On Certain Riccati Integral Equations and Second-Order Linear Oscillation

MAN KAM KWONG

Department of Mathematical Sciences,
Northern Illinois University, DeKalb, Illinois 60115

Submitted by Ky Fan

1. INTRODUCTION

That the oscillatory nature of the equation
\[ y'' + q(t) y(t) = 0, \quad t \in [0, \infty) \]  
(1.1)
and the existence of solutions to the Riccati equations
\[ r'(t) = r^2(t) + q(t), \quad t \in [a, \infty), \quad a > 0 \]  
(1.2)
or
\[ r(t) = r(a) + \int_a^t q(s) \, ds + \int_a^t r^2(s) \, ds \]  
(1.3)
are closely related is well known. Many important results in the oscillation theory of (1.1) are in fact established by studying either (1.2) or (1.3).

Recent studies suggest that a great deal can still be said about (1.2) and (1.3), thus leading to new results on (1.1). See [7–9]. Particularly useful in these studies is the theory of differential and integral inequalities. The present work further supports this viewpoint.

In this paper, we first give a comparison theorem (Section 2, Theorem 1) for Riccati integral equations of the form (1.3) involving monotonic rearrangements of the function \( \bar{Q}(t) = \int_0^t q(s) \, ds \). As a corollary, we establish an extension of the well-known oscillation criteria of Hille's [6] to more general potentials (Section 3, Theorem 4). In Section 4 we elaborate on a method of Hartman to derive a comparison-type result involving the square of the integral \( \int_0^t q(s) \, ds \) and then deduce a Lyapunov-type inequality for disfocality.

The preliminaries will be introduced in the appropriate sections.
2. A COMPARISON THEOREM FOR A RICCATI INTEGRAL EQUATION

We consider the Riccati integral equation in $r$

$$r(t) = S(t) + \int_0^t r^2(s) \, ds, \quad t \in [a, b),$$  \hspace{1cm} (2.1)

where $S$ is a non-negative piecewise continuous function defined on the interval $[a, b)$, $-\infty < a < b \leq \infty$. We call $S$ the potential of the equation. For the sake of having fewer symbols we may denote a solution of (2.1) by the same letter $r$, with the understanding that it may only exist in some subinterval $[a, c) \subset [a, b)$.

The usual method of successive integration yields the existence and uniqueness of a solution defined on a maximal subinterval $[a, c) \subset [a, b)$. Since $r(t) > 0$, the only reason why $r$ fails to be continuable at $c$ is that $\lim_{t \to c^-} r(t) = \infty$.

The following fact is well known:

Let $T$ be another piecewise continuous function defined on $[a, b)$ such that $0 \leq T(t) < S(t)$ for all $t$. Let $R$ be the solution of

$$R(t) = T(t) + \int_0^t R^2(s) \, ds.$$  

Then $R(t) \leq r(t)$ for all $t$ in the common domain of definition of $r$ and $R$.

This is a simple consequence of the general theory of integral inequalities (see, e.g., [3, 5]). It can also be easily established directly. For a generalization to the case where $S$ and $T$ may be negative, see [7].

Based on this fact, it is easy to prove the following Approximation Lemma. We omit the obvious proof.

**Lemma 1.** Let $\{S_n\}_{n=1}^\infty$ be a sequence of positive piecewise continuous functions on $[a, b]$ such that $S_n(t) \leq S(t)$, and $\lim_{n \to \infty} S_n(t) = S(t)$, for all $t \in [a, b)$. Let $r$ be the solution of (2.1) defined on $[a, c) \subset [a, b)$ and $r_n$ be the solutions of similar equations with $S$ replaced by $S_n$. Then $r_n$ can be extended to $[a, c)$ and $\lim_{n \to \infty} r_n(t) = r(t)$ for all $t \in [a, c)$.

Of course much more general results hold. For instance the approximation need not be from below. We restrict ourselves to the above simpler case, which is all we need below, to avoid the possible complication of $r_n$ being not extendable to the whole of $[a, c)$.

We define the monotonic increasing (decreasing) rearrangement $S_+ (S_-)$ of $S$ to be a monotonic increasing (decreasing) function defined on $[a, b)$ such that for all real numbers $\lambda$, the sets $\{t \in [a, b) : S(t) < \lambda\}$ and $\{t \in [a, b) : S_+(S_-(t)) < \lambda\}$ have the same measure. If $b$ is finite, the ranges
of $S_*$ and $S^*$ are the same as that of $S$. But if $b$ is infinite the range of $S_*$ contains only those $\lambda$ for which the set $\{t \in [a,b): S(t) > \lambda\}$ has finite measure. A similar remark applies to the range of $S^*$.

An obvious property of $S^*$ and $S_*$ is that

$$\int_a^b S^*(t) \, dt = \int_a^b S_*(t) \, dt = \int_a^b S(t) \, dt \quad (2.2)$$

for any $\gamma > 0$, provided that all three integrals are finite.

