two positive constants. However, we are unable to prove this at the moment, and anyway the estimates of ϕ are not the main aim of the present paper. Consequently, we state the result only for the b in (38), for which ϕ is explicitly given by a constant. (We notice that for self-similar fragmentation, the estimates on G fail for $\gamma = 0$, so this case is not included.)

On the other hand, the positivity condition (30) is strong, and makes this result not applicable to some cases of interest, such as the mitosis case mentioned at the end of Section 1. Notice that, as remarked at the end of Section 2 in [6], entropy–entropy dissipation inequalities are actually false in this case, which shows that a different method is required for this and similar situations where the fragmentation coefficient is “sparse” enough for the inequalities to fail.

1.8. Strategy

The main difficulty for establishing Theorem 1.9 lies in proving point 1, while point 2 is a consequence of point 1 and of a method for enlarging the functional space of decay estimates on semigroups from a “small” Hilbert space to a larger one recently obtained in [5].

The strategy we follow to prove inequality (17) is inspired by a paper of Diaconis and Stroock [2] which describes a technique to find bounds of the spectral gap of a finite Markov process. Eq. (1) can be interpreted as the evolution of the probability distribution of an underlying continuous Markov process, and as such can be thought of as a limit of finite Markov processes. The expression of the entropy and entropy dissipation has many similarities to the finite case, and hence the ideas used in [2] can be extended to our setting. The basic idea is that one may improve an inequality in the finite case by appropriately choosing, for each two points i and j in the Markov process, a chain of reactions from i to j which has a large probability of happening, measured in a particular way. In our case this corresponds to choosing a chain of reactions that can give particles of size x from particles of size y, and which has a better probability than just particles of size y fragmenting at rate

$b(y,x)$ to give particles of size x. In fact, we need the additional observation that one can choose many different paths from y to x, and then average among them to improve the probability. These heuristic ideas are made precise in the proof of point 1 in Theorem 1.9, given in Section 2. This suggests that the same techniques may be applicable to other linear evolution equations which can interpreted as the probability distribution of a Markov process.

In the context of the Boltzmann equation, an idea with some common points with this one was introduced in [1] to prove a spectral gap for the linearized Boltzmann operator. However, the inequality needed in that case does not have the same structure since the linearized Boltzmann operator does not come from a Markov process (it does not conserve positivity, for instance), and the "reactions" to be taken into account there are not jumps from one point to another, but collisions between two particles at given velocities, to give two particles at different velocities. Nevertheless, the geometric idea does bear some resemblance, and the connection to techniques developed for the study of finite Markov processes seems interesting.

Concerning the proof of point 2, we mainly have to show that the operator involved in the fragmentation equation decomposes as $\mathcal{A}+\mathcal{B}$ where $\mathcal{A}$ is a bounded operator and $\mathcal{B}$ is a coercive operator in the large Hilbert space $\mathcal{H}$. Then, for a such a kind of operator, the result obtained in [5] ensures that the operator in the large space $\mathcal{H}$ inherits the spectral properties and the decay estimates of the associated semigroup which are true in the small space H. The additional restriction $\gamma \in (0,1)$ in the extension of the spectral gap comes from the fact that we were unable to prove the coercivity of the operator $\mathcal{B}$ for $\gamma \in [1,2)$.

In the next section we prove the functional inequality (17). In Sections 3 and 4 we prove upper and lower bounds on the profiles G and ϕ, solution of the eigenproblem (5)–(6); Section 3 is dedicated to the self-similar fragmentation equation (the case $a(x)=x$), while Section 4 deals with the growth-fragmentation equation (the case $a(x)=1$). Finally, Section 5 uses these results and the techniques in [5] to complete the proof of Theorem 1.9.

2. Entropy dissipation inequalities

In order to study the speed of convergence to equilibrium, in terms of the entropy functional, we are interested in proving the entropy–entropy dissipation inequality (17):

$$H_2[g|G] \leqslant \frac{1}{2\beta} D^b[g|G].$$

We begin with the following basic identity:

Lemma 2.1. *Take nonnegative measurable functions $G, \phi : (0,\infty) \to \mathbb{R}_+$ such that*

$$\int_0^\infty \phi(x)G(x)\,dx = 1. \tag{39}$$

Defining

$$D_2[g|G] := \int_0^\infty \int_x^\infty \phi(x)G(x)\phi(y)G(y)\big(u(x)-u(y)\big)^2\,dy\,dx \tag{40}$$

with $u(x) = \frac{g(x)}{G(x)}$, there holds:

$$H_2[g|G] = D_2[g|G], \tag{41}$$

for any nonnegative measurable function $g : (0,\infty) \to \mathbb{R}$ such that $\int \phi g = 1$.

Proof. With the above notations, by a simple expansion of the squares, we find that

$$H_2[g|G] = \int_0^\infty \big(u(x)-1\big)^2 G(x)\phi(x)\,dx = \int_0^\infty u(x)^2 G(x)\phi(x)\,dx - 1,$$

while expanding $D_2[g|G]$ we find:

$$D_2[g|G] = \frac{1}{2}\int_0^\infty \int_0^\infty G(x)\phi(x)G(y)\phi(y)\big(u(x)-u(y)\big)^2\,dy\,dx$$
$$= \frac{1}{2}\left(2\int_0^\infty u(x)^2 G(x)\phi(x)\,dx - 2\right),$$

which gives the same result. □

Taking into account the above result, in order to prove (17) it is enough to prove the following inequality for some constant $C > 0$:

$$D_2[g|G] \leqslant C D^b[g|G]. \tag{42}$$

Of course, one gets (42) if one assumes that the function appearing below the integral signs in the functional $D_2[g|G]$ is pointwise bounded by the corresponding function in the functional $D^b[g|G]$, or more precisely if one assumes $G(x)\phi(y) \leqslant Cb(y,x)$ whenever $0 < x < y$. However, the range of admissible rates b for which such an inequality holds seems to be very narrow. For example, for the self-similar rate (32), one must impose $\gamma = 2$.

The cornerstone of the proof of the functional inequality (42) lies in splitting $D_2[g|G]$ in two terms: one for which the pointwise comparison holds and one where it does not. In the latter part, called $D_{2,2}[g|G]$ in the next lemma, the idea is to break $u(x) - u(y)$ into "intermediate reactions" in order to obtain a new expression of $D_2[g|G]$ for which the pointwise estimate applies. The main part of the proof of inequality (42) is then the bound for this second part of $D_2[g|G]$. And it is just the content of the following result:

Lemma 2.2. *Take measurable functions $\phi, G : (0, +\infty) \to \mathbb{R}_+$ with*

$$\int_0^\infty G(x)\phi(x)\,dx = 1, \tag{43}$$

and such that, for some constants $K, M > 0$ and $R > 1$ and some function $\zeta : (R,\infty) \to [1,\infty)$,

$$0 \leqslant G(x) \leqslant K \quad (x > 0), \tag{44}$$

$$\int_{Rx}^\infty G(y)\phi(y)\,dy \leqslant K G(x) \quad (x > M), \tag{45}$$

$$\frac{1}{\zeta(y)}\phi(y) \leqslant K\phi(z) \quad \big(\max\{2RM, Rz\} < y < 2Rz\big). \tag{46}$$

Defining:

$$D_{2,2}[g|G] := \int_0^\infty \int_{\max\{2Rx,2RM\}}^\infty \phi(x)G(x)\phi(y)G(y)\big(u(x)-u(y)\big)^2\,dy\,dx \tag{47}$$

there is some constant $C > 0$ such that

$$D_{2,2}[g|G] \leqslant C\int_0^\infty \int_{\max\{x,M\}}^\infty \zeta(y)y^{-1}\phi(x)G(y)\big(u(x)-u(y)\big)^2\,dy\,dx, \tag{48}$$

for all measurable functions $g : (0,\infty) \to \mathbb{R}_+$ (in the sense that if the right-hand side is finite, then the left-hand side is also, and the inequality holds).

The point of this lemma is that the right-hand side of (48) will be easily bounded by $D^b[g|G]$.

Remark 2.3. Inequality (48) is strongest when $\zeta(x) \equiv 1$. Moreover, we expect (46) to be true when ζ is constantly 1 (for ϕ the solution of the dual equation (6), under some additional conditions). The reason that we allow for a choice of ζ is that we are not able to prove that (46) holds with $\zeta \equiv 1$, but only for a function $\zeta = \zeta(x)$ which grows like a power x^ϵ, with ϵ as small as desired.

Proof. We will denote by C any constant which depends on G, ϕ, K, M, or R, but not on g.

