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Estimates for Solutions of Reduced Hyperbolic Equations of the Second Order with a Large Parameter*

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We consider solutions of inhomogeneous, reduced hyperbolic equations of the second order, with a large parameter multiplying the unknown function. These solutions are defined on the m-dimensional region outside a star-shaped body. They satisfy an "outgoing" radiation condition at infinity and a Dirichlet boundary condition.

We obtain a priori estimates for these solutions, at every point outside or on the surface of a two- or three-dimensional star-shaped body, that hold for sufficiently large values of the parameter. We prove that each solution is bounded by a linear combination of (i) the maximum norm of its prescribed boundary values, (ii) the L_2 norm of the prescribed values of its tangential derivative, (iii) an L_2 norm of the source term. This result is based on similar inequalities that we first obtain for a certain L_2 norm of the gradient, and of the normal derivative on the boundary, of each solution defined outside an m-dimensional star-shaped body.

For the special case of the reduced wave equation, Morawetz and Ludwig [1] have obtained similar estimates. Just as their results have been used in [3] to confirm the geometrical theory of diffraction, the estimates obtained in this paper can be used to establish the validity of certain formal asymptotic solutions of reduced hyperbolic equations.

1. INTRODUCTION

In this paper we establish a priori estimates for solutions of second order, uniformly elliptic partial differential equations of the form

$$L_\lambda u = (A(x) \cdot \nabla) \cdot \nabla u + a(x) \cdot \nabla u + \lambda u = f(x, \lambda),$$

where $A(x)$ is a symmetric matrix. These estimates are for solutions defined in the m-dimensional exterior of a smooth star-shaped body, that satisfy the radiation condition

$$\lim_{r' \to \infty} \int_{r=r'} r \left| \frac{\partial u}{\partial r} - i\lambda u + \frac{(m-1)}{2r} u \right|^2 dS = 0,$$

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and which reduce to a prescribed function on the boundary ∂V of the star-shaped body.

Our estimates are obtained under the hypothesis that

$$\lim_{r\to\infty} A(x) = I, \qquad \text{uniformly},$$

where I is the identity matrix, and that

$$\left| \frac{\partial}{\partial r} A(x) \right| \leqslant \frac{C_1}{r^{p+1}}, \qquad | \nabla \cdot A(x) - a(x)| \leqslant \frac{C_1}{r^{p+1}},$$

if $r \geqslant r_1 \gg 1$ where C_1 is a constant, and $p > 2$.

Let n be the outward unit normal to the boundary ∂V of the region V where the solution $u(x)$ is defined. We establish first that $\| u_n \|_{\partial V}$ (the L_2 norm of the normal derivative of $u(x)$ on ∂V) and $\| \nabla u/r \|_V$ (the L_2 norm of $\nabla u/r$), are bounded from above by a linear combination of $\| \nabla u - nu_n \|_{\partial V}$ (the L_2 norm of the tangential derivative of $u(x)$ on ∂V), the L_2 norm $\| rf \|_V$, and $\lambda \| u \|'_{\partial V}$ (the maximum of $\lambda \mid u(x)|$ on ∂V). The constants in this linear combination depend on $a(x)$, $A(x)$, and first derivatives of the elements of $A(x)$, but are independent of λ. These estimates hold as $\lambda \to \infty$ if

$$\max_{V(r_1)\cup\partial V} | r(\nabla \cdot A - a)| + \tfrac{1}{2} \max_{V(r_1)\cup\partial V} | r(\partial/\partial r) A(x)|$$
$$\leqslant \min_{V(r_1)\cup\partial V} (\min_{|\xi|=1} (\xi \cdot (A(x) \cdot \bar{\xi})))$$

where $V(r_1) = V \cap \{x: | x | \leqslant r_1\}$. (If $L_\lambda u = f$ is the reduced wave equation for an inhomogeneous medium, i.e., if $a(x) = 0$ and $A(x) = \kappa(x) I$, we require instead that

$$\max_{V(r_1)\cup\partial V} | r(\nabla\kappa(x) - (x/r)\,\kappa(x))| + \tfrac{1}{2} \max_{V(r_1)\cup\partial V} | r(\partial/\partial r)\,\kappa(x)| \leqslant \min_{V(r_1)\cup\partial V} \kappa(x).$$

Making use of the estimates for $\| u_n \|_{\partial V}$ and $\| \nabla u/r \|_V$ we subsequently obtain an inequality of similar form for $\lambda^{-(1+m)/2} \mid u(x)|$ $(m = 2, 3)$ that holds as $\lambda \to \infty$, uniformly on $(V \cup \partial V) \cap \{x: | x | \leqslant r_2 - \delta, 0 < \delta < r_2/2\}$ for every value of r_2 such that $\{x: | x | < r_2/2\} \supset \partial V$.

For solutions of the reduced wave equation our estimates reduce to those obtained by Morawetz and Ludwig [1]. They were able to establish the mathematical validity of the geometrical theory of optics by using them.

Our estimates can be applied in a similar way to establish the asymptotic character of formal series solutions that depend on λ in the same way as the expansions of geometrical optics, and also of certain diffraction expansions (cf. [2, 3]).

In Section 3 of this paper we present the inequality that is basic in obtaining our estimates for $\| u_n \|_{\partial V}$ and $\| \nabla u/r \|_V$. This inequality is derived from an identity that expresses the divergence of a certain vector with components that are quadratic forms in u, and the first derivatives of u, as the sum of quadratic forms in these quantities, and the product of $L_\lambda u$ with a linear combination of u and its first derivatives. The identity is derived in Appendix I.

In Section 4 we integrate the basic inequality over the region V exterior to the star-shaped body. The result is an inequality, which implies that under the above conditions $(\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V})^2$ and $(\| \nabla u/r \|_V)^2$ are each bounded from above by a linear combination of $(\| u \|_{\partial V})^2$, $(\| \nabla u - nu_n \|_V)^2$, $(\| rL_\lambda u \|_V)^2$ and $(\| u/r \|_V)^2$. This is true for all sufficiently large λ.

In Section 5 we first establish that the quantity $\lambda^2(\| u/r \|_V)^2$ is bounded from above by a linear combination (with coefficients independent of λ) of the quantities $(\| u \|_{\partial V})^2$, $(\| \nabla u - nu_n \|_{\partial V})^2$, $(\| rL_\lambda u \|_V)^2$, $(\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V})^2$, and $(\| \nabla u/r \|_V)^2$. The first three of these are known a priori, while the last two are not. This result, together with the estimates of Section 4, imply that if λ is sufficiently large, then both $\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V}$ and $\| \nabla u/r \|_V$ are bounded from above by a linear combination of quantities that are all known a priori, viz., $\| u \|_{\partial V}$, $\| \nabla u - nu_n \|_{\partial V}$ and $\| rL_\lambda u \|_V$.

In Section 6 we derive an estimate for $| u(x)|$ that holds uniformly on $(V \cup \partial V) \cap \{x: | x | \leqslant r_2 - \delta, 0 < \delta < r_2/2\}$. We establish for $m = 2, 3$ that $\lambda^{(1-m)/2} | u(x)|$ is bounded from above by a linear combination of (the known quantities) $\lambda \| u \|'_{\partial V}$, $\| \nabla u - nu_n \|_{\partial V}$, $\| rL_\lambda u \|_V$, and a linear combination of (the unknown quantities) $\lambda^2 \| u/r \|_V$, $\lambda \| \nabla u/r \|_V$, $\| u_n \|_{\partial V}$.

Finally, if the estimates of Sections 5 and 6 are combined, the result obtained is that $\lambda^{-(1+m)/2} | u(x)|$ is bounded from above by a constant multiple of the sum of $\lambda \| u \|'_{\partial V}$, $\| \nabla u - nu_n \|_{\partial V}$ and $\| rL_\lambda u \|_V$. The multiple is constant with respect to λ and depends only on norms of $A(x)$, $a(x)$, and first derivatives of the elements of $A(x)$.

2. Glossary

Notation

1. λ is a positive real number.
2. x is an m-dimensional row vector with components $x^1, x^2, x^2, \ldots, x^m$;

$$r = | x | = \left(\sum_{i=1}^{m} (x^i)^2 \right)^{1/2}.$$

3. $\rho(x)$, $\gamma(x)$, $\theta(x)$, $\nu(x)$, $\mu(x)$ and $\omega(x)$ are real valued functions of x.

4. $u(x)$ is a complex valued function of x; $|u| = (u\bar{u})^{1/2}$.

5. $a(x)$ and $b(x)$ are row vectors with components $a^1(x)$, $a^2(x)$, $a^3(x),\ldots,$ $a^m(x)$ and $b^1(x)$, $b^2(x)$, $b^3(x),\ldots,$ $b^m(x)$ that are real valued functions of x.

6. I is the $m \times m$ identity matrix. $A(x)$ is a matrix with rows $A^{1*}(x)$, $A^{2*}(x),\ldots,$ $A^{m*}(x)$ and columns $A^{*1}(x)$, $A^{*2}(x),\ldots,$ $A^{*m}(x)$. The elements of $A(x)$ are the real valued functions $A^{ij}(x)$, where $i, j = 1, 2, 3,\ldots, m$.

7. $\mathbf{B}(x)$ is a row vector with components $B^1(x)$, $B^2(x),\ldots,$ $B^m(x)$ that are $m \times m$ matrices. The elements of $B^k(x)$ are the real valued functions $B^{ijk}(x)$.