**Lemma 2.** Let $S_1$ and $S_2$ be non-negative step functions defined on an interval $[a, a + 2\delta]$:

$$S_1(t) = \alpha, \quad t \in [a, a + \delta]$$

$$= \beta, \quad t \in [a + \delta, a + 2\delta],$$

$$S_2(t) = \beta, \quad t \in [a, a + \delta],$$

$$= \alpha, \quad t \in [a + \delta, a + 2\delta]$$

with $\alpha > \beta > 0$. Let $r_1$ and $r_2$ be solutions of (2.1) with $S_1$ and $S_2$ in place of $S$, respectively, and suppose $r_1$ and $r_2$ are finite in $[a, a + 2\delta]$. Then

$$\int_a^b r_1^2(t) \, dt > \int_a^b r_2^2(t) \, dt$$

and

$$\int_a^b r_1^2(t) \, dt > \int_a^{a + 2\delta} r_1^2(t) \, dt.$$

**Proof:** The first inequality follows from the simple comparison principle. Let us prove the second inequality. A change of variables allows us to take $\delta = 1$. Direct computation gives

$$r_1(t) = \frac{\alpha}{1 - \alpha(t - a)}, \quad t \in [a, a + 1],$$

$$= \frac{\gamma + \beta}{1 - (\gamma + \beta)(t - a - 1)}, \quad t \in [a + 1, a + 2],$$

where $\gamma = \alpha^2/(1 - \alpha)$.

A similar formula holds for $r_2$. The assumption that $r_2(t) < \infty$ for all $t \in [a, a + 2]$ requires that the denominator of the second fraction in the formula for $r_1$ be positive for all $t$. In particular (take $t = a + 2$)

$$1 - \left(\frac{\alpha^2}{1 - \alpha} + \beta\right) > 0.$$

Thus $\alpha + \beta < 1 - \alpha^2/(1 - \alpha) + \alpha = (1 - 2\alpha^2)/(1 - \alpha)$. It is easy to see that the last fraction is less than 1 if $\alpha \geq \frac{1}{2}$. On the other hand, if $\alpha < \frac{1}{2}$, $\alpha + \beta < 2\alpha < 1$. Thus in any case $\alpha + \beta < 1$. 

409/85/2.3
The above formula for $r_1$ leads to

$$\int_a^{a+2} r_1^2(t) \, dt = r_1(a + 2) - S_1(a + 2) = \frac{\gamma + \beta}{1 - (\gamma + \beta)} - \beta. \quad (2.3)$$

Let us make some change of variables. Let $B = 1/(\alpha + \beta) > 1$ and $A = aB > \frac{1}{2}$. Then after multiplying both the numerator and the denominator of the fraction in (2.3) by $(1 - \alpha)B$, we have

$$\int_a^{a+2} r_1^2(t) \, dt = \frac{A^2 + (1 - A)(B - A)}{B(B - A) - A^2 - (1 - A)(B - A)} + \frac{A - 1}{B}$$

$$= \frac{B^2 + (2A^2 - 2A)B + (A^2 - 2A^3)}{B[B^2 - B + (A - 2A^2)]} - \frac{1}{B}. \quad (2.4)$$

Similarly we have (replacing $A$ by $1 - A$)

$$\int_a^{a+2} r_2^2(t) \, dt = \frac{B^2 - (2A - 2A^2)B - (1 - 4A + 5A^2 - 2A^3)}{B[B^2 - B - (1 - 3A + 2A^2)]} - \frac{1}{B}.$$ 

The rest of the proof, though formidably long, is elementary and straightforward.

$$\int_a^{a+2} [r_1^2(t) - r_2^2(t)] \, dt = \frac{1}{B} \left[ \frac{B^2 + (2A^2 - 2A)B + (A^2 - 2A^3)}{B^2 - B + (A - 2A^2)} \right.$$ 

$$\left. - \frac{B^2 - (2A - 2A^2)B - (1 - 4A + 5A^2 - 2A^3)}{B^2 - B - (1 - 3A + 3A^2)} \right].$$

After combining the two fractions together using a common denominator, the numerator of the new fraction turns out to be

$$B^2(-2A + 6A^2 - 4A^3) + B(-1 + 6A - 12A^2 + 8A^3)$$

$$+ (A - 7A^2 + 18A^3 - 20A^4 + 8A^5)$$

$$= 2B^2(1 - A)(2A - 1) + B(2A - 1)^3 + A(2A - 1)(1) - A(2A - 1)^3(1 - A)$$


The first term is positive since $\frac{1}{2} \leq A \leq 1$. The second term is also positive since $A(1 - A) \leq \frac{1}{4}$ and $B > 1$.

Together with the Approximation Lemma (Lemma 1), the above lemma easily implies the following comparison result.
THEOREM 1. Let \( r_* \) and \( r^* \) be solutions of

\[
r_*(t) = S_*(t) + \int_a^t r_*^2(s) \, ds
\]  \hspace{1cm} (2.5)

and

\[
r^*(t) = S^*(t) + \int_a^t r^{*2}(s) \, ds,
\]  \hspace{1cm} (2.6)

where \( S_* \) and \( S^* \) are monotonic rearrangements of the function \( S \) in (2.1). Then

\[
\int_a^t r_*^2(s) \, ds \leq \int_a^t r^2(s) \, ds \leq \int_a^t r^{*2}(s) \, ds
\]  \hspace{1cm} (2.7)

for all \( t \) in the common domain of definitions of the solutions.