First step. The idea is to break $u(x) - u(y)$ into "intermediate reactions": for $y > x$ and any $z \in [x, y]$, one obviously has:

$$u(x) - u(y) = u(x) - u(z) + u(z) - u(y).$$

We then average among a range of possible splittings. More precisely, for $y > 2Rx$,

$$u(x) - u(y) = \frac{2R}{y} \int_{y/(2R)}^{y/R} \big(u(x) - u(z) + u(z) - u(y)\big)\, dz,$$

and then by Cauchy–Schwarz's inequality,

$$\begin{aligned}
\big(u(x) - u(y)\big)^2 &\leqslant \frac{2R}{y} \int_{y/(2R)}^{y/R} \big(u(x) - u(z) + u(z) - u(y)\big)^2\, dz \\
&\leqslant \frac{4R}{y} \int_{y/(2R)}^{y/R} \big(u(x) - u(z)\big)^2\, dz + \frac{4R}{y} \int_{y/(2R)}^{y/R} \big(u(z) - u(y)\big)^2\, dz \\
&=: T_1(x, y) + T_2(y).
\end{aligned}$$

Using this, we have:

$$\begin{aligned}
D_{2,2}[g|G] \leqslant &\int_0^\infty \int_{\max\{2Rx, 2RM\}}^\infty \phi(x)\phi(y)G(x)G(y)T_1(x, y)\, dy\, dx \\
&+ \int_0^\infty \int_{\max\{2Rx, 2RM\}}^\infty \phi(x)\phi(y)G(x)G(y)T_2(y)\, dy\, dx.
\end{aligned}$$

We estimate these terms in the next two steps.

Second step. For the term with T_1,

$$\begin{aligned}
&\int_0^\infty \int_{\max\{2Rx, 2RM\}}^\infty \phi(x)\phi(y)G(x)G(y)T_1(x, y)\, dy\, dx \\
&\quad = C \int_0^\infty \int_{\max\{2Rx, 2RM\}}^\infty \phi(x)\phi(y)G(x)G(y)\frac{1}{y} \int_{y/(2R)}^{y/R} \big(u(x) - u(z)\big)^2\, dz\, dy\, dx \\
&\quad = C \int_0^\infty \int_{\max\{x, M\}}^\infty \phi(x)G(x)\big(u(x) - u(z)\big)^2 \int_{\max\{2Rx, Rz, 2RM\}}^{2Rz} \frac{\phi(y)}{y} G(y)\, dy\, dz\, dx \\
&\quad \leqslant C \int_0^\infty \int_{\max\{x, M\}}^\infty \phi(x)G(x)\big(u(x) - u(z)\big)^2 \frac{1}{Rz} \int_{Rz}^\infty \phi(y)G(y)\, dy\, dz\, dx
\end{aligned}$$

$$\leqslant C \int_0^\infty \int_{\max\{x,M\}}^\infty \phi(x)\big(u(x)-u(z)\big)^2 \frac{1}{z} G(z)\,dz\,dx,$$

where we have used (45) and the fact that G is uniformly bounded from (44). With this series of inequalities we obtain that the last one integral is bounded by the right-hand side in (48), as $\zeta(y) \geqslant 1$ for all y.

Third step. For the term with T_2, we use (43) and (46). Note that (46) was not used before this point in the proof: $\zeta(y) \leqslant 1$ was enough up to now, but this is not the case in the following calculation:

$$\int_0^\infty \int_{\max\{2Rx,2RM\}}^\infty \phi(x)\phi(y)G(x)G(y)T_2(y)\,dy\,dx$$
$$= C\int_0^\infty \int_{\max\{2Rx,2RM\}}^\infty \phi(x)\phi(y)G(x)G(y)\frac{1}{y}\int_{y/(2R)}^{y/R} \big(u(z)-u(y)\big)^2\,dz\,dy\,dx$$
$$= C\int_{2RM}^\infty \int_{y/(2R)}^{y/R} \frac{1}{y}\phi(y)G(y)\big(u(z)-u(y)\big)^2 \int_0^{y/(2R)} \phi(x)G(x)\,dx\,dz\,dy$$
$$\leqslant C\int_{2RM}^\infty \int_{y/(2R)}^{y/R} \frac{1}{y}\phi(y)G(y)\big(u(z)-u(y)\big)^2\,dz\,dy$$
$$= C\int_M^\infty \int_{\max\{2RM,Rz\}}^{2Rz} \frac{\zeta(y)}{y}\frac{\phi(y)}{\zeta(y)}G(y)\big(u(z)-u(y)\big)^2\,dy\,dz$$
$$\leqslant C\int_0^\infty \int_{\max\{2RM,Rz\}}^{2Rz} \frac{\zeta(y)}{y}\phi(z)G(y)\big(u(z)-u(y)\big)^2\,dy\,dz,$$

where the last integral is bounded by the right-hand side in (48) and this finishes the proof. □

Once we have controlled the "bad" term in $D_2[g|G]$ we can reach the objective of this section: to obtain an entropy–entropy dissipation inequality. It is shown in the following theorem:

Theorem 2.4. *Assume the conditions in Lemma 2.2, and also that the fragmentation coefficient b satisfies, for the constants K, M, R in Lemma 2.2,*

$$G(x)\phi(y) \leqslant Kb(y,x) \quad \big(0 < x < y < \max\{2Rx, 2RM\}\big), \tag{49}$$
$$\zeta(y)y^{-1} \leqslant Kb(y,x) \quad (y > M,\ y > x > 0). \tag{50}$$

Then there is some constant $C > 0$ such that

$$H_2[g|G] \leqslant C D^b[g|G] \tag{51}$$

for all measurable functions $g:(0,\infty) \to \mathbb{R}_+$ such that $\int_0^\infty g\phi = 1$ (in the sense that if the right-hand side is finite, then the left-hand side is also, and the inequality holds).

Proof. We split $D_2[g|G]$ as

$$D_2[g|G] = D_{2,1}[g|G] + D_{2,2}[g|G],$$

with

$$D_{2,1}[g|G] := \int_0^\infty \int_x^{\max\{2Rx, 2RM\}} \phi(x)G(x)\phi(y)G(y)\big(u(x)-u(y)\big)^2\, dy\, dx$$

and $D_{2,2}[g|G]$ defined by (47). On the one hand thanks to (49) we have:

$$\begin{aligned} D_{2,1}[g|G] &\leqslant \int_0^\infty \int_x^{\max\{2Rx, 2RM\}} Kb(y,x)\phi(x)G(y)\big(u(x)-u(y)\big)^2\, dy\, dx \\ &\leqslant KD^b[g|G]. \end{aligned} \tag{52}$$

On the other hand, thanks to inequality (48) in Lemma 2.2 and (50) we have:

$$\begin{aligned} D_{2,2}[g|G] &\leqslant C \int_0^\infty \int_{\max\{x,M\}}^\infty \frac{\zeta(y)}{y}\phi(x)G(y)\big(u(x)-u(y)\big)^2\, dy\, dx \\ &\leqslant CK \int_0^\infty \int_{\max\{x,M\}}^\infty b(y,x)\phi(x)G(y)\big(u(x)-u(y)\big)^2\, dy\, dx \\ &\leqslant CKD^b[g|G]. \end{aligned} \tag{53}$$

We conclude by gathering (52) and (53). □

This result will be the key to prove point 1 in Theorem 1.9, which will be done in Section 5.1. In order to prove that (44) and (49) hold in our context, we need to establish some upper bounds on G and ϕ for our model cases. For (45) and (46) we need to control the asymptotic behavior (bounds from above and below) of the functions $G(x)$ and $\phi(x)$ as $x \to \infty$. These upper and lower estimates on G and ϕ will be established in the next sections. Finally, condition (50) simply imposes some restrictions on the fragmentation rate (typically some restrictions on the value of γ for a self-similar fragmentation rate of the form (32)).

3. Bounds for the self-similar fragmentation equation

In order to apply Theorem 2.4 to the self-similar fragmentation equation (25) we need more precise bounds than those proved in [4,10,3]; in particular, we need L^∞ bounds on the self-similar profile G for condition (44) to hold. We actually prove the following accurate exponential growth estimate on the profile G.

Theorem 3.1. *Assume Hypotheses* 1.1–1.5 *on the fragmentation coefficient b. Call* $\Lambda(x) := \int_0^x \frac{B(s)}{s}\, ds$.

1. *For any* $\delta > 0$ *and any* $a \in (0, B_m/B_M)$, $a' \in (1, +\infty)$ *there exist constants* $C' = C'(a', \delta)$, $C = C(a, \delta) > 0$ *such that*

$$C' e^{-a'\Lambda(x)} \leqslant G(x) \leqslant C e^{-a\Lambda(x)} \quad \textit{for } x > \delta. \tag{54}$$

2. *Assume additionally Hypothesis* 1.6. *Then one may take C independent of* δ *in* (54)*, i.e.,*

$$G(x) \leqslant C e^{-a\Lambda(x)} \quad \textit{for } x > 0. \tag{55}$$

Remark 3.2. We notice that, due to Hypothesis 1.3, $(B_m/\gamma)x^\gamma \leqslant \Lambda(x) \leqslant (B_M/\gamma)x^\gamma$, so the bound (54) directly implies that for any $a_1 > B_M/\gamma$ and $a_2 < B_m^2/(\gamma B_M)$ one has:

$$C' e^{-a_1 x^\gamma} \leqslant G(x) \leqslant C e^{-a_2 x^\gamma} \quad \text{for } x > \delta, \tag{56}$$

and also the corresponding one instead of (55). Note that when $B_M = B_m$ the condition on a_1, a_2 becomes $a_2 < B_M/\gamma < a_1$.