Operations

1. If v and w are row vectors with scalar components $v^1, v^2,\ldots, v^m$ and $w^1, w^2,\ldots, w^m$, then

(i) $vw = (v^iw^j)_{m\times m}$,

(ii) $v \cdot w = \sum_{i=1}^m v^iw^i$,

(iii) $|v| = (v \cdot \bar{v})^{1/2}$.

2. If V is a row vector with components $V^1, V^2,\ldots, V^m$ that are row vectors with scalar components, then

(i) $|V| = \sum_{i=1}^m |V^i|$,

(ii) $V \cdot v = (V^1 \cdot v, V^2 \cdot v,\ldots, V^m \cdot v)$,
$v \cdot V = (v \cdot V^1, v \cdot V^2,\ldots, v \cdot V^m)$.

3. If M is a matrix with rows $M^{1*}, M^{2*},\ldots, M^{m*}$, then

(i) $|M| = (\sum_{i=1}^m |M^{i*}|^2)^{1/2}$,

(ii) $M \cdot v = (M^{1*} \cdot v, M^{2*} \cdot v,\ldots, M^{m*} \cdot v)$, $v \cdot M = \sum_{i=1}^m v^iM^{i*}$.

4. If $\mathbf{M}$ is a row vector with matrix components $M^1, M^2,\ldots, M^m$, then

(i) $|\mathbf{M}| = (\sum_{k=1}^m |M^k|^2)^{1/2}$

(ii) $v \cdot \mathbf{M} = (v \cdot M^1, v \cdot M^2,\ldots, v \cdot M^m)$,
$\mathbf{M} \cdot v = (M^1 \cdot v, M^2 \cdot v,\ldots, M^m \cdot v)$.

5. If $s(x)$ is a complex valued function of x, then $\nabla s(x)$ is a row vector with components $s_1(x), s_2(x),\ldots, s_m(x)$ where $s_k(x) = \partial s(x)/\partial x^k$.

6. If $v(x)$ is a row vector with components $v^1(x), v^2(x),\ldots, v^m(x)$, then

$$\nabla \cdot v(x) = \sum_{i=1}^m v_i^i(x).$$

7. If $M(x)$ is a matrix with rows $M^{1*}(x), M^{2*}(x),\ldots, M^{m*}(x)$, then

$$\nabla \cdot M(x) = \sum_{i=1}^m M_i^{i*}(x) \quad \text{and} \quad |\nabla \cdot M(x)| = \left(\sum_{i=1}^m |M_i^{i*}(x)|^2\right)^{1/2}$$

where

$$M_i^{i*}(x) = \partial M^{i*}(x)/\partial x^i.$$

8. If $\mathbf{M}$ is a row vector matrix components $M^1(x), M^2(x),..., M^m(x)$, then

$$\nabla \cdot \mathbf{M} = \sum_{k=1}^{m} M_k^k(x),$$

where

$$M_k^k(x) = \partial M^k(x)/\partial x^k.$$

9. If $F(x)$ is a complex valued function, a row vector with scalar components, a row vector with vector components, a matrix or a row vector with matrix components, then

(i) $\| F \|_D = \left(\int_{\bar{D}} | F(x)|^2 \, dx \right)^{1/2}$,

(ii) $\| F \|_D' = \max_{x \in \bar{D}} | F(x)|$,

(iii) $\| F \|_D'' = \int_{\bar{D}} | F(x)| \, dx$,

where $\bar{D}$ is the closure of D.

10. If $u(x)$ is a twice differentiable complex valued function of x, then

$$L_\lambda u = \nabla \cdot (A \cdot \nabla u) + (-(\nabla \cdot A) + a) \cdot \nabla u + \lambda^2 u.$$

3. The Basic Inequality

The a priori estimates derived in this paper are based on the following inequality:

$$\begin{aligned}
&- \nabla \cdot \operatorname{Re}[((\nabla u \cdot \mathbf{B}) + (-i\lambda\rho + \gamma) \, uA) \cdot \nabla \bar{u} + \lambda^2 (b/2) \, | \, u \, |^2] \\
&\quad \leqslant \left[\frac{\gamma(\rho^2 + 1)}{2 \, | \, c \, |^2} + \frac{\rho^2}{2\omega \, | \, c \, |^2} + \frac{| \, b \, |^2}{4\mu} + \frac{\rho^2}{16\nu \, | \, c \, |^2} \left(\frac{(b \cdot c) \, \sigma}{| \, c \, |^2 \, \rho} - 1 \right)^2 \right] | \, L_\lambda u \, |^2 \\
&\quad\quad + \left[\frac{\gamma \, | \, c \, |^2}{2(\rho^2 + 1)} + \frac{| \, d \, |^2}{\theta} \right] | \, u \, |^2 - \left(1 - \frac{\omega}{4} \right) \sigma \left| \frac{c \cdot \nabla u}{\sigma} - i\lambda u \right|^2 \\
&\quad\quad - \operatorname{Re} \nabla u \cdot ((\gamma A - ab + \nabla \cdot \mathbf{B} - (\tfrac{1}{2} + \theta + \nu) I) \cdot \nabla \bar{u}).
\end{aligned} \tag{1}$$

Here

$$2B^k = bA^{k*} + A^{k*}b - b^k A.$$

The above inequality holds under the assumption that A is a symmetric matrix whence

$$B^{ijk} = B^{jik}.$$

Also

$$\begin{aligned} 2d &= \nabla \cdot (\gamma A) - \gamma a, \\ 2c &= \nabla \cdot (\rho A) - \rho a, \\ \sigma &= (\nabla \cdot b)/2 - \gamma. \end{aligned}$$

The vector b and the scalars σ and γ must be chosen so that

$$\sigma \geqslant 2 \mid c \mid^2.$$

Finally, ν and μ must be chosen so that

$$\nu - \mu = -\tfrac{1}{2}.$$

Inequality (1) is derived as follows, from the identity:

$$\begin{aligned} &- \nabla \cdot \mathrm{Re}[((\nabla u \cdot \mathbf{B}) + (-i\lambda\rho + \gamma)\, uA) \cdot \nabla \bar{u} + \lambda^2(b/2) \mid u \mid^2] \\ &\quad = - \mathrm{Re}[b \cdot \nabla u + (-i\lambda\rho + \gamma)\, u]\, L_\lambda \bar{u} - 2\mathrm{Re}\, u(d \cdot \nabla \bar{u}) + \frac{\mid c \cdot \nabla u \mid^2}{\sigma} \\ &\qquad - \sigma \left| \frac{c \cdot \nabla u}{\sigma} - i\lambda u \right|^2 - \mathrm{Re}\, \nabla u \cdot ((\gamma A - ab + \nabla \cdot \mathbf{B}) \cdot \nabla \bar{u}). \qquad (2) \end{aligned}$$

(The derivation of (2) is given in Appendix I.)

We note first that

$$\begin{aligned} &\mathrm{Re}[b \cdot \nabla u + (-i\lambda\rho + \gamma)\, u]\, L_\lambda \bar{u} \\ &\quad = \mathrm{Re}\, \rho \left[\frac{c \cdot \nabla u}{\sigma} - i\lambda u\right] L_\lambda \bar{u} + \mathrm{Re}\left[\frac{b \cdot c}{\mid c \mid} - \frac{\rho \mid c \mid}{\sigma}\right] \frac{c \cdot \nabla u}{\mid c \mid} L_\lambda \bar{u} \\ &\quad + \mathrm{Re}\left[\frac{-(b \cdot c)}{\mid c \mid} \frac{c}{\mid c \mid} + b\right] \cdot \nabla u L_\lambda \bar{u} + \mathrm{Re}\, \gamma u L_\lambda \bar{u}. \end{aligned}$$

The following inequalities hold for the terms on the right side of the preceding equation.

$$\begin{aligned} &- \mathrm{Re}\left[- \frac{(b \cdot c)}{\mid c \mid} \frac{c}{\mid c \mid} + b\right] \cdot \nabla u L_\lambda \bar{u} \\ &\qquad \leqslant - \frac{\mu \mid c \cdot \nabla u \mid^2}{\mid c \mid^2} + \mu \mid \nabla u \mid^2 + \frac{\mid b \mid^2}{4\mu} \mid L_\lambda u \mid^2, \\ &- \mathrm{Re}\, \rho \left[\frac{c \cdot \nabla u}{\sigma} - i\lambda u\right] L_\lambda \bar{u} \\ &\qquad \leqslant \frac{\omega}{2} \mid c \mid^2 \left| \frac{c \cdot \nabla u}{\sigma} - i\lambda u \right|^2 + \frac{\rho^2}{2\omega \mid c \mid^2} \mid L_\lambda u \mid^2, \end{aligned}$$

$$-\operatorname{Re}\gamma u L_\lambda \bar{u} \leqslant \frac{\gamma}{2}\frac{|c|^2}{(\rho^2+1)}|u|^2 + \frac{\gamma}{2}\frac{(\rho^2+1)}{|c|^2}|L_\lambda u|^2,$$

$$-\operatorname{Re}\left[\frac{b\cdot c}{|c|} - \frac{\rho|c|}{\sigma}\right]\frac{c\cdot\nabla u}{|c|}L_\lambda\bar{u}$$
$$\leqslant \frac{\nu|c\cdot\nabla u|^2}{|c|^2} + \frac{(\rho|c|)^2}{4\nu\sigma^2}\left(\frac{(b\cdot c)\sigma}{|c|^2\rho} - 1\right)^2|L_\lambda u|^2.$$