Proof. We outline the ideas and omit the details. In view of Lemma 1, we need only prove the theorem for step functions \( S \) having equal step sizes. Furthermore it suffices to prove (2.7) only for \( t \) equal to the end points of the steps. Let us only consider the proof of the first inequality in (2.7), that of the second being similar. The rearrangement \( S^* \) can be obtained from \( S \) by successively interchanging pairs of neighboring steps, the one on the left in each pair being larger before the operation. Hence it suffices to show that each such operation lowers the integral \( \int r^2(s) \, ds \). Let the steps affected be the \( i \)th and the \((i + 1)\)th ones. The equations before and after the operation are identical in the interval of the first \( i - 1 \) steps. Hence the integral \( \int r^2(s) \, ds \) is not changed in the same interval. Notice that (2.1) can be rewritten in the following from using any \( \tilde{f} \in [a, b) \) as the initial point instead of \( a \):

\[
r(t) = \left[ S(t) + \int_a^t r^2(s) \, ds \right] + \int_{\tilde{f}}^t r^2(s) \, ds.
\]  \hspace{1cm} (2.8)

In particular if we choose \( \tilde{f} \) to be the left end point of the \( i \)th step, then Lemma 2 applies and the new integral over the \( i \)th and \((i + 1)\)th steps. Using (2.8) with \( \tilde{f} \) the end point of the \((i + 1)\)th step, we can show easily using the simple comparison principle that the new \( r \) is smaller than the original \( r \) over the rest of the steps. The required inequality then follows.

An immediate corollary is that the non-existence (existence) of a solution to (2.1) on the whole of \([a, b)\) can be deduced from the nonexistence (existence) of a solution to (2.5) (2.6)) on \([a, b)\). As applications, we have
COROLLARY 1. If, for some \( \lambda > 0 \), the set \( \{ t \in [a, b) : S(t) \geq \lambda \} \) has measure greater than \( \frac{1}{\lambda} \), then (2.1) has no solution on \( [a, b) \).

Proof. Since \( S_* \) satisfies the same condition, Theorem 1 allows us to assume that \( S \) is non-decreasing. The conclusion follows from comparing the equation under consideration with a similar one whose "potential" is a step function equal to zero in \( [a, b - 1/\lambda) \) and \( \lambda \) in \( [b - 1/\lambda, b) \).

COROLLARY 2. If for some fixed \( \lambda^* > 0 \) we have

\[
\lambda \mu \{ t \in [a, b) : S(t) \geq \lambda \} \geq \text{some constant } k > \frac{1}{4}
\]

for all \( \lambda > \lambda^* \), then (2.1) has no solution on \( [a, b) \), where \( \mu \) denotes the Lebesgue measure.

Proof. As in Corollary 1 we may assume that \( S \) is non-decreasing. Comparing the equation with one having potential \( T(t) = k/(b - t) \) yields the result.

Lemma 2 permits us to interchange two neighboring steps without increasing the integral of \( r^2 \). One may expect the same to be true in case the two steps are separated. Unfortunately this is false as the following simple example shows.

Let

\[
S(t) = \begin{cases} 
0.1, & t \in [0, 1) \\
0.57, & t \in [1, 2) \\
0.05, & t \in [2, 3]
\end{cases}
\]

and

\[
T(t) = \begin{cases} 
0.05, & t \in [0, 1) \\
0.57, & t \in [1, 2) \\
0.1, & t \in [2, 3]
\end{cases}
\]

Direct computation yields

\[
r(t) = \frac{0.1}{1 - (0.1)t}, \quad R(t) = \frac{0.05}{1 - (0.05)t}, \quad t \in [0, 1).
\]

Thus

\[
\gamma_1 = \int_0^1 r^2(s) \, ds = \lim_{t \to 1^-} (r(t) - S(t)) = 0.01111...,
\]

\[
\Gamma_1 = \int_0^1 R^2(s) \, ds = \lim_{t \to 1^-} (R(t) - T(t)) = 0.0026315... .
\]
Similarly

\[ r(t) = \frac{0.57 + \gamma_1}{1 - (0.57 + \gamma_1)(t - 1)}, \quad R(t) = \frac{0.57 + \Gamma_1}{1 - (0.57 + \Gamma_1)(t - 1)}, \quad t \in [1, 2), \]

\[ \gamma_2 = \int_0^2 r'(s) \, ds = 0.81726..., \quad \Gamma_2 = \int_0^2 R'(s) \, ds = 0.76990..., \]

\[ r(t) = \frac{0.05 + \gamma_2}{1 - (0.05 + \gamma_2)(t - 2)}, \quad R(t) = \frac{0.1 + \Gamma_2}{1 - (0.1 + \Gamma_2)(t - 2)}, \quad t \in [2, 3]. \]

Finally \( \gamma_3 = \int_0^3 r^2(s) \, ds = 6.48... < \Gamma_3 = \int_0^3 R^2(s) \, ds = 6.59... \).