The goal of this section is to give the proof of the above theorem, which we develop in several steps. We remark that in the particular case of p constant in (32) (so, $p \equiv 2$ due to the normalization $\int zp(z)\,dz = 1$) we can find an explicit expression for the equilibrium G (i.e., a solution of (5)). Indeed, $G(x) = \exp(-\int_0^x \frac{B(s)}{s}\,ds)$ satisfies (5). As a consequence, in the case $b(x,y) = 2x^{\gamma-1}$ (so $B(x) = x^\gamma$), the profile G is $e^{-\frac{x^\gamma}{\gamma}}$ for $\gamma > 0$. In the general case where b is not of the form (32) there is no explicit expression available for the self-similar profile G.

Lemma 3.3. *Assume that b satisfies Hypotheses* 1.1–1.5*, and consider G the unique self-similar profile given by Theorem* 1.8*. For any $0 < a < B_m/\gamma$ there exists a constant C_a such that*

$$\int_0^\infty e^{ax^\gamma} G(x)\,dx \leqslant C_a. \tag{57}$$

Proof. We denote by M_k the k-th moment of G. Multiply Eq. (5) by x^k with $k > 1$ to obtain:

$$(k-1)M_k \geqslant (1-p_k)\int_0^\infty x^k B(x)G(x)\,dx, \tag{58}$$

where we have taken into account that, using Corollary A.4,

$$\int_0^\infty x^k \int_x^\infty b(y,x)G(y)\,dy\,dx = \int_0^\infty G(y)\int_0^y x^k b(y,x)\,dx\,dy \leqslant p_k \int_0^\infty y^k B(y)G(y)\,dy.$$

By Hypothesis 1.3, and as $p_k < 1$ for $k > 1$, (58) implies that

$$(k-1)M_k \geqslant (1-p_k)B_m M_{k+\gamma}, \quad \text{or} \quad M_{k+\gamma} \leqslant \frac{k}{B_m(1-p_k)} M_k.$$

Applying this for integer $\ell \geqslant 1$ and $k := 1 + \ell\gamma$,

$$M_{1+(\ell+1)\gamma} \leqslant C_\ell M_{1+\ell\gamma}, \quad \text{where } C_\ell := \frac{1+\gamma\ell}{B_m(1-p_{1+\gamma\ell})}.$$

Solving the recurrence relation,

$$M_{1+\ell\gamma} \leqslant M_{\gamma+1}\prod_{i=1}^{\ell-1} C_i \quad (\ell \geqslant 1). \tag{59}$$

With this,

$$\int_0^\infty e^{ax^\gamma} G(x)\,dx = \sum_{i=0}^\infty \int_0^\infty \frac{a^i}{i!} x^{\gamma i} G(x)\,dx = \sum_{i=0}^\infty \frac{a^i}{i!} M_{\gamma i} \leqslant M_0 + M_\gamma + M_{1+\gamma}\sum_{i=1}^\infty \frac{a^i}{i!}\prod_{i=1}^{\ell-1} C_i.$$

The last expression in the sum is a power series in a, with radius of convergence equal to B_m/γ. This can be checked, for example, by noticing that

$$\frac{C_\ell}{\ell+1} \to \frac{\gamma}{B_m} \quad \text{as } \ell \to +\infty,$$

which corresponds to the quotient of two consecutive terms in the power series. This proves the lemma. □

With this result we can now prove Theorem 3.1.

Proof of Theorem 3.1. Taking $\delta > 0$, we give the proof in several steps.

First step (upper bound for $x \geqslant \delta$). From (5), we calculate as follows:

$$\begin{aligned}\partial_x\left(x^2 e^{a\Lambda} G\right) &= x e^{a\Lambda}(2G + x\partial_x G) + aBxe^{a\Lambda}G = xe^{a\Lambda}\mathcal{L}G + aBxe^{a\Lambda}G \\ &= xe^{a\Lambda}\int_x^\infty b(y,x)G(y)\,dy + (a-1)Bxe^{a\Lambda}G. \end{aligned} \tag{60}$$

Let us show that for $a < 1$ this expression is integrable. For the first term,

$$\begin{aligned}&\int_0^\infty xe^{a\Lambda(x)}\int_x^\infty b(y,x)G(y)\,dy\,dx \\ &\quad= \int_0^\infty G(y)\int_0^x e^{a\Lambda(x)} b(y,x)\,dx\,dy \leqslant \int_0^\infty ye^{a\Lambda(y)}G(y)B(y)\,dy \leqslant B_M\int_0^\infty y^{\gamma+1}e^{\frac{aB_M}{\gamma}y^\gamma}G(y)\,dy < +\infty,\end{aligned}$$

where the last inequality is due to Lemma 3.3, since $y^{\gamma+1}e^{\frac{aB_M}{\gamma}y^\gamma} \leqslant Ce^{by^\gamma}$, where $a < b < \frac{B_m}{\gamma}$ and C is a constant depending on a, b and γ (recall that $a < B_m/B_M$). For the same reason, the last term in (60) is integrable. Therefore, since $\partial_x(x^2e^{a\Lambda(x)}G(x))$ is bounded in L^1, we deduce that

$$x^2e^{ax^\gamma}G(x) \in BV(0,\infty) \subset L^\infty,$$

and in consequence

$$G(x) \leqslant C_a^1 e^{-a\Lambda(x)} \quad \text{for } x \geqslant \delta.$$

Second step (lower bound for $x > \delta$). Writing Eq. (60) for $a = 1$ gives that $x \mapsto x^2e^{\Lambda(x)}G(x)$ is a nondecreasing function, so

$$x^2e^{\Lambda(x)}G(x) \geqslant \delta^2e^{\Lambda(\delta)}G(\delta) \quad (x > \delta),$$

which implies the lower bound in (54) for any $a < 1$. Notice that $G(\delta) > 0$ by Theorem 1.8.

Third step (upper bound for $x < \delta$). As $x \mapsto x^2e^{x^\gamma/\gamma}G(x)$ has bounded variation, it must have a limit as $x \to 0$. As $x \mapsto xG(x)$ is integrable, it follows that this limit must be 0. Hence, writing (60) for $a = 1$, integrating between 0 and z, and using Hypothesis 1.6,

$$\begin{aligned}z^2e^{z^\gamma/\gamma}G(z) &= \int_0^z xe^{a\Lambda}\int_x^\infty b(y,x)G(y)\,dy\,dx \\ &\leqslant CB_M\int_0^z xe^{a\Lambda}\int_x^\infty y^{\gamma-1}G(y)\,dy\,dx \leqslant C'\int_0^z xe^{x^\gamma/\gamma}\,dx \leqslant C'z^2e^{z^\gamma/\gamma},\end{aligned}$$

which implies $G(z) \leqslant C'$ for all $z > 0$. This is enough to have (55) for $x < \delta$. We have used above that the moment of G of order $\gamma - 1$ is bounded, as given by Theorem 1.8, taking into account that (29) holds and $\gamma > 0$, so $\int_0^x y^{\gamma-1}b(x,y)\,dy \leqslant (C/\gamma)x^{\gamma-1}$. $\square$

4. Bounds for the growth-fragmentation equation

In this section we present some estimates by above and below for the functions G and ϕ, solutions to the eigenvalue problem (5) and (6). The aim, as in the previous section, is to obtain bounds which are accurate enough to apply Theorem 2.4, and then prove Theorem 1.9.

The main additional difficulty as compared to the self-similar fragmentation equation from previous section is that the dual eigenfunction ϕ is in general not explicit, which makes it necessary to have additional estimates for it.