Also, if $\sigma \geqslant 2|c|^2$, then the last term on the right side of the preceding inequality is obviously less than

$$\frac{\rho^2}{16\nu|c|^2}\left(\frac{(b\cdot c)\sigma}{|c|^2\rho} - 1\right)^2|L_\lambda u|^2.$$

Consequently, if $\sigma \geqslant 2|c|^2$, and $\nu - \mu = -\frac{1}{2}$, we have

$$\begin{aligned}&-\operatorname{Re}[b\cdot\nabla u + (-i\lambda\rho+\gamma)u]L_\lambda\bar{u}\\ &\leqslant\left[\frac{|b|^2}{4\mu} + \frac{\rho^2}{2\omega|c|^2} + \frac{\gamma(\rho^2+1)}{|c|^2} + \frac{\rho^2}{16\nu|c|^2}\left|\frac{(b\cdot c)\sigma}{|c|^2\rho} - 1\right|^2\right]|L_\lambda u|^2 \qquad (3)\\ &\quad -\frac{|c\cdot\nabla u|^2}{2|c|^2} + \frac{\gamma|c|^2}{2(\rho^2+1)}|u|^2 + \mu|\nabla u|^2 + \omega\frac{\sigma}{4}\left|\frac{c\cdot\nabla u}{\sigma} - i\lambda u\right|^2.\end{aligned}$$

Furthermore, if θ is any scalar function, and $\sigma \geqslant 2|c|^2$, we have

$$-2\operatorname{Re}u(d\cdot\nabla\bar{u}) \leqslant \frac{|d|^2}{\theta}|u|^2 + \theta|\nabla u|^2,$$
$$\frac{|c\cdot\nabla u|^2}{\sigma} \leqslant \frac{1}{2}\frac{|c\cdot\nabla u|^2}{|c|^2}. \qquad (4)$$

We finally get inequality (1) by using (3) and (4) to estimate the first three terms on the right-hand side of identity (2).

4. Integration of the Basic Inequality

Assume now that

$$|A_r(x)| \leqslant \frac{C_1}{r^{p+1}}, \qquad \text{and} \qquad |\nabla\cdot A(x) - a(x)| \leqslant \frac{C_1}{r^{p+1}}$$

if $r \geqslant r_1 \gg 1$, where C_1 is a constant, and $p > 2$. Assume also that

$$\lim_{r\to\infty} A(x) = I,$$

uniformly with respect to the angular variables.

Suppose that ∂V is star-shaped, and that the radiation condition

$$\lim_{r'\to\infty} \int_{r=r'} r \left| \frac{x}{r} \cdot \nabla u - i\lambda u + \frac{(m-1)}{2r} u \right|^2 dS = 0$$

is satisfied.

In (1) we set

$$b = x\Gamma,$$

with

$$\Gamma = \begin{cases} \epsilon'^{-1} + 1 & \text{if} \quad r_0 \leqslant r \leqslant r_3, \quad r_3 > r_1, \\ -\left(\left(\frac{r_3}{r}\right) - 1\right)^2 \epsilon'^{-1} + \epsilon'^{-1} + 1 & \text{if} \quad r \geqslant r_3. \end{cases}$$

We define γ by the equation

$$\sigma = \frac{\nabla \cdot b}{2} - \gamma = 2 \mid c \mid^2.$$

This choice of γ is obviously consistent with the requirement that $\sigma \geqslant 2 \mid c \mid^2$.

Under the assumption that ∂V is star-shaped, we have

$$\min_{\partial V} n \cdot b \geqslant \beta = \min_{\partial V} n \cdot x > 0.$$

Next, we choose the scalar function ρ so that $\mid c \mid^{-2}$ is uniformly bounded on $\partial V \cup V$, and so that, as $r \to \infty$, we have

$$\rho = r + O\left(\frac{1}{r}\right), \quad \nabla \rho = \frac{x}{r}\left(1 + O\left(\frac{1}{r^2}\right)\right), \quad \nabla \mid \nabla \rho \mid = O\left(\frac{1}{r^3}\right),$$

uniformly in the angular variables. (See Appendix II, where we show how to construct a function ρ with these properties.)

As for the remaining scalar functions in (1) we set

$$\nu = \epsilon/3r^2, \quad \theta = \nu/2, \quad \mu = (1 + 2\nu)/2, \quad \omega = 2.$$

Under the above conditions, integration of (1) over $\partial V \cup V$ leads to the following inequality:

$$\begin{aligned} &\frac{(1-\epsilon)}{2} \beta \int_{\partial V} n \cdot (A \cdot n) \mid u_n \mid^2 dS \\ &\quad - \int_{\partial V} \left[\mid \mathbf{B} \mid + \frac{\mid A \mid \mid b \mid^2}{(n \cdot b)\, \epsilon} + \mid A \mid \mid b \mid \epsilon \right] \mid \nabla u - n u_n \mid^2 dS \\ &\quad - \int_{\partial V} \left[\frac{3}{2} \frac{(\lambda\rho + \gamma)^2}{(n \cdot b)\, \epsilon} \mid A \mid - \lambda^2 \frac{n \cdot b}{2} \right] \mid u \mid^2 dS \end{aligned} \tag{5}$$

$$\leqslant -\int_V \operatorname{Re} \nabla u \cdot ((\gamma A - ab + \nabla \cdot \mathbf{B} - (\tfrac{1}{2} + \theta + \nu) I) \cdot \nabla \bar{u})\, dV$$
$$- \int_V \left(1 - \frac{\omega}{4}\right) \sigma \left| \frac{c \cdot \nabla u}{\sigma} - i\lambda u \right|^2 dV + \int_V \left[\frac{\gamma \mid c \mid^2}{2(\rho^2 + 1)} + \frac{\mid d \mid^2}{\theta}\right] \mid u \mid^2 dV$$
$$+ \int_V \left[\frac{\gamma(\rho^2 + 1)}{2 \mid c \mid^2} + \frac{\rho^2}{2\omega \mid c \mid^2} + \frac{\mid b \mid^2}{4\mu} + \frac{\rho^2}{16\nu \mid c \mid^2} \left(\frac{(b \cdot c)\, \sigma}{\mid c \mid^2 \rho} - 1\right)^2\right]$$
$$\times \mid L_\lambda u \mid^2 dV. \tag{5}$$

Here ϵ is any positive constant less than one.

To establish (5) we first integrate (1) over the region outside ∂V, and inside the sphere $S_{r'} = \{x: \mid x \mid = r'\}$. The left side of (1) integrates into the difference $I_1 - I_2$, where

$$I_j = \int_{S_j} \operatorname{Re} \left[n_j \cdot \left(\nabla u \cdot \mathbf{B} + (-i\lambda\rho + \gamma)\, uA) \cdot \nabla \bar{u} + \lambda^2 \frac{b \mid u \mid^2}{2}\right)\right] dS,$$

with $S_1 = \partial V$, $n_1 = n$ (outward unit normal to ∂V); $S_2 = S_{r'}$ and $n_2 = x/r$.

Under the conditions imposed above it can be shown that

$$\lim_{r' \to \infty} I_2 = 0.$$

We get (5) by letting $r' \to \infty$, and then making use of the inequality

$$I_1 \geqslant \frac{(1 - \epsilon)}{2} \beta \int_{\partial V} n \cdot (A \cdot n) \mid u_n \mid^2 dS$$
$$- \int_{\partial V} \left[\mid \mathbf{B} \mid + \frac{\mid A \mid \mid b \mid^2}{(n \cdot b)\, \epsilon} + \mid A \mid \mid b \mid \epsilon\right] \mid \nabla u - n u_n \mid^2 dS \tag{6}$$
$$- \int_{\partial V} \left[\frac{3(\lambda\rho + \gamma)^2}{2(n \cdot b)\, \epsilon} \mid A \mid - \lambda^2 \frac{(n \cdot b)}{2}\right] \mid u \mid^2 dS.$$

This inequality is obtained in a straightforward way, once it is established that

$$\begin{aligned} &\operatorname{Re} n \cdot ((\nabla u \cdot \mathbf{B}) \cdot \nabla \bar{u}) \\ &\quad = \tfrac{1}{2}\, n \cdot (A \cdot n)\, (n \cdot b) \mid u_n \mid^2 + n \cdot (A \cdot n) \operatorname{Re} \bar{u}_n (\nabla u - n u_n) \cdot b \\ &\qquad + n \cdot [((\nabla u - n u_n) \cdot \mathbf{B}) \cdot (\nabla \bar{u} - n \bar{u}_n)], \end{aligned}$$

and that

$$\begin{aligned} &\operatorname{Re}(-i\lambda\rho + \gamma)\, u(A \cdot \nabla \bar{u}) \\ &\quad = \operatorname{Re}(-i\lambda\rho + \gamma)\, un \cdot (A \cdot n)\, \bar{u}_n + \operatorname{Re}(-i\lambda\rho + \gamma)\, un \cdot (A \cdot (\nabla \bar{u} - n \bar{u}_n)). \end{aligned}$$

We obtain (6) by first integrating both sides of the last two identities over ∂V, and then making use of the inequalities

$$2 \mid v \cdot w \mid \leqslant \mid v \mid^2/\epsilon + \epsilon \mid w \mid^2,$$
$$2 \mid v \cdot \mathbf{M} \mid \leqslant \mid v \mid^2/\epsilon + \epsilon \mid \mathbf{M} \mid^2,$$
$$\mid v \cdot \mathbf{M} \mid \leqslant \mid v \mid \mid \mathbf{M} \mid .$$

It follows from our assumptions about the asymptotic behavior of A, A_r, and $\nabla \cdot A - a$, as $r \to \infty$, and the definition of Γ, that

$$c \sim \frac{\nabla \rho}{2}, \qquad \sigma = 2 \mid c \mid^2 \sim \frac{\mid \nabla \rho \mid^2}{2}, \qquad \gamma \sim \frac{m}{2} - \frac{\mid \nabla \rho \mid^2}{2},$$
$$d \sim -\frac{\nabla \mid \nabla \rho \mid^2}{2} = - \mid \nabla \rho \mid (\nabla \mid \nabla \rho \mid),$$

as $r \to \infty$.