We choose to state Theorem 1 in its present form because of its simplicity and because this is all we need in the sequel. It is obvious that the following more general result follows from Lemma 2 in exactly the same way as Theorem 1 does.

Let \([a, b)\) be partitioned into disjoint intervals, say, with points \(a = a_0 < a_1 < a_2 < a_3 < \cdots\). Suppose that in each \((a_i, a_{i+1})\) either \(S(t) = T(t)\) or \(T'(t)\) is the monotonic increasing rearrangement of \(S\) (restricted to \((a_i, a_{i+1})\)), then \(\int_a^b R'(s) \, ds \leq \int_a^b r^2(s) \, ds\) for all \(t\) in the common domain of definitions of \(r\) and \(R\), where \(R\) is the solution of the Riccati equation with potential \(T\).

The inequality is reversed when monotonic increasing rearrangements are replaced by monotonic decreasing rearrangements.

An immediate application of Theorem 1 and its corollaries is the derivation of oscillation criteria for second-order linear differential equations. As an example, consider the equation on \([0, 1]\):

\[ y''(t) + q(t) \, y(t) = 0. \]

The function \(r(t) = -y'(t)/y(t)\) satisfies (1.1) with \(S(t) = r(0) + \int_0^t q(s) \, ds\). Suppose \(\int_0^t q(s) \, ds \geq 0\) for all \(t \in [0, 1]\), and \(y'(0) = 0\). Then \(y\) has a zero in \([0, 1]\) if the solution \(z\) of

\[ z''(t) + q_1(t) \, z(t) = 0 \]

with \(z'(0) = 0\) has a zero in \([0, 1]\) where \(q_1(t) \geq 0\) is a function such that \(\int_0^t q_1(s) \, ds \leq\) the increasing rearrangement of \(\int_0^t q(s) \, ds\). In the next two sections we will further expound this idea.

The following Riccati integral equation also occurs in oscillation theory and can be studied using similar techniques:

\[ r(t) = Q(t) + \int_t^b r^2(s) \, ds, \quad (2.9) \]
where $b \leq \infty$ and $Q \geq 0$ is piecewise continuous on $[a, b)$. In fact we may regard (2.9) as in a sense dual to (2.1). Since $b$ is a singular point, a solution need not exist in any neighborhood of $b$, and even if one does exist, it need not be the unique one. For instance, when $b = \infty$, $Q(t) = 1$ there exists no solution in the whole of any neighborhood of $\infty$. When $Q(t) = 0, r(t) = (t + \alpha)^{-1}$ is a solution to (2.9) in $[0, \infty)$ for any $\alpha > 0$. The concept of a minimal solution as introduced in the general theory of differential and integral inequality can be used, but we prefer to present a complete treatment here.

**Theorem 2.** If (2.9) has a solution defined on $(c, b)$ for some $c \in [a, b)$, then it has a minimal solution $r$ defined on $(c, b)$ such that $r(t) \leq \bar{r}(t)$ for all $t \in (c, b)$, where $\bar{r}$ is any other solution defined on $(c, b)$. Equation (2.9) has a minimal solution defined on $(c, b)$ if and only if every equation of the following family

$$
r_e(t) = Q(t) + \int_c^e r^2_e(s) \, ds, \quad e \in (c, b),
$$

has a solution defined on $(c, e)$.

**Proof.** First observe that a reflection transforms (2.9) and (2.10) into a Riccati equation of the form (3.1) except for the fact that the left end point may be singular. Thus all results concerning (2.1) can be modified to apply to (2.9) and (2.10). In particular the simple comparison principle implies that if $r_1$ and $r_2$ are solutions of (2.9) on $(c, b)$ and if for some $t_0$, $r_1(t_0) < r_2(t_0)$, then $r_1(t) < r_2(t)$ for all $t$. For any $t \in (c, b)$ define $r(t) = \inf \{ \bar{r}(t) : \bar{r}$ is a solution of (2.9) $\}$. Choose a fixed $t_0 \in (c, b)$. By definition, there exists a sequence of solutions of (2.1), $\{\bar{r}_n\}_{n=1}^\infty$ such that $\lim_{n \to \infty} \bar{r}_n(t_0) = r(t_0)$ monotonically. A continuity argument using the monotonicity property of the solutions shows that $\lim_{n \to \infty} \bar{r}_n(t) = r(t)$ for all $t \in (c, b)$. The dominated convergence theorem of Lebesgue integration then implies that $r$ is a solution of (2.9). The minimal property of $r$ is obvious from its definition.