In the rest of this section we will give the proof of the following result:

Theorem 4.1. *Assume Hypotheses* 1.1–1.5. *Call*

$$\Lambda(x) := \lambda x + \int_0^x B(x)\,dx.$$

1. *For any $a \in (0, B_m/B_M)$ there exists $K_a > 0$ such that*
$$\forall x \geqslant 0 \quad G(x) \leqslant K_a e^{-a\Lambda(x)}. \tag{61}$$
If we also assume Hypothesis 1.6 *and $\gamma > 0$, then this can be strengthened to,*
$$\forall x \geqslant 0 \quad G(x) \leqslant K_a \min\{1, x\} e^{-a\Lambda(x)}. \tag{62}$$
2. *For any $\delta > 0$ there exists $K_\delta > 0$ such that*
$$\forall x \geqslant \delta \quad G(x) \geqslant K_\delta e^{-\Lambda(x)}. \tag{63}$$
3. *Assume additionally that*
$$\frac{B(x)}{x^{\mu+1}} \to 0 \quad \text{and} \quad B(x) \to +\infty \quad \text{as } x \to +\infty. \tag{64}$$
(Here μ is the one in Hypothesis 1.4.*) There exist $C_0, C_1 \in (0, \infty)$ and for any $k \in (0,1)$ there exists $C_k \in (0, \infty)$ such that*
$$\forall x \geqslant 0 \quad C_0 + C_k x^k \leqslant \phi(x) \leqslant C_1(1 + x). \tag{65}$$

Remark 4.2. With (61), (63) and (27), in the case $\gamma > 0$ it is easy to see that for any $\delta > 0$, $a_1 > B_M/(\gamma+1)$ and $a_2 < B_m^2/(B_M(\gamma+1))$ there exist $C_1, C_2 > 0$ such that

$$C_1 e^{-a_1 x^{\gamma+1}} \leqslant G(x) \leqslant C_2 e^{-a_2 x^{\gamma+1}} \quad (x > \delta). \tag{66}$$

In the case $\gamma = 0$, from (61) and (63) one sees that for any $\delta > 0$ and $a > 1$ there exist $C_1, C_2 > 0$ such that

$$C_1 e^{-(\lambda + B_M)x} \leqslant G(x) \leqslant C_2 e^{-a(\lambda + B_m)x} \quad (x > \delta). \tag{67}$$

4.1. Bounds for G

Lemma 4.3. *Assume Hypotheses* 1.1–1.5. *Call*

$$\Lambda_m(x) := \lambda x + \frac{B_m}{\gamma+1} x^{\gamma+1}.$$

For each $a < 1$,

$$\int_0^\infty e^{a\Lambda_m(x)} G(x)\,dx < +\infty. \tag{68}$$

Proof. We will first do the proof for $\gamma > 0$, and leave the case $-1 < \gamma \leqslant 0$ for later. Multiply Eq. (5) by x^k with $k > 1$, and integrate to obtain:

$$kM_{k-1} - \lambda M_k + \int_0^\infty G(y) \int_0^y x^k b(y,x)\,dx\,dy - \int_0^\infty B(x) x^k G\,dx = 0,$$

which gives, using (92) and then (27) (noting $p_k < 1$ for $k > 1$),

$$(1 - p_k) B_m M_{\gamma+k} \leqslant k M_{k-1} - \lambda M_k \leqslant k M_{k-1}. \tag{69}$$

Applying this for $k = \ell(\gamma+1) - \gamma$, with $\ell \geqslant 2$ an integer,

$$M_{\ell(\gamma+1)} \leqslant C_\ell M_{(\ell-1)(\gamma+1)}, \quad \text{with } C_\ell := \frac{\ell(\gamma+1) - \gamma}{(1 - p_{\ell(\gamma+1)-\gamma}) B_m}.$$

Solving this recurrence relation gives, for $\ell \geqslant 2$,

$$M_{\ell(\gamma+1)} \leqslant M_{\gamma+1} \prod_{i=2}^{\ell} C_\ell \quad (\ell \geqslant 2).$$

Now, following an analogous calculation to the one in Lemma 3.3,

$$\int_0^\infty e^{a x^{\gamma+1}} G(x)\, dx \leqslant M_0 + M_{\gamma+1} \sum_{i=2}^{\infty} \frac{a^i}{i!} \prod_{i=2}^{\ell} C_\ell. \tag{70}$$

Again as in Lemma 3.3, one can check that the above power series in a has radius of convergence $B_m/(\gamma+1)$ (using that $p_k \to 0$ when $k \to +\infty$, from Corollary A.4 in Appendix A). This proves the lemma for $\gamma > 0$ (note that the dominant term in Λ in this case is $x^{\gamma+1}$, as then $x \leqslant C_\epsilon + \epsilon x^{\gamma+1}$ for any $\epsilon > 0$ and some $C_\epsilon > 0$).

When $\gamma = 0$, from (69) we obtain:

$$\big((1 - p_k) B_m + \lambda\big) M_k \leqslant k M_{k-1} \quad (k > 1).$$

We can then follow the same reasoning as above, with the only difference that now,

$$C_\ell := \frac{k}{(1 - p_\ell) B_m + \lambda} \quad (\ell \geqslant 2).$$

Now the power series in (70) has radius of convergence $B_m + \lambda$, which proves the lemma also in this case.

In the case $\gamma \in (-1, 0)$, from (69) we obtain the inequality,

$$\forall k > 1 \quad M_k \leqslant \frac{k}{\lambda} M_{k-1},$$

from which we deduce thanks to an iterative argument as before that

$$\forall k \in \mathbb{N}^* \quad M_k \leqslant \frac{k!}{\lambda^k} (\lambda M_1).$$

Hence, for $a < 1$,

$$\int_0^\infty e^{a\lambda x} G(x)\, dx = M_0 + \sum_{k=1}^{\infty} \frac{a^k \lambda^k}{k!} M_k \leqslant M_0 + \lambda M_1 \sum_{k=1}^{\infty} a^k < +\infty.$$

The dominant term in Λ in this case is λx, so this finishes the proof. $\square$

Proof of points 1–2 in Theorem 4.1. With the previous lemma we are ready to prove our bounds on G.

First step (upper bound). Take $0 \leqslant a \leqslant 1$, and calculate the derivative of $G(x)e^{a\Lambda(x)}$:

$$\big(G e^{a\Lambda}\big)' = (a-1)(B+\lambda) G e^{a\Lambda} + e^{a\Lambda} \mathcal{L}_+(G). \tag{71}$$

For $a < 1$ one can see that the right-hand side is integrable on $(0, +\infty)$: for the last term,

$$\begin{aligned}
\int_0^\infty e^{a\Lambda} \mathcal{L}_+(G) &= \int_0^\infty \mathcal{L}_+^*\big(e^{a\Lambda}\big) G = \int_0^\infty G(x) \int_0^x e^{a\Lambda(y)} b(x,y)\, dy\, dx \\
&\leqslant \int_0^\infty G(x) e^{a\Lambda(x)} \int_0^x b(x,y)\, dy\, dx = \kappa \int_0^\infty B(x) G(x) e^{a\Lambda(x)}\, dx < +\infty,
\end{aligned}$$

where we have used (26). The last expression is finite for $a < B_m/B_M$ due to Lemma 4.3 and the fact that

$$\Lambda(x) = \lambda x + \int_0^x B(y)\,dy \leqslant \lambda x + B_M \int_0^x y^\gamma\,dy$$
$$= \lambda x + \frac{B_M}{\gamma+1}x^{\gamma+1} \leqslant \frac{B_M}{B_m}\left(\lambda x + \frac{B_m}{\gamma+1}x^{\gamma+1}\right) = \frac{B_M}{B_m}\Lambda_m(x),$$

using the upper bound in (27). The other term in (71) is also integrable for similar reasons, and we deduce that $e^{a\Lambda}G \in BV(0,\infty) \subset L^\infty$, which proves (61).

In order to get (62) we need to prove that additionally, $G(x) \leqslant Cx$ for x small (say, $x \leqslant 1$). For this it is enough to notice that $G(0) = 0$ due to the boundary condition (5b), and also that the right-hand side of (71) is bounded for $x \in (0,1)$, as B and $e^{a\Lambda}G$ are, and

$$\mathcal{L}_+(G) = \int_x^\infty b(y,x)G(y)\,dy \leqslant C\int_0^\infty y^{\gamma-1}G(y)\,dy < +\infty,$$

due to Hypotheses 1.6 and 1.3. The last integral is finite because $\gamma > 0$ and we already know G is bounded. This finishes the proof.

Second step (lower bound). Writing Eq. (71) for $a = 1$ we obtain:

$$\left(Ge^{\Lambda}\right)' = e^{\Lambda}\mathcal{L}_+(G) \geqslant 0,$$

and hence

$$G(x)e^{\Lambda(x)} \geqslant G(\delta)e^{\Lambda(\delta)} \quad (x \geqslant \delta).$$

This proves the result, as $G(\delta) > 0$ by Theorem 1.8. □

4.2. Bounds for ϕ

First, in the case $B(x) = B$ constant, the first eigenvalue λ of the operator $-\partial_x + \mathcal{L}$ is explicitly given by $\lambda = B(\kappa - 1)$ (under Hypotheses 1.1–1.2), and $\phi(x) = C$ constant is a solution of (6), for some appropriate $C > 0$ determined by the normalization (6b).