Recalling how we choose ρ, μ, ν and b, we have, as $r \to \infty$,

$$\frac{\rho^2}{\mid c \mid^2} \sim \frac{4\rho^2}{\mid \nabla \rho \mid^2} = O(r^2),$$
$$\frac{\gamma(\rho^2 + 1)}{2 \mid c \mid^2} \sim \frac{1}{2} (m - \mid \nabla \rho \mid^2) \frac{(\rho^2 + 1)}{\mid \nabla \rho \mid^2} = O(r^2),$$
$$\frac{\mid b \mid^2}{4\mu} \sim \frac{r^2}{2(1 + \epsilon/2r^2)} = O(r^2),$$
$$\frac{\rho^2}{16\nu \mid c \mid^2} \left(\frac{(b \cdot c)\, \sigma}{\mid c \mid^2 \rho} - 1\right)^2 \sim \frac{r^2}{\epsilon} \frac{\rho^2}{\mid \nabla \rho \mid^2} \left(\frac{x \cdot \nabla \rho}{\rho} - 1\right)^2 = O(r^2).$$

So the coefficient of $\mid L_\lambda u \mid^2$ in (5) is $O(r^2)$, as $r \to \infty$.

Furthermore, as $r \to \infty$, we have

$$\frac{\gamma \mid c \mid^2}{2(\rho^2 + 1)} \sim \frac{(m - \mid \nabla \rho \mid^2) \mid \nabla \rho \mid^2}{16(\rho^2 + 1)} = O\left(\frac{1}{r^2}\right),$$
$$\frac{\mid d \mid^2}{\theta} \sim \frac{8r^2}{\epsilon} \mid \nabla \rho \mid^2 \mid \nabla \mid \nabla \rho \mid \mid^2 = O\left(\frac{1}{r^2}\right).$$

The coefficient of $\mid u \mid^2$ in (5) is, therefore, $O(1/r^2)$ as $r \to \infty$.

Finally, it follows directly from (5) that

$$\frac{(1-\epsilon)}{2} (\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V})^2 + (\min_{V \cup \partial V} r^2 q(\xi_0, x)) \left(\left\| \frac{\nabla u}{r} \right\|_V\right)^2$$
$$\leqslant \alpha(\| rL_\lambda u \|_V)^2 + \left(\frac{3}{2\lambda^2} \left\| \frac{(\lambda\rho + \gamma)^2}{n \cdot b} \right\|'_{\partial V} \| A \|'_{\partial V} - \min_{\partial V} \frac{(n \cdot b)}{2}\right) (\lambda \| u \|_{\partial V})^2$$
$$+ \left(\| \mathbf{B} \|'_{\partial V} + \frac{1}{\epsilon} \left\| \frac{A \mid b \mid^2}{n \cdot b} \right\|'_{\partial V} + \epsilon \| A \mid b \mid \|'_{\partial V}\right) (\| \nabla u - n u_n \|_{\partial V})^2$$
$$+ \left(\left\| r^2 \left(\frac{\mid d \mid^2}{\theta} + \frac{\gamma \mid c \mid^2}{2(\rho^2 + 1)}\right) \right\|'_V\right) \left(\left\| \frac{u}{r} \right\|_V\right)^2, \tag{7}$$

where

$$\alpha(r) = \left\| \frac{1}{r^2} \left(\frac{\gamma(\rho^2+1)}{2\,|\,c\,|^2} + \frac{\rho^2}{2\omega\,|\,c\,|^2} + \frac{|\,b\,|^2}{4\mu} + \frac{\rho^2}{16\nu\,|\,c\,|^2} \left(\frac{(b\cdot c)\,\sigma}{|\,c\,|^2\rho^2} - 1 \right)^2 \right) \right\|_V',$$

$$q(\xi, x) = \operatorname{Re} \xi \cdot ((\gamma A + \nabla \cdot \mathbf{B} - ab - (\tfrac{1}{2} + \theta + \nu) I) \cdot \bar{\xi}),$$

and

$$q(\xi_0, x) = \min_{|\xi|=1} q(\xi, x).$$

5. Estimates for the Norms $\| u/r \|_V$, $\|\nabla u/r\|_V$ and $\| u_n \|_{\partial V}$

In this section we establish that the following inequality holds as $\lambda \to \infty$:

$$\begin{aligned}
&\left(\frac{1}{2} \min_{V\cup\partial V} r^2 q(\xi_0 x)\right) \left(\left\| \frac{u}{r} \right\|_V\right)^2 \\
&\quad \leqslant \frac{12}{\lambda^2} \left(1 + \left\| \frac{1}{r^2} \right\|_{\partial V}' + \left\| \frac{1}{r} \right\|_V' + \left(\left\| \frac{1}{r^2} \right\|_V'\right)^2\right) \\
&\quad \times \max\left(\min_{V\cup\partial V} \frac{r^2 q(\xi_0, x)}{2}, \| A \|_V', \| \nabla \cdot A - a \|_V'\right) \\
&\quad \times \max(1, \| A \|_{\partial V}') \Bigg[(1 + 2\alpha) (\| rL_\lambda u \|_V)^2 \qquad (8) \\
&\quad + 2 \left(\frac{3}{2\lambda^2 \epsilon} \left(\left\| \frac{(\lambda\rho + \gamma)^2}{n \cdot b} \right\|_{\partial V}'\right) (\| A \|_{\partial V}') - \min_{\partial V} \frac{(n \cdot b)}{2} + \frac{1}{2}\right) (\lambda \| u \|_{\partial V})^2 \\
&\quad + 2 \left(\frac{1}{\epsilon} \left\| \frac{A\,|\,b\,|^2}{n \cdot b} \right\|_{\partial V}' + \| \mathbf{B} \|_{\partial V}' + \frac{1}{2} + \epsilon \| A\,|\,b\,| \|_{\partial V}'\right) \\
&\quad \times (\| \nabla u - n u_n \|_{\partial V})^2\Bigg].
\end{aligned}$$

In deriving (8) we obtain similar inequalities for

$$\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V} \qquad \text{and} \qquad \| \nabla u/r \|_V,$$

viz., (13).

The above inequality is derived from (7), and the identity

$$\begin{aligned}
&\nabla \cdot \left(\frac{\bar{u}}{r^2} (A \cdot \nabla u)\right) \\
&\quad = \nabla \left(\frac{\bar{u}}{r^2}\right) \cdot (A \cdot \nabla u) + \frac{\bar{u}}{r^2} (\nabla \cdot A) \cdot \nabla u + \frac{\bar{u}}{r^2} (L_\lambda u - a \cdot \nabla u - \lambda^2 u). \qquad (9)
\end{aligned}$$

The argument we use to derive (8) requires that the coefficient of $(\| u/r \|_V)^2$ be positive. In Appendix III we show that this requirement will be satisfied if the differential operator $L_0 - a \cdot \nabla$ is uniformly elliptic, and

$$\min_{V(r_1)\cup\partial V}(\min_{|\xi|=1}(\xi \cdot (A \cdot \bar{\xi}))) - \tfrac{1}{2} \| rA_r \|'_{V(r_1)} - \| r(\nabla \cdot A - a)\|'_{V(r_1)} > 0. \tag{10}$$

In the special case that $A = \kappa I$ (i.e., if $L_\lambda u = f$ is the reduced wave equation for a nonhomogeneous medium), the coefficient of $(\| u/r \|_V)^2$ will be positive if

$$\min_{V(r_1)\cup\partial V} \kappa(x) - \frac{1}{2} \| r\kappa_r \|'_{V(r_1)} - \left\| r\left(\nabla\kappa - \frac{x}{r}\kappa_r\right)\right\|'_{V(r_1)} > 0. \tag{11}$$