Suppose a minimal solution of (2.9) exists. By the simple comparison principle, $r(t) \geq r_e(t)$. Thus the solutions $r_e$ exists on the whole of $(c, e)$ for all $e$. On the other hand, suppose that all the $r_e$'s exists on $(c, e)$. Choose a sequence $\{e_n\}_{n=1}^\infty$ that converges monotonically to $b$. Let $r_n = r_{e_n}$ be extended to $[c, b)$ by defining it to be zero on $[e, b)$. The comparison principle shows that $r_n$ form an increasing sequence of functions on $[c, b)$. For any $i \in (c, b)$, $\int_i^b r^2_n(s) \, ds$ must be bounded above otherwise by rewriting (2.9) using $i$ as the right initial point, we see that $r_n$ would tend to $\infty$ before reaching $c$ from the right. Using the familiar convergence theorems of Lebesgue integrals, we easily see that $\lim_{n \to \infty} \int_i^b r^2_n(s) \, ds$ exists and is finite for every $i$. It then follows that $\lim_{n \to \infty} r_n(i)$ exists. It is not difficult to see that the limit function is the minimal solution.
The proof of the following is analogous to that of Theorem 1.

**Theorem 3.** Let $r_*$ and $r^*$ be the minimal solutions of

$$r_*(t) = Q_*(t) + \int_t^b r_*^2(s) \, ds,$$  
and

$$r^*(t) = Q^*(t) + \int_t^b r^*^2(s) \, ds,$$

respectively. Then

$$\int_t^b r_*^2(s) \, ds \geq \int_t^b r^2(s) \, ds \geq \int_t^b r^*^2(s) \, ds$$

for all $t$ in the common domain of definitions of the solutions.

An immediate corollary is that the existence of a solution on $[a, b)$ to (2.9) can be deduced from the existence of a solution to (2.11) and the non-existence of a solution on $[a, b)$ to (2.9) from the non-existence of a solution to (2.12).

Corollaries similar to those following Theorem 1 also hold.

**Corollary 3.** If for some $\lambda$, the set $\{t \in [a, b) : Q(t) > \lambda\}$ has measure greater than $1/\lambda$, then (2.9) has no solution.

**Corollary 4.** If for some fixed $\lambda^* > 0$, we have

$$\lambda\mu\{t \in [a, b) : Q(t) > \lambda\} \geq some constant k > \frac{1}{4}$$

for all $\lambda \in (0, \lambda^*)$, then (2.9) has no solution.

3. Oscillation Criteria—An Extension of Hille's Theorem

In this section we are going to extend a result due to Hille.

The equation

$$y''(t) + q(t) y(t) = 0, \quad t \in [0, \infty),$$

where $q(t)$ is locally integrable, is said to be oscillatory if each of its solutions has an infinite number of zeros. Many sufficient conditions for oscillation are known. Among these we mention the method of Coles and
Willett [2] and the very recent results of Kwong and Zettl [9]. Both approaches make use of the function $\tilde{Q}(t) = \int_{0}^{t} q(s) \, ds$ and each gives a fairly general classification of (3.1). The former approach involves the existence of a weighted mean with respect to certain positive functions chosen from a class of admissible weights. The second approach, which will be described below, depends on a pair of numbers intrinsic to the function $\tilde{Q}$. For a large class of $q$ the two classifications are equivalent. The same question about general $q$ has not been investigated in detail.

We define

$$\alpha = \sup \{ a \in (-\infty, \infty) : [\tilde{Q} - a]_{-} \in L^{2}[0, \infty) \}$$

and

$$\overline{\alpha} = \inf \{ a \in (a, \infty) : [\tilde{Q} - a]_{+} \in L^{2}[0, \infty) \},$$

where $[\tilde{Q} - a]_{-}$ is the function $[\tilde{Q} - a]_{-}(t) = \max\{a - \tilde{Q}(t), 0\}$ and $[\tilde{Q} - a]_{+}$ is the function $[\tilde{Q} - a]_{+}(t) = \max\{\tilde{Q}(t) - a, 0\}$. The function $\tilde{Q}$ is said to be asymptotically oscillatory if $\alpha \neq \overline{\alpha}$, asymptotically large (negatively) if $\alpha = \overline{\alpha} = \infty$ ($-\infty$), and asymptotically constant or asymptotically close to $a$ if $\alpha = \alpha = \overline{\alpha} \neq \infty$.

The main result established in [9] is that if $\tilde{Q}$ is asymptotically large or if $\tilde{Q}$ is asymptotically oscillatory and satisfies a "boundedness below" restriction, then (3.1) is oscillatory. Instead of stating the general "boundedness below" condition, we just mention the special case: for some $\lambda \in (-\infty, \infty)$ the measure of the set $\{ t \in [0, \infty) : \tilde{Q}(t) \geq \lambda \}$ is infinite. This is satisfied in particular if $\alpha > -\infty$. For the class of asymptotically constant $\tilde{Q}$, additional information is needed to determine the oscillatory nature of (1.1). A subclass of such functions has long been studied. These are functions $\tilde{Q}$ for which $\lim_{t \to \infty} \tilde{Q}(t)$ exists. For recent results and a survey of known ones, see [10]. All known criteria in this category are formulated in terms of the function $Q(t) = \lim_{s \to \infty} \tilde{Q}(s) - \tilde{Q}(t)$. In [9] it is shown that all such criteria apply to the general class of asymptotically constant $\tilde{Q}$ when the function $Q$ is defined to be $\alpha - \tilde{Q}(t)$, where $\alpha = a = \overline{\alpha}$.