In general the solution ϕ of the eigenproblem (6) is not explicit, and for its study we will use the following truncated problem: given $L > 0$, consider,

$$-\partial_x\phi_L + \big(B(x) + \lambda_L\big)\phi_L(x) = \mathcal{L}_+^*(\phi_L)(x) \quad (0 < x < L), \tag{72a}$$

$$\phi \geqslant 0, \quad \phi_L(L) = 0, \quad \int_0^L G(x)\phi_L(x)\,dx = 1. \tag{72b}$$

This approximated problem is slightly different from the one considered in [3], in that we are considering the first eigenvector G in the normalization (72b), and not an approximation G_L obtained by solving a similar truncated version of Eq. (5). However, this modification is not essential, and the results in [3] show the following (see also the truncated problems in [8,12]):

Lemma 4.4. *Assume Hypotheses* 1.1–1.4. *There exists $L_0 > 0$ such that for each $L \geqslant L_0$ the problem* (72) *has a unique solution (λ_L, ϕ_L), with $\lambda_L > 0$ and $\phi_L \in W^{1,\infty}_{\mathrm{loc}}$. In addition,*

$$\lambda_L \xrightarrow{L\to+\infty} \lambda, \tag{73}$$

$$\text{for every } A > 0, \quad \phi_L \xrightarrow{L\to+\infty} \phi \quad \text{uniformly on } [0,A), \tag{74}$$

where (λ, ϕ) is the unique solution of (6).

In the rest of this section we always consider $L \geqslant L_0$, so that Lemma 4.4 ensures the existence of a solution.

4.2.1. Upper bounds

In order to obtain bounds for ϕ we use a comparison argument, valid for each truncated problem on $[0, L]$, and then pass to the limit, as the bounds we obtain are independent of L. Let us do this. The function ϕ_L is a solution of the equation,

$$\mathcal{S}\phi_L(x) = 0 \quad \big(x \in (0, L)\big),$$

where $\mathcal{S}$ is the operator given by,

$$\mathcal{S}\phi(x) := -\phi'(x) + \big(\lambda_L + B(x)\big)\phi(x) - \int_0^x b(x, y)\phi(y)\,dy, \tag{75}$$

defined for all $\phi \in W^{1,\infty}(0, L)$, and for $x \in (0, L)$. The operator $\mathcal{S}$ satisfies the following *maximum principle*:

Definition 4.5. We say that $w \in W^{1,\infty}(0, L)$ is a *supersolution* of $\mathcal{S}$ on the interval $I \subseteq (0, L)$ when

$$\mathcal{S}w(x) \geqslant 0 \quad (x \in I).$$

Lemma 4.6 *(Maximum principle for $\mathcal{S}$). Assume Hypothesis* 1.1. *Take $A \geqslant 1/\lambda_L$. If w is a supersolution of $\mathcal{S}$ on (A, L), $w \geqslant 0$ on* $[0, A]$ *and $w(L) \geqslant 0$ then $w \geqslant 0$ on* $[A, L]$.

Proof. We will prove the lemma when $w \in \mathcal{C}^1([0, L])$, and then one can prove it for $w \in W^{1,\infty}(0, L)$ by a usual approximation argument. Assume the contrary: there exists $x_0 \in (A, L)$ such that $w(x_0) < 0$ and $w(x)/x$ attains a minimum, i.e.,

$$w(x_0) < 0, \tag{76}$$

$$\frac{w(x_0)}{x_0} \leqslant \frac{w(x)}{x} \quad \big(x \in (0, L)\big). \tag{77}$$

Then, because of (77), we have $w'(x_0) = w(x_0)/x_0$ and hence

$$\begin{aligned}
\mathcal{S}(x_0) &= -\frac{w(x_0)}{x_0} + \big(\lambda_L + B(x_0)\big)w(x_0) - \int_0^{x_0} b(x_0, y)w(y)\,dy \\
&\leqslant -\frac{w(x_0)}{x_0} + \big(\lambda_L + B(x_0)\big)w(x_0) - \frac{w(x_0)}{x_0}\int_0^{x_0} b(x_0, y)y\,dy \\
&= w(x_0)\left(\lambda_L - \frac{1}{x_0}\right) < 0,
\end{aligned}$$

which contradicts that w is a supersolution on (A, L). □

One can easily check that $v(x) = x$ is a supersolution of $\mathcal{S}$ on $(1/\lambda_L, L)$. A useful variant of that fact is the following:

Lemma 4.7. *Assume Hypotheses* 1.1–1.4*, and also that*

$$\frac{B(x)}{x^{\mu+1}} \to 0 \quad \text{as } x \to +\infty, \tag{78}$$

where μ is that in Hypothesis 1.4. *Take a smooth function $\eta : [0, +\infty) \to [0, 1]$, with compact support contained on* $[0, R]$. *Then, there exists $A > 0$ and $L_* > A$ such that the function*

$$v(x) = x + \eta(x)$$

is a supersolution of $\mathcal{S}$ on (A, L) for any $L > L_*$.

Proof. We calculate, using (4),

$$\mathcal{S}v(x) = -1 + \lambda_L x - \eta'(x) + \big(\lambda_L + B(x)\big)\eta(x) - \int_0^x b(x,y)\eta(y)\,dy,$$

which for $x > R$ becomes,

$$\mathcal{S}v(x) = -1 + \lambda_L x - \int_0^R b(x,y)\eta(y)\,dy \geqslant -1 + \lambda_L x - CR^{\mu}\frac{B(x)}{x^{\mu+1}},$$

as η is bounded by 1, and using Hypothesis 1.4. Due to (78), this is positive for all x greater than a certain number A which depends only on λ_L. As $\lambda_L \to \lambda$ when $L \to +\infty$ (see Lemma 4.4), one can choose A to be independent of L. □

Proposition 4.8. *Assume the hypotheses of Lemma* 4.7. *The solution ϕ of* (6) *satisfies,*

$$\phi(x) \leqslant C(1+x) \quad (x \geqslant 0), \tag{79}$$

for some $C > 0$.

Proof. Due to the uniform convergence (74) we have the bound,

$$\phi(x) \leqslant K(A) \quad \big(x \in [0, A]\big), \tag{80}$$

for some constant $K = K(A)$ which does not depend on L. This proves the bound on $(0, A]$ for a fixed A.

To prove the bound on all of $(0, +\infty)$ consider the function $v(x) = x + \eta(x)$ from Lemma 4.7 with $\eta = \mathbf{1}_{[0,1]}$, which is a supersolution on (A, L) for some $A > 0$. Then, for any $C > 0$, the function

$$w(x) := Cv(x) - \phi(x)$$

is a supersolution on (A, L). Since ϕ is bounded above on $(0, A)$ by (80), uniformly in L, we may choose $C \geqslant \|\phi\|_{L^\infty(0,A)}$ independently of L, such that

$$\phi(x) \leqslant Cv(x) \quad \big(x \in [0, A]\big),$$

or equivalently $w \geqslant 0$ on $[0, A]$. As $\phi(L) = 0$, so that $w(L) \geqslant 0$, Lemma 4.6 shows that $w \geqslant 0$ on $[0, L]$, which is the bound we wanted. □

4.2.2. Lower bounds

Let us look now for subsolutions.

Lemma 4.9. *Assume Hypotheses* 1.1–1.5*, and also that $B(x) \to +\infty$ as $x \to +\infty$. Let $\varphi : (-\infty, 0) \to [0, 1]$ be a decreasing $\mathcal{C}^1$ function which is* 1 *on $(-\infty, -\epsilon)$,* 0 *on $(-\epsilon/2, 0)$, and satisfies $|\varphi'(x)| \leqslant 4/\epsilon$ for $x \in (-\infty, 0)$. Take* $0 \leqslant k < 1$. *There is a number A which is independent of L for which $v(x) := x^k\varphi(x - L)$ is a subsolution of $\mathcal{S}$ on* (A, L).

Proof. First, from Corollary A.4 we have:

$$\int_0^x y^k b(x,y)\,dy \geqslant p'_k x^k B(x) \quad (x > 0)$$

for some $p'_k > 1$. Hence, by Lemma A.3, we may choose $\epsilon_* > 0$ such that

$$\int_0^{(1-\epsilon_*)x} y^k b(x,y)\,dy \geqslant C_k x^k B(x) \quad (x \geqslant 1), \tag{81}$$

with $C_k > 1$. Now, define $\varphi_L(x) = \varphi(x - L)$. We have, for $x > \max\{R_k, \epsilon/\epsilon_*\}$, and using (81),

$$\begin{aligned}
\mathcal{S}v(x) &:= -kx^{k-1}\varphi_L(x) - x^k\varphi_L'(x) + \big(\lambda_L + B(x)\big)x^k\varphi_L(x) - \int_0^x b(x,y)y^k\varphi_L(x)\,dy \\
&\leqslant \frac{4}{\epsilon}x^k + \big(\lambda_L + B(x)\big)x^k - \int_0^{x-\epsilon} b(x,y)y^k\,dy \\
&\leqslant x^k\left(\frac{4}{\epsilon} + \lambda_L - B(x)(C_k - 1)\right),
\end{aligned}$$

which is negative for x greater than some number A which depends only on λ_L and k. In order to be able to apply (81) we have also used that $x - \epsilon \geqslant (1 - \epsilon_*)x$ for $x \geqslant \epsilon/\epsilon_*$. □