First, by an argument similar to the one used to derive (5) from (1), we obtain from (9) the preliminary result that, as $\lambda \to \infty$

$$\begin{aligned}
\left(\left\|\frac{u}{r}\right\|_V\right)^2 \leqslant{}& \frac{2}{\lambda^2}\left(\left(1 + \left\|\frac{1}{r}\right\|'_V\right)(\| A \|'_V) + \| \nabla \cdot A - a \|'_V\right)\left(\left\|\frac{\nabla u}{r}\right\|_V\right)^2 \\
&+ \frac{2}{\lambda^2}\left(\left\|\frac{1}{r^2}\right\|'_{\partial V}\right) \| A \|'_{\partial V}((\| u \|_{\partial V})^2 + (\| \nabla u - nu_n \|_{\partial V})^2) \\
&+ \frac{1}{\lambda^2}\left(\left\|\frac{1}{r^2}\right\|'_{\partial V}\right)(\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V})^2 \\
&+ \frac{1}{2\lambda^2}\left(\left\|\frac{1}{r^2}\right\|'_V\right)^2 (\| rL_\lambda u \|_V)^2.
\end{aligned} \tag{12}$$

Using (12) to estimate the last term in (7), we find that, as $\lambda \to \infty$

$$\begin{aligned}
&\left.\begin{matrix}(\frac{1}{2} - \epsilon)^{1/2} \beta^{1/2} \|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V} \\ \frac{1}{\sqrt{2}}(\min_{V\cup\partial V} r^2 q(\xi_0, x))^{1/2} \left\|\frac{\nabla u}{r}\right\|_V\end{matrix}\right\} \leqslant 2^{1/2} \alpha^{1/2} \| rL_\lambda u \|_V \\
&\quad 2^{1/2}\left(\frac{3}{2\lambda^2\epsilon}\left\|\frac{(\lambda\rho + \gamma)^2}{n \cdot b}\right\|'_{\partial V} \| A \|'_{\partial V} - \min_{\partial V}\frac{(n \cdot b)}{2}\right)^{1/2} (\lambda \| u \|_{\partial V}) \\
&\quad + 2^{1/2}\left(\| \mathbf{B} \|'_{\partial V} + \frac{1}{\epsilon}\left\|\frac{A \mid b \mid^2}{n \cdot b}\right\|'_{\partial V} + \epsilon \| A \mid b \mid \|'_{\partial V}\right)^{1/2} \| \nabla u - nu_n \|_{\partial V}.
\end{aligned} \tag{13}$$

Finally, using (13) to estimate $\| \nabla u/r \|_V$, and $\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V}$ in (12), we obtain (8).

6. A Pointwise estimate for the Solution

In this section we obtain the following pointwise estimate for $u(x)$. If $m = 2$ or 3, and $\lambda \gg 1$, then

$$
\begin{aligned}
| u(x)| \leqslant{} & O(\lambda^{(m-1)/2})\, A^{00}(x) \Big[\Big(\Big\| \frac{1}{A^{00}} \Big\|'_{V} \Big) (\| rL_\lambda u \|_V) \\
& + \lambda^2 \max \Big(1, \Big\| \frac{A(x)}{A^{00}(x)} \Big\| \Big) \\
& \times \Big(\Big\| r'^2 \Big(\frac{1}{A^{00}(x')} - \frac{1}{A^{00}(x)} \Big) \Big\|'_{V(r_2+\delta)} \Big) \Big(\Big\| \frac{u}{r} \Big\|_V \Big) \\
& + \lambda \max \Big(1, \Big\| r'^2 \Big(\frac{A(x')}{A^{00}(x')} - \frac{A(x)}{A^{00}(x)} \Big) \Big\|'_{V(r_2+\delta)} \Big) \\
& \times \max \Big(\Big\| \frac{r'^2}{| x - x' |} \Big(\frac{A(x')}{A^{00}(x')} - \frac{A(x)}{A^{00}(x)} \Big) \Big\|'_{V(r_2+\delta)}, \\
& \Big\| r'^2 \Big(\nabla' \cdot \Big(\frac{A}{A^{00}} \Big) - \frac{a}{A^{00}} \Big) \Big\|'_{V(r_2+\delta)} \Big) \Big(\Big\| \frac{\nabla u}{r} \Big\|_V \Big) \\
& + O(\lambda^{(m-1)/2})\, A^{00}(x) \Big[\lambda \Big| \frac{A(x)}{A^{00}(x)} \Big| (\| u \|'_{\partial V}) \\
& + \Big(\Big\| \frac{1}{A^{00}} \Big\|'_{\partial V} \Big) ((\| A - I \|'_{\partial V})(\| \nabla u - n u_n \|_{\partial V}) + \| (n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V}) \Big]
\end{aligned} \tag{14}
$$

Here $A^{00}(x)$ is any positive function in $C^1(V \cup \partial V)$,

$$V(r_2 + \delta) = V \cap \{x \colon | x | \leqslant r_2 + \delta\},$$

and $0 < \delta < r_2/2$. This inequality holds uniformly on $V(r_2 - \delta) \cup \partial V$ for every value of r_2 such that $V(r_2/2) \supset \partial V$ where

$$V(r_2 - \delta) = V \cap \{x \colon | x | \leqslant r_2 - \delta\}.$$

If $m \geqslant 4$ a more complicated argument is needed to obtain a pointwise estimate for $u(x)$. This is because our derivation of (14) requires the existence of the integrals

$$\| h(x)/r' \|_{V(r_2+\delta)} \qquad \text{and} \qquad \| \, | x' - x | \, \nabla h(x)/r' \|_{V(r_2+\delta)} ,$$

where $h(x) = h(x, x')$ is the fundamental solution of

$$(A(x) \cdot \nabla') \cdot \nabla' h + \lambda^2 h = 0,$$

that satisfies the same radiation condition as $u(x)$, if $A(x) \equiv I$. These integrals do not exist if $m \geqslant 4$.

To get (14) we start with the identity

$$\begin{aligned}
\frac{u(x)}{A^{00}(x)} &= \int_{V(r_2+\delta)} \eta h(x) \frac{L_\lambda u}{A^{00}}\, dV' + \lambda^2 \int_{V(r_2+\delta)} \eta h(x) \left(\frac{1}{A^{00}(x)} - \frac{1}{A^{00}}\right) u\, dV' \\
&\quad + \int_{V(r_2+\delta)} \eta \Big((\nabla' h(x)) \cdot \Big(\frac{A}{A^{00}} - \frac{A(x)}{A^{00}(x)}\Big)\Big) \cdot \nabla' u\, dV' \\
&\quad + \int_{V(r_2+\delta)} \eta h(x) \Big(\nabla' \cdot \Big(\frac{A}{A^{00}}\Big)\Big) \cdot \nabla' u\, dV' - \int_{V(r_2+\delta)} \eta h(x) \frac{a \cdot \nabla' u}{A^{00}}\, dV' \\
&\int_{\Delta V(r_2)} h(x) \Big(\nabla' \eta \cdot \Big(\frac{A}{A^{00}} - \frac{A(x)}{A^{00}(x)}\Big)\Big) \cdot \nabla' u\, dV' \\
&\quad - 2 \int_{\Delta V(r_2)} \nabla' \eta \cdot (A(x) \cdot \nabla' h(x)) \frac{u}{A^{00}(x)}\, dV' \\
&\quad - \int_{\Delta V(r_2)} ((A(x) \cdot \nabla') \cdot \nabla' \eta)\, h(x) \frac{u}{A^{00}(x)}\, dV' \\
&\quad + \int_{\partial V} h(x)\, n \cdot \Big(\frac{A}{A^{00}} \cdot \nabla' u\Big)\, dS' - \int_{\partial V} n \cdot \Big(\frac{A(x)}{A^{00}(x)} \cdot \nabla' h(x)\Big)\, u\, dS', \\
&\qquad\qquad x \in V(r_2),
\end{aligned} \tag{15}$$

where

$$\Delta V(r_2) = V(r_2 + \delta) - V(r_2).$$

The fundamental solution $h(x) = h(x, x')$ in (15) is given explicitly by the equation

$$\begin{aligned}
h(x, x') &= C_0 \frac{\lambda^{(m-1)/2}}{[\det \Lambda(x)]^{1/2}} [\lambda((x' - x) \cdot (\Lambda^{-1}(x) \cdot (x' - x)))^{1/2}]^{-(m-2)/2} \\
&\quad \times H^{(1)}_{(m-1)/2}(\lambda(x' - x) \cdot (\Lambda^{-1}(x) \cdot (x' - x))^{1/2}),
\end{aligned} \tag{16}$$

where $\Lambda(x)$ is the diagonal matrix whose entries are the eigenvalues of $A(x)$, $H^{(1)}_{(m-2)/2}$ is the Hankel function of the first kind of order $(m - 2)/2$, and

$$C_0 = \frac{i}{4(2\pi)^{(m-2)/2}}.$$

Equation (15) is derived under the assumption that $\eta = \eta(x, r_2)$ is a function in $C^2(V(r_2 + \delta) \cup \partial V)$ with the following properties: $\eta(x, r_2) \leqslant 1$ if $x \in V(r_2 + \delta) \cup \partial V$, $\eta(x, r_2) \equiv 1$ if $x \in V(r_2) \cup \partial V$, $\eta(x, r_2)$, $\nabla\eta(x, r_2) \equiv 0$ if $x \in S(r_2 + \delta) = \{x : |x| = r_2 + \delta\}$.