Among these criteria is the following result due to Hille [6]:

If $\lim_{t \to \infty} tQ(t) > \frac{1}{4}$, then (3.1) is oscillatory. If, furthermore, $q$ is positive, then $\lim_{t \to \infty} tQ(t) > 1$ also implies that (3.1) is oscillatory.

The first part was originally proved by Hille with the additional condition $q \geq 0$. The improvement is due to Opial. The criteria are rather restrictive since, for instance, the case in which $Q(t)$ assumes negative values for large $t$ has to be excluded.

An immediate extension can be obtained using the telescoping principle introduced in [8]:
Let \( \{(a_n, b_n)\}_{n=1}^{\infty} \) be a sequence of open intervals \( a_n < b_n < a_{n+1} < b_{n+1}, \) \( a_n \to \infty, \) such that \( \tilde{Q}(a_n) = \tilde{Q}(b_n) \) for all \( n. \) By shrinking each of the intervals \((a_n, b_n)\) to a point we obtain a new equation defined on a finite or infinite interval. If the new equation is oscillatory, so is the original one.

The intervals \((a_n, b_n)\) can be chosen optimally so as to cut out any undesirable parts, for example, those for which \( Q(t) \) is negative, so that the telescoped equation may be covered by any known oscillation criterion. This allows more cases to be covered by say Hille's result.

The results obtained in Section 2 can be used to further extend the applicability of Hille's criteria.

Let us first consider the case \( a > -\infty. \) If \( \bar{a} > a, \) we already have oscillation. So let \( \bar{a} = a. \) By the result in [9], (3.1) is non-oscillatory if and only if the following Riccati integral equation has a solution on \([a, \infty)\) for some \( \alpha > 0:\)

\[
r(t) = Q(t) + \int_{t}^{\infty} r^2(s) \, ds.
\]

(3.2)

We assume that \( Q \) is sufficiently positive so that the set \( \{t \in [0, \infty): Q(t) > 0\} \) is of infinite measure. Let \( \mu(\lambda) \) be the measure of the set

\[
\{t \in [0, \infty): Q(t) \geq \lambda\}.
\]

(3.3)

The following conditions on \( Q \) are extensions of those of Hille:

\[
\lim_{\lambda \to 0^+} \lambda \mu(\lambda) < \frac{1}{2} \quad \text{or} \quad \lim_{\lambda \to 0^+} \lambda \mu(\lambda) > 1.
\]

(3.4)

In particular if \( Q \) satisfies Hille's conditions, it satisfies (3.4). In view of the telescoping principle, we may assume without loss of generality that \( Q(t) \geq 0. \) Now the results of the last section apply. In particular Corollaries 3 and 4 show that (3.4) implies that (3.2) has no solution, whence (3.1) is oscillatory. We have thus proved:

**Theorem 4.** Let \( Q(t) = a - \int_{0}^{t} q(s) \, ds. \) Define \( \mu(\lambda) \) by (3.3). If condition (3.4) is satisfied, then (3.1) is oscillatory.

More generally the idea of the proof of Theorem 4 underlies the following comparison theorem.

**Theorem 5.** Let \( q_1(t) \geq 0, \ t \in [a, b) \) be such that the equation

\[
z''(t) + q_1(t) \, z(t) = 0
\]

is oscillatory and \( \int_{0}^{\infty} q_1(t) \, dt < \infty. \) Define \( Q_1(t) = \int_{0}^{t} q_1(s) \, ds. \) If \( Q_1(t) \leq Q^*(t) \) for all \( t \) large enough, then (3.1) is also oscillatory.
Notice that here $Q^*$ is defined just as in Section 2 although $Q$ may assume negative values.

The following is an immediate consequence of Theorem 4.

**Corollary 5.** Let $\tilde{Q}(t) = \min \{Q_+(t), 1\} = \min \{\max \{Q(t), 0\}, 1\}$. If for some $\gamma > 1$, $\int_0^\infty \tilde{Q}'(t) \, dt = \infty$, then (3.1) is oscillatory.

**Proof.** We may assume without loss of generality that $Q$ is non-increasing and that $Q(t) \leq 1$ so that $\tilde{Q}(t) = Q(t)$. It suffices to show that $Q$ satisfies the second condition of Hille. If $\lim_{t \to \infty} tQ(t) < 1$, then $Q(t) \leq 2/t$ for large $t$. It follows that $Q' \leq 2/t'$ is integrable, contradicting the hypothesis.

Let us now look at the case $a = -\infty$. If for some $\lambda < (-\infty, \infty)$, $\{t \in [0, \infty): \tilde{Q}(t) = \int_0^t q(s) \, ds \geq \lambda\}$ has infinite measure, then the "boundedness below" condition mentioned earlier is satisfied. Furthermore $a \geq \lambda \neq a$. It follows from the result in [9] that (3.1) is oscillatory. Hence the more interesting case is that the set $\{t \in [0, \infty): \tilde{Q}(t) = \int_0^t q(s) \, ds \geq \lambda\}$ has finite measure for all $\lambda$. The following result may be thought of as a dual of Theorem 4.