Lemma 4.10. *Assume Hypotheses* 1.1–1.5*, and also that $B(x) \to +\infty$ as $x \to +\infty$. For any $0 \leqslant k < 1$ there is some constant $C_k > 0$ such that*

$$\phi(x) \geqslant C_k x^k \quad (x > 0).$$

Proof. Take φ as in Lemma 4.9, and let A be the one given there. The function,

$$w(x) = \phi_L(x) - Cx^k\varphi(x - L),$$

is a supersolution on (A, L) for any choice of $C > 0$. Now, the uniform convergence of $\{\phi_L\}$ from Lemma 4.4 together with the positivity of ϕ from Theorem 1.8 imply that there exists C_A such that for L large enough

$$\phi_L(x) \geqslant C_A \quad \big(x \in [0, A]\big),$$

which in turn implies

$$\phi_L(x) \geqslant \big(A^{-k}C_A\big)x^k \quad \big(x \in [0, A]\big).$$

With $C := A^{-k}C_A$ we have $w \geqslant 0$ on $[0, A]$, $w(L) = 0$, and we conclude using the maximum principle from Lemma 4.6, and again the locally uniform convergence of $\{\phi_L\}$ from Lemma 4.4. □

5. Proof of the main theorem

Finally, we are ready to complete the proof of Theorem 1.9. We give the proof of point 1 in the next subsection, and that of point 2 in the following one.

5.1. Exponential convergence to the asymptotic profile

Now that the inequality in Theorem 2.4 and the bounds on the profiles G and ϕ have been shown, we can use them to prove the first point in Theorem 1.9.

Proof for self-similar fragmentation. Let us show that Eqs. (44), (45), (46), (49) and (50) hold for G, ϕ. Then, as a direct application of Theorem 2.4, point 1 of Theorem 1.9 follows.

- The bound (44) is an immediate consequence of (56). With $\zeta(y) \equiv 1$ and whatever M and R be, (46) is satisfied due to the fact that $\phi(y) = y$ for the self-similar fragmentation model.
- Using (56) for any $a_1 > B_M/\gamma$ and $a_2 < B_m^2/(\gamma B_M)$ (with $0 < \gamma \leqslant 2$) we have for any $x \geqslant M$, $R > 1$ and for some constants denoted by C,

$$\int_{Rx}^{\infty} yG(y)\,dy \leqslant C\int_{Rx}^{\infty} ye^{-a_2y^\gamma}\,dy \leqslant C\int_{Rx}^{\infty} y^{\gamma-1}e^{-a'y^\gamma}\,dy$$
$$= Ce^{-a'R^\gamma x^\gamma} \leqslant Ce^{-a_1x^\gamma} \leqslant CG(x),$$

where we consider a' such that: $\frac{a_1}{R^\gamma} < a' < a_2(< a_1)$ (which is possible since $R > 1$), and that proves (45).

- We split the proof of (49) in two steps:
  - For $y < 2RM$: On one hand, we have $G(x)\phi(y) \leqslant Cy$. On the other hand, Hypotheses 1.7 and 1.3 show that $b(y,x) \geqslant Cy^{\gamma-1}$, so for $y < 2RM$ (49) holds, as $\gamma \leqslant 2$.
  - For $2RM \leqslant y \leqslant 2Rx$: We have, again using (56),

$$G(x)\phi(y) \leqslant Cye^{-a_2x^\gamma} \leqslant Cye^{-a_2y^\gamma/(2^\gamma R^\gamma)} \leqslant Cy^{\gamma-1},$$

  and we conclude as in the previous case, by means of Hypotheses 1.7 and 1.3.
- Finally, considering $\zeta(y) = 1$ and Hypothesis 1.7, we obtain $\zeta(y)y^{-1} = y^{-1} \leqslant Cy^{\gamma-1}$ for any $y \geqslant M$ because $\gamma \geqslant 0$, and therefore (50) holds. □

Proof for growth-fragmentation. As in the self-similar fragmentation case, we only need to show that for growth-fragmentation model, K, M, R can be chosen appropriately so that Eqs. (44), (45), (46), (49) and (50) hold for G, ϕ. In this way, as a direct application of Theorem 2.4, point 1 of Theorem 1.9 holds in that case.

First, for the case $b(x,y) = 2B_b/x$ with $B_b > 0$ a constant, the bound (67) holds, and $\phi(x) = C_\phi$ for some constant $C_\phi > 0$, as remarked at the beginning of Section 4.2. In this simpler case, (44) is a consequence of (61), (45) is a consequence of (67), and (46) obviously holds with $\zeta(y) \equiv 1$, since ϕ is a constant. Similarly, (49) is obtained from (61), and (50) is true with $\zeta(y) \equiv 1$ and $K = 2B_b$. This allows us to apply Theorem 2.4 and prove point 1 in Theorem 1.9 in this case.

Let us consider now the case $\gamma > 0$. Note that the requirements in Eq. (64) hold, as due to (29) one may take $\mu = 1$ in Hypothesis 1.4, and we have $\gamma \in (0,2)$. Hence, all the bounds in Theorem 4.1 are valid here.

- The bound (44) is an immediate consequence of (61). With $\zeta(y) = y^\epsilon$, where $0 < \epsilon < 1$ and whatever M and R are, (46) is satisfied due to the fact that $\phi(y)$ verifies (65): for $Rz < y < 2Rz$,

$$\frac{\phi(y)}{\zeta(y)} \leqslant C\bigl(1 + y^{1-\epsilon}\bigr) \leqslant C\bigl(C_0 + C_\epsilon z^{1-\epsilon}\bigr) \leqslant C\phi(z).$$

- Using (27), (65) and (66) we have for any $x \geqslant M$ and for some constants denoted by C,

$$\int_{Rx}^{\infty} \phi(y)G(y)\,dy \leqslant C\int_{Rx}^{\infty} (1+y)e^{-a_2y^{\gamma+1}}\,dy$$
$$\leqslant C\int_{Rx}^{\infty} y^\gamma e^{-a'y^{\gamma+1}}\,dy = Ce^{-a'(Rx)^{\gamma+1}} \leqslant CG(x),$$

  which holds by taking $0 < a' < a_2 < (B_m)^2/(B_M(\gamma+1))$ and $R > 1$ such that

$$a'R^{\gamma+1} > \frac{B_M}{\gamma+1}.$$

  This proves (45).
- As in the self-similar fragmentation case, we split the proof of (49) in two steps:
  - On the one hand for $y \leqslant 2RM$ and $x < y$, using (65) and (62) we have $G(x)\phi(y) \leqslant Cy(1+y)$ and using Hypothesis 1.7 one sees (49) holds because $0 < \gamma \leqslant 2$.
  - On the other hand for $y \geqslant 2RM$ and $x \geqslant \frac{y}{2R}$ (which falls in the case $\max\{2RM, 2Rx\} = 2Rx$) we have, again using (65), (66),

$$G(x)\phi(y) \leqslant C(1+y)e^{-a_2x^{\gamma+1}} \leqslant Cy^{\gamma-1},$$

  and we conclude due to Hypothesis 1.7.

- Finally, considering $\zeta(y) = y^\epsilon$ with $0 < \epsilon < \min\{\gamma, 1\}$ and Hypothesis 1.7, we obtain $\zeta(y)y^{-1} = y^{\epsilon-1} \leqslant Cy^{\gamma-1}$ for any $y \geqslant M$, and therefore (50) holds. □

5.2. *Spectral gap in L^2 space with polynomial weight*

Gathering the first point of Theorem 1.9 with some recent result obtained in [5] (see also [11] for the first results in that direction) we may enlarge the space in which the spectral gap holds and prove part 2 of Theorem 1.9.

We will make use of the following result:

Theorem 5.1. *(See [5].) Consider two Hilbert spaces H and $\mathcal{H}$ such that $H \subset \mathcal{H}$ and H is dense in $\mathcal{H}$. Consider two unbounded closed operators with dense domain L on H, Λ on $\mathcal{H}$ such that $\Lambda|_H = L$. On H assume that*:

1. *There is $G \in H$ such that $LG = 0$ with $\|G\|_H = 1$.*
2. *Defining $\psi(f) := \langle f, G\rangle_H G$, the space $H_0 := \{f \in H;\ \psi(f) = 0\}$ is invariant under the action of L.*
3. *$L - \alpha$ is dissipative on H_0 for some $\alpha < 0$, in the sense that*
$$\forall g \in D(L) \cap H_0 \quad \big((L-\alpha)g, g\big)_H \leqslant 0,$$
where $D(L)$ denotes the domain of L in H.
4. *L generates a semigroup e^{tL} on H.*
Assume furthermore on $\mathcal{H}$ that
5. *there exists a continuous linear form $\Psi : \mathcal{H} \to \mathbb{R}$ such that $\Psi|_H = \psi$;*
and Λ decomposes as $\Lambda = \mathcal{A} + \mathcal{B}$ with
6. *$\mathcal{A}$ is a bounded operator from $\mathcal{H}$ to H.*
7. *$\mathcal{B}$ is a closed unbounded operator on $\mathcal{H}$ (with same domain as $D(\Lambda)$ the domain of Λ) and satisfying the dissipation condition*
$$\forall g \in D(\Lambda) \quad \big((\mathcal{B}-\alpha)g, g\big)_{\mathcal{H}} \leqslant 0.$$

Then, for any $a \in (\alpha, 0)$ there exists $C_a \geqslant 1$ such that for any $g_{in} \in \mathcal{H}$ there holds:
$$\forall t \geqslant 0 \quad \big\|e^{t\Lambda} g_{in} - \Psi(g_{in})G\big\|_{\mathcal{H}} \leqslant C_a e^{at} \big\|g_{in} - \Psi(g_{in})G\big\|_{\mathcal{H}}.$$

Proof of part 2 in Theorem 1.9. We split the proof into several steps.