If $x \in V(r_2 - \delta) \cup \partial V$, the preceding identity implies the estimate

$$\frac{|u(x)|}{A^{00}(x)} \leqslant \left(\left\| \frac{h(x)}{r'} \right\|_{V(r_2+\delta)}\right) \left(\left\| \frac{1}{A^{00}} \right\|'_{V(r_2+\delta)}\right) (\| rL_\lambda u \|_V)$$

$$+ \left[\lambda^2 \left(\left\| \frac{h(x)}{r'} \right\|_{V(r_2+\delta)}\right) \left(\left\| r'^2 \left(\frac{1}{A^{00}} - \frac{1}{A^{00}(x)}\right) \right\|'_{V(r_2+\delta)}\right)\right.$$

$$+ 2(\| r'^2 \nabla \eta \|'_{\Delta V(r_2)}) \left(\left\| \frac{\nabla h(x)}{r'} \right\|_{\Delta V(r_2)}\right) \left\| \frac{A(x)}{A^{00}(x)} \right\|$$

$$\left. + (\| r'^2 (A(x) \cdot \nabla') \cdot \nabla' \eta \|'_{\Delta V(r_2)}) \left(\frac{1}{A^{00}(x)}\right) \left(\left\| \frac{h(x)}{r'} \right\|_{\Delta V(r_2)}\right)\right] \left(\left\| \frac{u}{r} \right\|_V\right)$$

$$+ \left[\left(\left\| | x' - x | \frac{\nabla h(x)}{r'} \right\|_{V(r_2+\delta)}\right) \left(\left\| \frac{r'^2}{| x' - x |} \left(\frac{A}{A^{00}} - \frac{A(x)}{A^{00}(x)}\right) \right\|'_{V(r_2+\delta)}\right)\right.$$

$$+ \left(\left\| \frac{h(x)}{r'} \right\|_{V(r_2+\delta)}\right) \left(\left(\left\| r'^2 \left(\nabla' \cdot \left(\frac{A}{A^{00}}\right) - \frac{a}{A^{00}}\right) \right\|'_{V(r_2+\delta)}\right)\right.$$

$$+ (\| \nabla \eta \|'_{\Delta V(r_2)}) \left(\left\| \frac{h(x)}{r'} \right\|_{\Delta V(r_2)}\right)$$

$$\left.\times \left(\left\| r'^2 \left(\frac{A}{A^{00}} - \frac{A(x)}{A^{00}(x)}\right) \right\|'_{\Delta V(r_2)}\right)\right] \left(\left\| \frac{\nabla u}{r} \right\|_V\right)$$

$$+ (\| h(x) \|_{\partial V}) \left(\left\| \frac{1}{A^{00}} \right\|'_{\partial V}\right) (\| (n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V}$$

$$+ (\| A - I \|'_{\partial V}) (\| \nabla u - n u_n \|_{\partial V})) + \left| \frac{A(x)}{A^{00}(x)} \right| (\| \nabla h(x) \|''_{\partial V}) (\| u \|'_{\partial V}). \tag{17}$$

It follows in turn from this estimate that, for every x in $V(r_2 - \delta) \cup \partial V$,

$$\frac{|u(x)|}{A^{00}(x)} \leqslant \tau_1(x) \left[\left(\left\| \frac{1}{A^{00}} \right\|'_{V(r_2+\delta)}\right) (\| rL_\lambda u \|_V)\right.$$

$$+ \lambda^2 \max \left(\left\| r'^2 \left(\frac{1}{A^{00}} - \frac{1}{A^{00}(x)}\right) \right\|'_{V(r_2+\delta)},\right.$$

$$\left.\frac{1}{\lambda^{1/2}} \| \nabla \eta \|'_{\Delta V(r_2)}, \frac{1}{\lambda} \max_{1 \leqslant i \leqslant m} \| \nabla \eta_i \|'_{\Delta V(r_2)}\right) \max \left(1, \left| \frac{A(x)}{A^{00}(x)} \right|\right)$$

$$\times \left(1 + \frac{2(r_2 + \delta)^2}{\lambda^{1/2}} + \frac{(r_2 + \delta)^2 m^{1/2}}{\lambda}\right) \left(\left\| \frac{u}{r} \right\|_V\right)$$

$$+ \lambda \max \left(\left\| \frac{r'^2}{|x' - x|} \left(\frac{A}{A^{00}} - \frac{A(x)}{A^{00}(x)} \right) \right\|'_{V(r_2+\delta)} , \right.$$

$$\left. \left\| r'^2 \left(\nabla' \cdot \left(\frac{A}{A^{00}} \right) - \frac{a}{A^{00}} \right) \right\|_{V(r_2+\delta)} , \frac{1}{\lambda^{1/2}} \| \nabla \eta \|'_{\Delta V(r_2)} \right)$$

$$\times \max \left(1, \left\| r'^2 \left(\frac{A}{A^{00}} - \frac{A(x)}{A^{00}(x)} \right) \right\|'_{\Delta V(r_2)} \right) \left(1 + \frac{2}{\lambda} + \frac{1}{\lambda^{1/2}} \right) \left(\left\| \frac{\nabla u}{r} \right\|_V \right)$$

$$+ \tau_2(x) \left[\left(\left\| \frac{1}{A^{00}} \right\|'_{\partial V} \right) (\| (n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V} \right.$$

$$\left. + (\| A - I \|'_{\partial V}) (\| \nabla u - n u_n \|_{\partial V}) \Big) + \lambda \left(\left\| \frac{A(x)}{A^{00}(x)} \right\| \right) (\| u \|'_{\partial V}) \right] . \quad (18)$$

Here

$$\tau_1(x) = \max \left(\left\| \frac{h(x)}{r'} \right\|_{V(r_2+\delta)} , \frac{1}{\lambda} \left\| |x' - x| \frac{\nabla h(x)}{r'} \right\|_{V(r_2+\delta)} , \right.$$

$$\left. \left\| \frac{h(x)}{r'} \right\|_{\Delta V(r_2)} , \frac{1}{\lambda} \left\| \frac{\nabla h(x)}{r'} \right\|_{\Delta V(r_2)} \right),$$

and

$$\tau_2(x) = \max \left(\| h(x) \|_{\partial V} , \frac{1}{\lambda} \| \nabla h(x) \|''_{\partial V} \right) .$$

To derive (14) from (18) we assume $\lambda \gg 1$, and make use of the following asymptotic formulas:

$$| h(x, x') | = \begin{cases} O(1) (\det \Lambda(x))^{-1/2} (\Lambda^1(x) \, | x - x' |)^{-1}, & m = 3, \\ O(1) (\det \Lambda(x))^{-1/2} \ln(1 + (\lambda \Lambda^1(x) \, | x - x' |)^{-1/2}), & m = 2, \end{cases}$$

$$| \nabla h(x, x') | = O(1) \, \lambda^{(m-1)/2} (\det \Lambda(x))^{-1/2} (\Lambda^1(x) \, | x - x' |)^{(1-m)/2}$$

$$+ O(1) (\det \Lambda(x))^{-1/2} (\Lambda^1(x) \, | x - x' |)^{1-m}, \qquad m = 2, 3,$$

where

$$\Lambda^1(x) = \max_{|\xi|=1} (\bar{\xi} \cdot (A \cdot \xi)).$$

These inequalities hold uniformly in x and x', for all $x, x' \in V \cup \partial V$.

A laborious but straightforward calculation based on these formulas leads to the conclusion that, as $\lambda \to \infty$,

$$\tau_1(x), \tau_2(x) = O(1) \, \lambda^{-(3-m)/2}, \quad (19)$$

uniformly in x, $x \in V(r_2 - \delta) \cup \partial V$.

Inequality (14) follows directly from (18) by virtue of (19) as $\lambda \to \infty$.

7. Conclusion

Using Inequalities (8) and (13) to estimate the quantities

$$\|(n \cdot (A \cdot n))^{1/2} u_n \|_{\partial V}, \qquad \| u/r \|_V, \qquad \text{and} \qquad \| \nabla u/r \|_V$$

in (14), we obtain a linear combination of $\| \nabla u - nu_n \|_{\partial V}$, $\lambda \| u \|'_{\partial V}$, and $\| rL_\lambda u \|_V$ that is greater than $\lambda^{-(1+m)/2} | u(x)|$, for all $x \in V(r_2 - \delta) \cup \partial V$. Denoting the largest constant in this linear combination by C, we finally obtain the pointwise estimate

$$| u(x)| \leqslant C\lambda^{(1+m)/2}(\| rL_\lambda u \|_V + \lambda \| u \|'_{\partial V} + \| \nabla u - nu_n \|_{\partial V}).$$

C is independent of λ and x. This estimate holds as $\lambda \to \infty$, for all $x \in V(r_2 - \delta) \cup \partial V$.

Appendix I

To establish (2) we remark first that if $u(x) \in C^2(V \cup \partial V)$, and $B^{ijk} = \bar{B}^{jik}$, then

$$\operatorname{Re} \sum_{k=1}^{m} \left(\sum_{i=0}^{m} \sum_{j=0}^{m} B^{ijk} u_i \bar{u}_j \right)_k$$
$$= \operatorname{Re} \sum_{k=1}^{m} \sum_{i=0}^{m} \sum_{j=0}^{m} (B^{ijk})_k \, u_i \bar{u}_j + \operatorname{Re} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=1}^{m} 2B^{ijk} \bar{u}_{jk} u_i , \tag{I.1}$$

where $u_0 = u$.