**Theorem 6.** Suppose $a = -\infty$. Define $\bar{\mu}(\lambda)$ to be the measure of the set $\{t \in [0, \infty): \tilde{Q}(t) > \lambda\}$. If

$$\lim_{\lambda \to \infty} \lambda \bar{\mu}(\lambda) > \frac{1}{2} \quad \text{or} \quad \lim_{\lambda \to \infty} \lambda \bar{\mu}(\lambda) > 1, \tag{3.5}$$

then (3.1) is oscillatory.

**Proof.** If $q$ satisfies the hypotheses of the theorem, then the restriction of $q$ on a subinterval $[a, \infty)$ also satisfies the hypotheses. Thus it suffices to show that every solution of (3.1) has at least one zero. The function $r(t) = -y'(t)/y(t)$ satisfies the Riccati integral equation

$$r(t) = r(0) + \bar{Q}(t) + \int_0^t r^2(s) \, ds. \tag{3.6}$$

Suppose $y$ does not have a zero in $[0, \infty)$, then $r$ is defined throughout $[0, \infty)$. It is easy to see that $\int_0^\infty r^2(s) \, ds = \infty$. In fact if $\int_0^\infty r^2(s) \, ds < \infty$, then, from (3.6), $r(t) \leq \int_0^t r^2(s) \, ds + \bar{Q}(t)$ and hence $\int_0^\infty [\bar{Q}(t) + r(0) + \int_0^\infty r^2(s) \, ds]^2 \, dt = \infty$ (the last equality follows from the assumption that $q = -\infty$), a contradiction.

Let $a \in [0, \infty)$ be such that $r(0) + \int_0^a r^2(s) \, ds \geq 0$. Then

$$r(t) \geq \bar{Q}(t) + \int_a^t r^2(s) \, ds.$$
The telescoping principle allows us to shrink the intervals in which $\overline{Q}(t) < 0$, leaving us with an equation with a positive potential defined on a finite interval $[a, b)$. The conclusion then follows from Corollaries 1 and 2.

The following corollary is proved in the same way as Corollary 5.

**Corollary 6.** Let $q = -\infty$. If $\int_0^\infty \overline{Q}_1(t) \, dt = \infty$ for some $0 < \gamma < 1$, then (3.1) is oscillatory.

4. **A Comparison Theorem and a Lyapunov-Type Inequality**

Hartman in [4] proved that if (3.1) is oscillatory, so is the equation

$$z''(t) + 4Q^2(t)z(t) = 0.$$  

In other words, if the latter is non-oscillatory, so is (3.1). His idea can be refined to give a slightly stronger result. In the following we first formulate the result in terms of Riccati integral equations, and then derive a comparison theorem and a Lyapunov-type inequality for disfocality.

Consider the Riccati integral equation

$$r(t) = S(t) + \int_0^t r^2(s) \, ds, \quad t \in [a, b). \quad (4.1)$$

This is similar to (2.1) but we do not require $S$ to be non-negative.

Let $N = \{t \in [a, b): S(t) < 0\}$. Being an open set, $N$ is the union of open intervals. By shrinking these intervals to half their lengths while keeping the complement of $N$ intact, we obtain a shorter interval $[a, b)$. On this distorted interval we define the function $p$ such that $p(\tau) = 2S^2(\tau)$ when $\tau$ is in one of the shortened intervals that corresponds to one of the open intervals of $N$, and $\tau$ corresponds to the point $t$ in the original interval $[a, b)$. More precisely, if $(a_n, b_n)$ is one of the intervals of $N$ and the corresponding subinterval of the distorted interval $[a, b)$ is $(\tilde{a}_n, \tilde{b}_n)$, then $p(\tau) = 2S^2(a_n + 2(\tau-a_n))$. On the complement, $p(\tau) = 4S^2(t)$. Finally define $T(\tau) = \int_0^\tau p(s) \, ds$, and let $R(\tau)$ be the solution of the Riccati equation

$$R(\tau) = T(\tau) + \int_0^\tau R^2(s) \, ds, \quad \tau \in [a, b). \quad (4.2)$$

**Theorem 7.** If (4.1) has no solution on $[a, b)$, then (4.2) has no solution on $[a, b)$.  


Proof. Let \( U(t) = \int_0^t r^2(s) \, ds \). Then \( U \) satisfies the Riccati differential equation

\[
U'(t) - S^2(t) + 2S(t) U(t) + U^2(t) \leq \begin{cases} 
2S^2(t) + 2U^2(t) & \text{when } S(t) \geq 0 \\
S^2(t) + U^2(t) & \text{when } S(t) < 0.
\end{cases}
\]  

(4.3)

The change of variables \( W(\tau) = 2U(t) \) for \( t \in N \) and \( W(\tau) = U(t) \) for \( t \in N \), where the correspondence between \( \tau \) and \( t \) are defined as above, reduces the Riccati inequality to

\[
W'(\tau) \leq p(\tau) + W^2(\tau).
\]

On integration this gives

\[
W(\tau) \leq T(\tau) + \int_a^\tau W^2(s) \, ds.
\]

A comparison with (4.2) shows that \( W(\tau) \leq R \). If (4.1) has no solution, then \( r(t) \to \infty \) before \( t \) reaches \( b \). It follows that \( W(\tau) \to \infty \) and hence \( R(\tau) \to \infty \) before \( \tau \) reaches \( b \).