Step 1. Under the hypotheses of Theorem 1.9, take G, ϕ solutions of (5), (6), respectively. We define $H := L^2(\phi G^{-1}\,dx)$ and $\mathcal{H} := L^2(\theta\,dx)$ with $\theta = \phi(x) + x^k$, $k \geqslant 1$. Due to the bounds of G and ϕ proved above, one can see that $H \subseteq \mathcal{H}$.

We define:
$$\Lambda g := -a(x)\partial_x g - \big(\lambda + B(x)\big)g + \mathcal{L}_+ g$$
on $\mathcal{H}$ and $L := \Lambda|_H$ on H. We also define:
$$\Psi(g) := (g, G)_H = \int_0^\infty g\phi\,dx \quad (g \in \mathcal{H}),$$
and $\psi := \Psi|_H$. From part 1 in Theorem 1.9 it is clear that L satisfies points 1–4. Moreover, Ψ is correctly defined and continuous on $\mathcal{H}$ as soon as $k > 3$, so that $\mathcal{H} \subset L^1(\phi\,dx)$ thanks to Cauchy–Schwarz's inequality. To finish proving point 5, for given $M, R > 0$, we define $\chi := M\mathbb{1}_{[0,R]}$,
$$(\mathcal{A}g)(x) := g(x)\chi(x),$$
and
$$\mathcal{B}g := \big[-g\chi - a(x)\partial_x g - \lambda g\big] + \big[\mathcal{L}_+ g - B(x)g\big] = \Lambda(g) - g\chi,$$

so that $\Lambda = \mathcal{A} + \mathcal{B}$ and clearly $\mathcal{A}$ satisfies point 6. In order to conclude we have to establish that $\mathcal{B}$ satisfies point 7 for some well chosen k, M and R. Let us prove this separately for the cases $a(x) = x$ and $a(x) = 1$.

Step 2. The self-similar fragmentation equation. For $a(x) = x$, one has $\phi(x) = x$ and we may easily compute, for $m \geqslant 1$,

$$(\mathcal{B}g, g)_{L^2(x^m\,dx)} = T_1 + T_2 + T_3 + T_4 + T_5,$$

with

$$T_1 := \int_0^\infty (-x\partial_x g) g x^m = \int_0^\infty \frac{\partial_x(x^m x)}{2} g^2 = \frac{m+1}{2} \int g^2 x^m\,dx,$$

$$T_2 := \int_0^\infty (-2g) g x^m = -2 \int g^2 x^m\,dx,$$

$$T_3 := -\int_0^\infty B(x) g x^m g \leqslant -B_m \int g^2 x^{m+\gamma},$$

$$T_4 := -\int_0^\infty g^2 x^m \chi,$$

$$T_5 := \int_0^\infty (\mathcal{L}_+ g) x^m g.$$

Introducing the notation $\mathcal{G}(x) = \int_x^\infty |g(y)| y^{\gamma-1}\,dy$, we compute, using (29),

$$\begin{aligned} T_5 &\leqslant \int_0^\infty x^m \big|g(x)\big| \left(P_M B_M \int_x^\infty \big|g(y)\big| y^{\gamma-1}\,dy \right) dx \\ &= -\frac{P_M B_M}{2} \int_0^\infty 2\mathcal{G}\mathcal{G}' x^{m+1-\gamma}\,dx \\ &= \frac{P_M B_M}{2} \int_0^\infty \mathcal{G}^2 \partial_x\big(x^{m+1-\gamma}\big)\,dx. \end{aligned}$$

Thanks to Cauchy–Schwarz inequality we have for any $a > 1$

$$\mathcal{G}^2(x) \leqslant \int_x^\infty y^{2\gamma-2+a} g^2(y)\,dy \int_x^\infty y^{-a}\,dy \leqslant \frac{x^{1-a}}{a-1} \int_x^\infty y^{2\gamma-2+a} g^2(y)\,dy.$$

We then deduce

$$\begin{aligned} T_5 &\leqslant \frac{P_M B_M}{2} \frac{m+1-\gamma}{a-1} \int_0^\infty y^{2\gamma-2+a} g^2(y) \int_0^y x^{1-a} x^{m-\gamma}\,dx\,dy \\ &\leqslant \nu \int_0^\infty y^{\gamma+m} g^2(y)\,dy, \end{aligned}$$

with

$$\nu = \nu(a,m) = \frac{P_M B_M}{2}\mu(a,m), \quad \mu(a,m) := \frac{(m+1)-\gamma}{(m+1)-\gamma-(a-1)} \times \frac{1}{a-1},$$

provided that $m+2-\gamma-a>0$. In particular, we notice that for $m=1$ and $a^* = 2-\gamma/2 \in (1,2)$ we have $\nu(a^*,1) = 2P_M B_M/(2-\gamma)$, so that for $m=1$

$$T_5 \leqslant \frac{2P_M B_M}{2-\gamma} \int_0^\infty x^{1+\gamma} g^2\,dx =: C_0 \int_0^\infty x^{1+\gamma} g^2\,dx.$$

We also need to use the above calculation for $m=k$. We can find a and k such that

$$k>3,\ 1<a<k+2-\gamma, \quad \nu(a,k) < \frac{B_m}{2}.$$

To see this, take

$$a = 1 + \frac{2P_M B_M}{B_m}, \quad \text{so that} \quad \frac{1}{a-1} \leqslant \frac{B_m}{2p_m B_M}$$

and then take k large enough so that

$$\frac{k+1-\gamma}{k+1-\gamma-a+1} \leqslant 2.$$

Putting together the preceding estimates we have proved

$$(\mathcal{B}g,g)_{\mathcal{H}} \leqslant \int_0^\infty x g^2(x)\bigl\{-1-\chi+(C_0-1)x^\gamma\bigr\}\,dx + \int_0^\infty x^k g^2(x)\left\{\frac{k-3}{2} - \chi - \frac{B_m}{2}x^\gamma\right\}dx.$$

Recalling the definition of χ, for any $C>0$ we can find R and M large enough so that

$$(\mathcal{B}g,g)_{\mathcal{H}} \leqslant -C\int_0^\infty \theta g^2\,dx,$$

and that proves that assumption (7) in Theorem 5.1 is fulfilled with $\alpha = C = \beta$. The conclusion of Theorem 5.1 provides the conclusion in Theorem 1.9.

Step 3. The growth-fragmentation equation. In this case we have $a(x)=1$ and ϕ is the solution to the dual eigenvalue problem (6). We first compute,

$$(\mathcal{B}g,g)_{L^2(\phi\,dx)} = T_{123} + T_4 + T_5,$$

with

$$\begin{aligned} T_{123} &:= \int_0^\infty \{-\partial_x g - \lambda g - Bg\} g\phi \\ &= \int_0^\infty \left\{\frac{1}{2}\partial_x\phi - \lambda\phi - B\phi\right\} g^2 \\ &= -\frac{1}{2}\int_0^\infty \bigl\{\lambda\phi + B\phi + \mathcal{L}^{+*}\phi\bigr\} g^2 \leqslant 0, \\ T_4 &:= -\int_0^\infty g^2\chi\phi, \end{aligned}$$

$$
\begin{aligned}
T_5 &:= \int_0^\infty (\mathcal{L}_+ g)\phi g \\
&\leqslant \int_0^\infty B_M C_1 (1+x)\big|g(x)\big| \left(\int_x^\infty \big|g(y)\big| y^{\gamma-1}\, dy \right) dx \\
&\leqslant C_2 \int_0^\infty g^2 x^\gamma (1+x)\, dx \\
&\leqslant C_3 \int_0^\infty g^2 \big(\phi(x) + x^{1+\gamma}\big)\, dx,
\end{aligned}
$$

where as in the previous step we define $C_2 = (B_M C_1/2)(\mu(a,0) + \mu(a,1))$ for some $1 < a < 2 - \gamma$ (recall that here we have made the hypothesis $\gamma \in (0,1)$) and C_3 comes from the fact that ϕ is uniformly lower bounded by a positive constant. Following the computation of the previous step we easily get an estimate on $(\mathcal{B}g, g)_{L^2(x^k\,dx)}$, choosing k as before. Putting all together we obtain:

$$
\begin{aligned}
(\mathcal{B}g, g)_{\mathcal{H}} &= \int_0^\infty (\mathcal{B}g) g \big(\phi(x) + x^k\big)\, dx \\
&\leqslant \int_0^\infty \big\{ C_3 \big(\phi(x) + x^{1+\gamma}\big) - \chi\phi \big\} g^2\, dx + \int_0^\infty \left\{ \frac{k}{2} x^{k-1} - (\lambda + \chi) x^k - \frac{B_m}{2} x^{k+\gamma} \right\} g^2\, dx.
\end{aligned}
$$

Again, for any $C > 0$ we can find R and M large enough so that

$$
(\mathcal{B}g, g)_{\mathcal{H}} \leqslant -C \int_0^\infty \theta g^2\, dx,
$$

and we conclude as in the previous step. $\square$

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Appendix A

The following results are useful for dealing with weak conditions on the fragmentation coefficient $b(x, y)$.