Focusing our attention on the right side of (I.1), we note that

$$2 \operatorname{Re} \sum_{i=0}^{m} \sum_{j=0}^{m} \sum_{k=1}^{m} B^{ijk} \bar{u}_{jk} u_i$$
$$= 2 \operatorname{Re} \sum_{i=0}^{m} \left(\sum_{j=1}^{m-1} \sum_{k=j+1}^{m} (B^{ikj} + B^{ijk}) \, \bar{u}_{jk} + B^{ijj} \bar{u}_{jj} \right) u_i \tag{I.2}$$
$$+ 2 \operatorname{Re} \sum_{k=1}^{m} B^{00k} \bar{u}_k u + 2 \operatorname{Re} \sum_{i=1}^{m} \sum_{k=1}^{m} B^{i0k} \bar{u}_k u_i ,$$

and that

$$\operatorname{Re} \sum_{k=1}^{m} \sum_{i=0}^{m} \sum_{j=0}^{m} (B^{ijk})_k \, u_i \bar{u}_j$$
$$= \operatorname{Re} \sum_{k=1}^{m} B_k^{00k} \, | u |^2 + \operatorname{Re} \sum_{i=0}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} B_k^{ijk} u_i \bar{u}_j + 2 \operatorname{Re} \sum_{i=1}^{m} \sum_{k=1}^{m} B_k^{i0k} u_i \bar{u}. \tag{I.3}$$

Turning to the left side of (I.1) we have

$$\operatorname{Re}\sum_{k=1}^{m}\left(\sum_{i=0}^{m}\sum_{j=0}^{m}B^{ijk}u_i\bar{u}_j\right)_k = \operatorname{Re}\sum_{k=1}^{m}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}B^{ijk}u_i\bar{u}_j + 2\sum_{j=1}^{m}B^{j0k}\bar{u}_j u + B^{00k}\mid u\mid^2\right)_k. \tag{I.4}$$

We now set

$$\begin{aligned}
B^{ikj} + B^{ijk} &= b^i A^{jk} && (i = 0, 1, 2,..., m; j = 1, 2,..., m-1,\ j+1 \leqslant k \leqslant m),\\
2B^{ijj} &= b^i A^{jj} && (i = 0, 1, 2,..., m; j = 1, 2,..., m),\\
2B^{j0k} &= (-i\lambda\rho + \gamma)\, A^{kj} && (i, k = 1, 2,..., m),\\
2B^{00k} &= b^k\lambda^2 && (k = 1, 2, 3,..., m).
\end{aligned} \tag{I.5}$$

Assuming that $\bar{b}^i = b^i$ for $i = 1, 2,..., m$ it follows immediately from (I.1)–(I.4) and Eqs. (I.5) that

$$\begin{aligned}
\operatorname{Re}\sum_{k=1}^{m}&\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{m}B^{ijk}u_i + (-i\lambda\rho+\gamma)\,uA^{kj}\right)\bar{u}_j + b^k\lambda^2\mid u\mid^2/2\right]_k\\
&= \operatorname{Re}\left(\sum_{i=1}^{m}b^iu_i + (-i\lambda\rho+\gamma)\,u\right)\left(\sum_{j=1}^{m}\sum_{k=1}^{m}A^{jk}\bar{u}_{jk} + \sum_{j=1}^{m}a^j\bar{u}_j + \lambda^2\bar{u}\right)\\
&\quad + \operatorname{Re}\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\gamma A^{ij} + \sum_{k=1}^{m}B^{ijk}_k - b^ia^j\right)u_i\bar{u}_j\\
&\quad + \operatorname{Re} u\sum_{j=1}^{m}\left(\sum_{k=1}^{m}((-i\lambda\rho+\gamma)\,A^{kj})_k - (-i\lambda\rho+\gamma)\,a^j\right)\bar{u}_j\\
&\quad + \left(\sum_{k=1}^{m}b_k{}^k/2 - \gamma\right)\lambda^2\mid u\mid^2.
\end{aligned} \tag{I.6}$$

In vector notation (cf. glossary) this becomes

$$\begin{aligned}
\nabla\cdot\operatorname{Re}&[(\nabla u\cdot\mathbf{B} + (-i\lambda\rho+\gamma)\,uA)\cdot\nabla\bar{u} + \lambda^2 b\mid u\mid^2/2]\\
&= \operatorname{Re}(b\cdot\nabla u + (-i\lambda\rho+\gamma)\,u)\,L_\lambda\bar{u} + 2\operatorname{Re} u(-i\lambda c + d)\cdot\nabla\bar{u}\\
&\quad + \operatorname{Re}\nabla u\cdot((\gamma A - ab + \nabla\cdot\mathbf{B})\cdot\nabla\bar{u}) + \lambda^2\sigma\mid u\mid^2.
\end{aligned} \tag{I.7}$$

If σ is positive, then (I.7) can be rewritten as (2). For if σ is positive, then

$$\begin{aligned} &2 \operatorname{Re} u(-i\lambda c + d) \cdot \nabla \bar{u} + \lambda^2 \sigma \mid u \mid^2 \\ &= -\frac{\mid c \cdot \nabla u \mid^2}{\sigma} + 2 \operatorname{Re} u(d \cdot \nabla \bar{u}) + \sigma \left| \frac{c \cdot \nabla u}{\sigma} - i\lambda u \right|^2. \end{aligned} \tag{I.8}$$

Finally, the equations for the B^{ijk} have the solution

$$2B^k = bA^{k*} + A^{k*}b - b^k A, \qquad k = 1, 2,..., m. \tag{I.9}$$

Appendix II

Our choice of ρ is motivated by the fact that

$$2 \mid c \mid \geqslant \chi_1 \mid \rho_r \mid - \chi_2(r) \frac{\mid \rho \mid}{r}, \tag{II.1}$$

where

$$\chi_1 = \min_{V \cup \partial V} (\min_{|\xi|=1} \mid \xi \cdot A \mid),$$

and

$$\chi_2(r) = \Psi(r) \| r(\nabla \cdot A - a) \|'_{V(r_1)} + (1 - \Psi(r)) \, C_1/r^p.$$

The function $\Psi(r)$ is a continuously differentiable, monotonic nonincreasing function of r, that equals one if $r_0 \leqslant r \leqslant r_1$, and which vanishes if $r \geqslant r_3$, $r_3 > r_1$. Also $V(r_1) = V \cap \{x: \mid x \mid \leqslant r_1\}$, and r_0 is a positive number such that the sphere $r = r_0$ lies inside ∂V.

By hypothesis we have

$$\mid r(\nabla \cdot A - a) \mid \leqslant C_1/r^p, \qquad p > 2, \tag{II.2}$$

if $r \geqslant r_1$ where C_1 is a constant. Inequality (II.1) follows from the inequality

$$2 \mid c \mid = |(\nabla \rho) \cdot A + (\nabla \cdot A - a) \rho \mid \geqslant \mid \nabla \rho \cdot A \mid - r \mid \nabla \cdot A - a \mid \rho/r,$$

if (II.2) holds.

In view of (II.1) we stipulate that $\rho(r)$ be a positive solution of the ordinary differential equation

$$\chi_1 \rho_r - \chi_2(r) \frac{\rho}{r} = \chi_1 \left(1 - \frac{\epsilon}{r^2}\right) \exp \left\{ -\frac{1}{\chi_1} \int_r^\infty \frac{\chi_2(S)}{S} \, dS \right\}, \tag{II.3}$$

where ϵ is any positive number such that $1 - \epsilon/r^2 > 0$ on $V \cup \partial V$. (Note that the assumed uniform ellipticity of the differential operator $L_0 - a \cdot \nabla$ assures that $\chi_1 \neq 0$.) For if $\rho(r)$ is a positive solution of (II.3), then

$$\begin{aligned} 2 \mid c \mid &\geqslant \chi_1 \rho_r - \chi_2(r) \rho \\ &\geqslant \chi_1 \left(1 - \epsilon \left\| \frac{1}{r^2} \right\|_V' \right) \exp \left\{ - \frac{\| r(\nabla \cdot A - a) \|_{V(r_1)}'}{\chi_1} \ln \frac{r_3}{r_0} \right\} \\ &\quad \times \exp \left\{ - \frac{C_1}{p \chi_1} \left(\frac{1}{r_1{}^p} - \frac{1}{r_3{}^p} \right) \right\} \exp \left\{ - \frac{C_1}{\chi_1} \frac{1}{p r_3{}^p} \right\} > 0. \end{aligned}$$

So $| c |^{-1}$ is uniformly bounded on $V \cup \partial V$, as required in Section 4.

To get $\rho(r)$ and its derivatives to behave for large r, as required in Section 4, we set

$$\rho(r) = \left(r + \frac{\epsilon}{r} \right) \exp \left\{ - \frac{1}{\chi_1} \int_r^\infty \frac{\chi_2(S)}{S} \, dS \right\},$$

which is a positive solution of (II.3).