An analogous dual result holds for Riccati integral equations of the form (2.9). We omit the details.

If we are willing to settle for a weaker result we can do without the change of independent variable.

Corollary 7. Let

\[
\tilde{p}(t) = 4S^2(t), \quad t \in N
\]

and \( \tilde{T}(t) = \int_0^t \tilde{p}(t) \, dt \). If (4.1) with \( \tilde{T} \) in place of \( S \) has no solution on \((a, b)\), so does (4.1).

Proof. Replace (4.3) by the weaker inequality

\[
U'(t) \leq 2S^2(t) + 2U^2(t), \quad t \in N
\]

\[
\leq S^2(t) + 2U^2(t), \quad t \in N
\]

and then use the change of variable \( \tilde{W}(t) = 2U(t) \).

Using the dual of Theorem 7 we have the following improvement of Hartman’s result.
THEOREM 8. Let \( \tilde{Q} \) be asymptotically close to the constant \( a \) and define \( Q(t) = \alpha - \tilde{Q}(t) \). Let \( p(t) = 4Q^2(t) \) when \( Q(t) > 0 \) and \( p(t) = 2Q^2(t) \) when \( Q(t) \leq 0 \) where the relation between \( \tau \) and \( t \) are defined as before Theorem 7. If the equation

\[ z''(\tau) + p(\tau) z(\tau) = 0 \]

is nonoscillatory, so is (3.1).

The classical Lyapunov inequality states that if a solution of (3.1) has two zeros at \( a \) and \( b \), then \( \int_a^b q_+(t) \, dt > \frac{4}{(b - a)} \). This result has found many applications in the study of eigenvalues and stability. It has been strengthened in various ways. The following inequality is due to Cohn [1]. Let \( y \) be a solution to (3.1). If at two points \( a < c \), \( y(a) = y'(c) = 0 \), then

\[ \int_a^c \max_{s \in [a,c]} \{Q(s)\} \, dt > 1, \]

where \( Q_+(t) = \max \{0, Q(t)\} \). This result has been further improved in [7]. For simplicity we just state the following special case. If \( \tilde{Q}(t) = \int_0^t q(s) \, ds \geq 0 \), define \( Q^*(t) = \max_{s \in [0,t]} \{Q(s)\} \). If \( y(a) = y'(c) = 0 \) as before, we must have

\[ \int_a^c Q^*(t) \, dt > 1. \quad (4.4) \]

Although in many cases (4.4) reduces to Cohn's inequality, it is a true improvement in some interesting examples. One unsatisfactory point about this result is that \( Q^* \) and not \( \tilde{Q} \) has to be used in the inequality. Hence it does not distinguish between the case in which \( \tilde{Q} \) increases to large values quickly near \( a \) and then stays large and the case in which \( \tilde{Q} \) increases to large values at first and then decreases rapidly. The following theorem may be of assistance in such situations.

Let \( N = \{ t \in [a, c) : \tilde{Q}(t) = \int_a^t q(t) \, dt < 0 \} \) and shrink the intervals of \( N \) into half their lengths as before, thus defining a change of variable, namely, from \( t \in [a, c) \) to \( \tau \in [a, \bar{c}) \). Define \( P(\tau) = 4\tilde{Q}^2(\tau) \) when \( \tilde{Q}(\tau) \geq 0 \) and \( P(\tau) = 2\tilde{Q}^2(\tau) \) when \( \tilde{Q}(\tau) < 0 \).

THEOREM 9. If (3.1) has a solution \( y \) such that \( y(a) = y'(c) = 0 \), \( a < c \), then

\[ \int_a^c P(\tau)(\tau - a) \, d\tau > 1. \quad (4.5) \]

Proof. This is an immediate consequence of Theorem 7 and Cohn's inequality.
Corollary 8. Under the same hypotheses of Theorem 9, we have

\[ \int_a^c Q(t)(t - a) \, dt > \frac{1}{4}. \] (4.6)

The following examples compare (4.6) with Cohn's inequality. Let \([a, c] = [0, 1]\), and

\[
q(t) = \begin{cases} 
\alpha, & t \in [0, \frac{1}{2}] \\
-\alpha, & t \in (\frac{1}{2}, 1]. 
\end{cases}
\]

Cohn's inequality gives \(\alpha > \frac{3}{\pi}\) for focality while (4.6) gives \(\alpha > \sqrt{6}\) which is weaker. Suppose on the same interval

\[
q(t) = \begin{cases} 
\alpha, & [0, \frac{1}{2}] \\
-\alpha, & (\frac{1}{2}, \frac{3}{2}] \\
0, & (\frac{3}{2}, 1]. 
\end{cases}
\]

Cohn's inequality gives \(\alpha > 3.6\), while (4.6) gives \(\alpha > \sqrt{243/8} > 5.511\).

References