Lemma A.1. *Let* $\{f_i\}_{i \in I}$ *be a family of nonnegative finite measures on* $[0,1]$*, indexed in some set* I*, and take* $k > 0$ *fixed. The following two statements are equivalent*:

$$
\exists \epsilon, \delta \in (0,1): \quad \int_{[1-\epsilon,1]} f_i \leqslant \delta \int_{[0,1]} f_i \quad \textit{for all } i \in I, \tag{82}
$$

$$
\exists P \in (0,1): \quad \int_{[0,1]} x^k f_i(x)\, dx \leqslant P \int_{[0,1]} f_i \quad \textit{for all } i \in I. \tag{83}
$$

Proof. First, assume (82) holds for some $\epsilon, \delta \in (0,1)$. Observe that (82) is easily seen to be equivalent to

$$\int_{[0,1-\epsilon)} f_i \geqslant (1-\delta) \int_{[0,1]} f_i \quad \text{for all } i \in I. \tag{84}$$

Using this,

$$\begin{aligned}
\int_{[0,1]} x^k f_i(x)\,dx &= \int_{[0,1-\epsilon)} x^k f_i(x)\,dx + \int_{[1-\epsilon,1]} x^k f_i(x)\,dx \\
&\leqslant (1-\epsilon)^k \int_{[0,1-\epsilon)} f_i(x)\,dx + \int_{[1-\epsilon,1]} f_i(x)\,dx \\
&= \int_{[0,1]} f_i(x)\,dx - \big(1-(1-\epsilon)^k\big) \int_{[0,1-\epsilon)} f_i(x)\,dx \\
&\leqslant \big(1-\big(1-(1-\epsilon)^k\big)(1-\delta)\big) \int_{[0,1]} f_i(x)\,dx \\
&= \big(\delta + (1-\delta)(1-\epsilon)^k\big) \int_{[0,1]} f_i(x)\,dx,
\end{aligned} \tag{85}$$

where (84) was used in the last step. This proves (83) with $P := \delta + (1-\delta)(1-\epsilon)^k < 1$.

Now, let us prove (82) assuming (83) by contradiction. Pick $\epsilon, \delta \in (0,1)$, and take $i \in I$ such that (82) is contradicted for these ϵ, δ. Then,

$$\begin{aligned}
\int_{[0,1]} x^k f_i(x)\,dx &\geqslant \int_{[1-\epsilon,1]} x^k f_i(x)\,dx \\
&\geqslant (1-\epsilon)^k \int_{[1-\epsilon,1]} f_i(x)\,dx \geqslant (1-\epsilon)^k \delta \int_{[0,1]} f_i(x)\,dx.
\end{aligned} \tag{86}$$

Hence, choosing δ close to 1 and ϵ close to 0 gives an $i \in I$ such that (83) is contradicted. □

Lemma A.2. *Let $\{f_i\}_{i \in I}$ be a family of nonnegative finite measures on* $[0,1]$, *indexed in some set I. The following two statements are equivalent*:

$$\forall \delta > 0,\ \exists \epsilon > 0\colon \quad \int_{[1-\epsilon,1]} f_i \leqslant \delta \int_{[0,1]} f_i \quad \textit{for all } i \in I. \tag{87}$$

There exists a strictly decreasing function $k \mapsto p_k$

$$\left.\begin{aligned}
&\textit{with} \quad 0 < p_k < 1, \quad \lim_{k \to +\infty} p_k = 0 \\
&\textit{and} \quad \int_{[0,1]} x^k f_i(x)\,dx \leqslant p_k \int_{[0,1]} f_i \quad \textit{for all } i \in I.
\end{aligned}\right\} \tag{88}$$

Proof. Let us first prove (88) assuming (87). Eq. (85) holds here also, so

$$\int_{[0,1]} x^k f_i(x)\,dx \leqslant \big(\delta + (1-\delta)(1-\epsilon)^k\big) \int_{[0,1]} f_i(x)\,dx.$$

Choosing δ small enough, and then k large enough, one can take p_k so that (88) holds.

Now, let us prove the other implication by contradiction. Assume (88) does not hold, so there is some $\delta > 0$ such that, for every $\epsilon > 0$, (88) fails at least for some $i \in I$. With the same calculation as in (86), choosing $\epsilon = 1 - (1/2)^{1/k}$, we have that for every $k \geqslant 1$ there is some $i \in I$ such that

$$\int_{[0,1]} x^k f_i(x)\,dx \geqslant \frac{\delta}{2} \int_{[0,1]} f_i(x)\,dx.$$

This contradicts (88). □

Lemma A.3. *Consider a fragmentation coefficient b satisfying Hypotheses* 1.1*,* 1.2 *and* 1.5*, and take* $0 \leqslant k \leqslant 1$*. For every $\delta > 0$ there exists an $\epsilon > 0$ such that*

$$\int_0^{(1-\epsilon)x} y^k b(x,y)\,dy \geqslant (1-\delta) \int_0^x y^k b(x,y)\,dy \quad (x > 0). \tag{89}$$

Proof. Equivalently, we need to prove that for every $\delta > 0$ there exists $\epsilon > 0$ such that

$$\int_{(1-\epsilon)x}^{x} y^k b(x,y)\,dy \leqslant \delta \int_0^x y^k b(x,y)\,dy \quad (x > 0). \tag{90}$$

Using Hypothesis 1.5, take $\epsilon > 0$ such that (28) holds with δ/κ instead of δ, where κ is the one in Hypothesis 1.2. Then,

$$\begin{aligned}\int_{(1-\epsilon)x}^{x} y^k b(x,y)\,dy &\leqslant x^k \int_{(1-\epsilon)x}^{x} b(x,y)\,dy \leqslant x^k \frac{\delta}{\kappa} \int_0^x b(x,y)\,dy \\ &= x^k \delta B(x) = x^k \delta \int_0^x \frac{y}{x} b(x,y)\,dy \leqslant \delta \int_0^x y^k b(x,y)\,dy. \qquad \square\end{aligned}$$

Corollary A.4. *Consider a fragmentation coefficient b satisfying Hypotheses* 1.1, 1.2 *and* 1.5*. Then there exists a strictly decreasing function $k \mapsto p_k$ for $k \geqslant 0$ with* $\lim_{k\to+\infty} p_k = 0$*,*

$$p_k > 1 \quad \textit{for } k \in [0,1), \qquad p_1 = 1, \qquad 0 < p_k < 1 \quad \textit{for } k > 1, \tag{91}$$

and such that

$$\int_0^x y^k b(x,y)\,dy \leqslant p_k x^k B(x) \quad (x > 0,\ k > 0). \tag{92}$$

Also, for each $0 \leqslant k < 1$ there exists $p'_k > 1$ such that

$$\int_0^x y^k b(x,y)\,dy \geqslant p'_k x^k B(x) \quad \bigl(x > 0,\ k \in [0,1)\bigr). \tag{93}$$

Proof. Apply Lemma A.2 to the set of measures $\{f_x\}_{x>0}$ given by:

$$f_x(z) := b(x, xz) \quad \bigl(z \in [0,1]\bigr),$$

for which Hypothesis 1.5 gives precisely (87). Then, by a change of variables and using Hypothesis 1.2, (88) is exactly (92).

For the second part, fix $0 \leqslant k < 1$. Applying Lemma A.2 to the set of measures $\{z^k b(x,xz)\}_{x>0}$ gives p'_k so that (93) holds, as this set also satisfies (87) (by Lemma A.3). □

Remark A.5. One can omit Hypothesis 1.2 in the previous corollary and still get the result for $k \geqslant 1$ by taking $f_x(z) := zb(x, xz)$ in the proof.

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