If $r \geqslant r_3 > r_1$, with $p > 2$ we have

$$\begin{aligned} \rho(r) &= \left(r + \frac{\epsilon}{r} \right) \exp \left\{ - \frac{C_1}{\chi_1} \frac{1}{p r^p} \right\} \\ &= \left(r + \frac{\epsilon}{r} \right) \left(1 + O\left(\frac{1}{r^p} \right) \right) = r \left(1 + O\left(\frac{1}{r^2} \right) \right), \\ \nabla \rho(r) &= \frac{x}{r} \left[\left(1 - \frac{\epsilon}{r^2} \right) + \left(r + \frac{\epsilon}{r} \right) \frac{C_1}{\chi_1 r^{p+1}} \right] \exp \left\{ - \frac{C_1}{\chi_1} \frac{1}{p r^p} \right\} \\ &= \frac{x}{r} \left(1 + O\left(\frac{1}{r^2} \right) \right), \end{aligned}$$

and

$$\begin{aligned} \nabla \mid \nabla \rho(r) \mid &= \frac{x}{r} \left[\frac{2\epsilon}{r^3} - \frac{p C_1}{\chi_1 r^{p+1}} - \frac{(p+2) \epsilon C_1}{\chi_1 r^{p+3}} \right. \\ &\quad \left. + \frac{C_1}{\chi_1 r^{p+1}} \left(\left(1 - \frac{\epsilon}{r^2} \right) + \left(r + \frac{\epsilon}{r} \right) \frac{C_1}{\chi_1 r^{p+1}} \right) \right] \exp \left\{ - \frac{C_1}{\chi_1} \frac{1}{p r^p} \right\} \\ &= \frac{x}{r} \left(\frac{2\epsilon}{r^3} + O\left(\frac{1}{r^{p+1}} \right) \right) \left(1 + O\left(\frac{1}{r^p} \right) \right) \\ &= \frac{x}{r} \left(\frac{2\epsilon}{r^3} \right) \left(1 + O\left(\frac{1}{r^{p-2}} \right) \right) = O\left(\frac{1}{r^3} \right). \end{aligned}$$

Appendix III

To establish (10) and (11) we first consider the quantity

$$\min_{V(r_3) \cup \partial V} r^2 q(\xi_0, x),$$

where $V(r_3) = \{x: |x| \geqslant r_3, r_3 > r_1\}$. By hypothesis

$$|A_r| \leqslant \frac{C_1}{r^{p+1}}, \qquad |\nabla \cdot A - a| \leqslant \frac{C_1}{r^{p+1}} \qquad \text{and} \qquad |A - I| \leqslant \frac{C_1}{r^p},$$

$p > 2$, if $r \geqslant r_3 > r_1$. Also if $r \geqslant r_3$ we set

$$\Gamma = -\left(\left(\frac{r_3}{r}\right) - 1\right)^2 \frac{1}{\epsilon'} + \left(\frac{1}{\epsilon'} + 1\right),$$

which implies that $|\Gamma - 1| \leqslant 2r_3/\epsilon' r$ if $r \geqslant r_3$.

Consequently,

$$\begin{aligned}
\operatorname{Re}[\bar{\xi}_0 \cdot (A \cdot \xi_0) + (\bar{\xi}_0 \cdot x)((\nabla \cdot A) \cdot \xi_0) - \bar{\xi}_0 \cdot (rA_r \cdot \xi_0)/2 - (\bar{\xi}_0 \cdot a)(x \cdot \xi_0)]& \\
\geqslant \bar{\xi}_0 \cdot (A \cdot \xi_0) - r|\nabla \cdot A - a| - r|A_r|/2& \quad \text{(III.1)}\\
= (1 - 5C_1/2r^p),&
\end{aligned}$$

$$\operatorname{Re}(\bar{\xi}_0 \cdot x)(\nabla\Gamma \cdot A) \cdot \xi_0 = \operatorname{Re}(\bar{\xi}_0 \cdot x)(\nabla\Gamma \cdot \xi_0) + O(1/r^p),$$

and

$$-2|c|^2 (\bar{\xi}_0 \cdot (A \cdot \xi_0)) \geqslant -|\nabla\rho|^2 + O(1/r^p),$$

with

$$|\nabla\rho| = \left(1 - \frac{\epsilon}{r^2} + O\left(\frac{1}{r^p}\right)\right).$$

It follows that

$$\begin{aligned}
&\min_{V(r_3)\cup\partial V} r^2 q(\xi_0, x)\\
&\quad\geqslant \min_{V(r_3)\cup\partial V} r^2 \left(\Gamma\left(1 - \frac{5C_1}{2r^p}\right) + \operatorname{Re}(\bar{\xi}_0 \cdot x)(\nabla\Gamma \cdot \xi_0) + O\left(\frac{1}{r^p}\right)\right.\\
&\qquad\left. - \frac{1}{2}\left(1 - \frac{\epsilon}{r^2} + O\left(\frac{1}{r^p}\right)\right)^2 - \frac{1}{2} - \frac{\epsilon}{2r^2}\right)\\
&\quad= \min_{V(r_3)\cup\partial V} r^2 \left(\frac{\epsilon}{2r^2} - \frac{\epsilon^2}{2r^4} + \Gamma - 1 + \operatorname{Re}(\bar{\xi}_0 \cdot x)(\nabla\Gamma \cdot \xi_0) + O\left(\frac{1}{r^p}\right)\right).
\end{aligned}$$

With Γ as defined above, the quantity $\Gamma - 1 + \operatorname{Re}(\bar{\xi}_0 \cdot x)(\nabla\Gamma \cdot \xi_0)$ is non-negative, so that

$$\min_{V(r_3)\cup\partial V} r^2 q(\xi_0, x) \geqslant \left(\frac{\epsilon}{2} + O\left(\frac{1}{r_3^{p-2}}\right)\right).$$

The quantity on the left is therefore positive if r_3 is sufficiently large, and $p > 2$.

If $r_1 \leqslant r \leqslant r_3$ inequality (III.1) still holds, and we set $\Gamma = (1/\epsilon' + 1)$. Consequently,

$$\begin{aligned}
\min_{r_1 \leqslant |x| \leqslant r_3} r^2 q(\xi_0, x) \\
&\geqslant r_1^2 \min_{r_1 \leqslant |x| \leqslant r_3} \left(\Gamma\left(1 - \frac{5C_1}{2r^p}\right) - 2\,|\,c\,|^2\,(\bar{\xi}_0 \cdot (A \cdot \xi_0)) - \frac{1}{2} - \frac{\epsilon}{2r^2}\right) \\
&\geqslant r_1^2 \left(\Gamma\left(1 - \frac{5C_1}{2r_1^p}\right) - 2 \max_{r_1 \leqslant |x| \leqslant r_3} |\,c\,|^2\,(\bar{\xi}_0 \cdot (A \cdot \xi_0)) - \frac{1}{2} - \frac{\epsilon}{2r_1^2}\right).
\end{aligned}$$

Since $-2\,|\,c\,|^2\,(\bar{\xi}_0 \cdot (A \cdot \xi_0))$ is independent of ϵ', the quantity on the left of this inequality is positive if ϵ' is sufficiently small, and we assume (without loss of generality) that $r_1 > 5^{1/p} C_1^{1/p}$.

Finally, consider the quantity

$$\begin{aligned}
\min_{V(r_1) \cup \partial V} r^2 q(\xi_0, x) = \min_{V(r_1) \cup \partial V} r^2 \operatorname{Re}\Big[\Gamma(\bar{\xi}_0 \cdot (A \cdot \xi_0) + (\bar{\xi}_0 \cdot x)\,((\nabla \cdot A) \cdot \xi_0) \\
- \tfrac{1}{2}\,\bar{\xi}_0 \cdot ((x \cdot \nabla A) \cdot \xi_0) - (\bar{\xi}_0 \cdot x)\,(a \cdot \xi_0)) \\
- 2\,|\,c\,|^2\,(\bar{\xi}_0 \cdot (A \cdot \xi_0)) - \frac{1}{2} - \frac{\epsilon}{2r^2}\Big],
\end{aligned}$$

with Γ still equal to $(1/\epsilon' + 1)$.

This quantity will be positive if

$$\begin{aligned}
\min_{V(r_1) \cup \partial V} \operatorname{Re}(\bar{\xi}_0 \cdot (A \cdot \xi_0) + (\bar{\xi}_0 \cdot x)\,((\nabla \cdot A) \cdot \xi_0) \\
- \tfrac{1}{2}\,\bar{\xi}_0 \cdot ((x \cdot \nabla A) \cdot \xi_0) - (\bar{\xi}_0 \cdot x)\,(a \cdot \xi_0))
\end{aligned} \tag{III.2}$$

is positive, since $-2\,|\,c\,|^2\,(\bar{\xi}_0 \cdot (A \cdot \xi_0)) - (1 + \epsilon/r^2)/2$ is independent of ϵ', and Γ can be made arbitrarily large by taking ϵ' sufficiently small.

In view of the above we conclude that

$$\min_{V \cup \partial V} r^2 q(\xi_0, x)$$

will be positive if (III.2) is positive. This will be the case if

$$\min_{V(r_1) \cup \partial V} (\min_{|\xi|=1} (\bar{\xi} \cdot (A \cdot \xi))) - \tfrac{1}{2}\,\|\, r A_r \,\|'_{V(r_1)} - \|\, r(\nabla \cdot A - a) \|'_{V(r_1)} > 0. \tag{III.3}$$

Note that if $A = \kappa I$, with $a = 0$, then $\min_{V \cup \partial V} r^2 q(\xi_0, x)$ will be positive if

$$\min_{V(r_1) \cup \partial V} \kappa(x) - \frac{1}{2}\,\|\, r\kappa_r \,\|'_{V(r_1)} - \left\|\, r\left(\nabla\kappa - \frac{x}{r}\,\kappa_r\right)\right\|'_{V(r_1)} > 0. \tag{III.4}$$

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