Extremal functions for the Caffarelli–Kohn–Nirenberg inequalities: A simple proof of the symmetry

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Abstract

In this article, we give a simple proof of the result due to Lin and Wang ensuring the foliated Schwarz symmetry of the extremal functions for the Caffarelli–Kohn–Nirenberg inequalities. This new proof uses a direct and powerful method due to Bartsch, Weth and Willem using polarizations.

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1. Introduction

In this paper we are interested by the problem

\[-\Delta_\sigma v + \lambda v = f(\sigma, v), \quad v \in H^1_0(\Omega), \quad (P)\]

where \(\Omega\) is an open subset of the cylinder \(\mathcal{C} = S^{N-1} \times \mathbb{R}\), \(\lambda\) is a non-negative real number and \(f: \Omega \times \mathbb{R} \to \mathbb{R}\) is a continuous function, and where \(\Delta_\sigma\) is the Laplace–Beltrami operator on the cylinder. We shall consider the functional \(\phi: H^1_0(\Omega) \to \mathbb{R}\) defined by

\[\phi(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) - \int_\Omega F(\sigma, v),\]

where \(F(\sigma, v) := \int_0^v f(\sigma, t) \, dt\), and the Nehari manifold \(\mathcal{N}\) defined by

\[\mathcal{N} = \{ v \in H^1_0(\Omega) \setminus \{0\} : \langle \phi'(v), v \rangle = 0 \} = \{ v \in H^1_0(\Omega) \setminus \{0\} : \int_\Omega (|\nabla v|^2 + v^2) = \int_\Omega f(\sigma, v)v \}\].

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We shall be concerned with the ground-states of problem \((P)\), i.e., functions \(v\) which minimize \(\phi\) on \(N\). The classical hypothesis ensuring that the functional \(\phi\) is well defined on \(H^1_0\) and that the ground-states are solutions of \((P)\) will be recalled later.

When \(\Omega\) and the function \(f\) are invariant with respect to the group \(G\) of rotations in \(\mathbb{R}^{N+1}\) following the \(N\) first variables (i.e. when the functional \(\phi\) is invariant with respect to the action of this group on \(H^1_0\)), we can ask whether ground-states \(v\) of problem \((P)\) are also invariants with respect to this action. In general the answer is negative, but the main result of this paper, Theorem 3.4, is that, under some additional assumptions on \(f\), ground-states have the so-called foliated Schwarz symmetries. That means the existence of a vector \(\xi \in S^{N-1} \times \{0\}\) and a function \(\tilde{v}\) such that

\[
\forall \sigma = (\theta, t) \in \Omega: \quad v(\sigma) = \tilde{v}(\theta \cdot \xi, t)
\]

with \(\tilde{v}\) non-increasing with respect to the first variable.

This theorem is the version on \(C\) of a theorem due to Bartsch, Weth and Willem in [1] concerning least energy nodal solutions of problem \((P)\) with \(\Omega \subseteq \mathbb{R}^N\). Let us notice that this result has been extended for weak solutions by Van Schaftingen and Willem in [2].

The paper is organized as follows. We recall the main tools in Section 2. Then we prove Theorem 3.4 in Section 3. Finally, we apply this theorem in Section 4 to the function \(f(\sigma, v) = |v|^{p-1}v\) with \(p\) superquadratic and subcritical to prove that extremal functions of the Caffarelli–Kohn–Nirenberg inequalities are foliated Schwarz symmetric.

2. Main tools

2.1. The cylinder and the Laplace–Beltrami operator

Let us denote by \(C\) the cylinder \(S^{N-1} \times \mathbb{R}\). We shall use the generic notations

\[
\sigma := (\theta, t) := (\theta_1, \ldots, \theta_N, t),
\]

for a point of \(C\), and \(v\) for a function from \(C\) to \(\mathbb{R}\). If a function \(v\) is of class \(C^1\) or \(C^2\), we shall denote by \(\nabla_\sigma v\) its gradient and by \(\Delta_\sigma v\) its Laplace–Beltrami operator on the cylinder. If \(v_1\) and \(v_2\) are two functions of class \(C^2\) defined from \(\Omega\) to \(\mathbb{R}\) with a support contained in \(\Omega\), then we have the integration by parts formula:

\[
- \int_{\Omega} (\Delta_\sigma v_1) v_2 = \int_{\Omega} \nabla_\sigma \cdot v_1 \nabla_\sigma v_2 = - \int_{\Omega} v_1 \Delta_\sigma v_2. \tag{1}
\]

Finally, let us recall the Strong Maximum Principle on the cylinder, similar to the Classical Strong Maximum Principle on an open subset of \(\mathbb{R}^N\). The reader can find a proof of a more general version of this theorem in [3].

**Theorem 2.1 (Strong Maximum Principle on the cylinder).** If

1. \(V \subset C\) is an open connected set, possibly unbounded,
2. \(c \in C(V)\) is a non-negative function,
3. \(v \in C^2(V)\) is such that \(-\Delta_\sigma v + c(\sigma)v \leq 0,\)
4. \(\sup_V v \geq 0\) when \(c \neq 0,\)

then

\[
\left( \exists \sigma_0 \in V: \sup_V v = v(\sigma_0) \right) \Rightarrow \left( v \equiv v(\sigma_0) \right).
\]

2.2. Useful spaces

Let us consider \(\Omega\) an open subset of \(C\). The following spaces will be important:

\[
D(\Omega) := \{ v \in C^{\infty}(\Omega): \text{Supp } v \text{ is a compact of } \Omega \}, \quad H^1_0(\Omega) := \overline{D(\Omega)}.
\]
where \( \mathcal{D}(\Omega) \) is equipped with the inner product \( (v|w) := \int_\Omega vw + \int_\Omega \nabla_\sigma v \cdot \nabla_\sigma w \) and the induced norm \( \| \cdot \| \). By regularization, the integration by parts formula remains true in \( H^1(\Omega) \times \mathcal{D}(\Omega) \):

\[
\forall v \in H^1_0(\Omega), \; \forall w \in \mathcal{D}(\Omega): \quad -\int_\Omega v \Delta_\sigma w = \int_\Omega \nabla_\sigma v \nabla_\sigma w.
\]

Let us recall the Sobolev injections for this space with \( \Omega = C \):

**Theorem 2.2.** The following embedding are continuous:

\[
\begin{align*}
H^1_0(C) & \subset L^p(C), \quad 2 \leq p < \infty, \; N = 2, \\
H^1_0(C) & \subset L^p(C), \quad 2 \leq p \leq 2^* := \frac{2N}{N - 2}, \; N \geq 3.
\end{align*}
\]

The reader can find a proof of a more general version of this theorem in [3].

2.3. Foliated Schwarz symmetrization and polarization

In this section we shall work in \( \mathbb{R}^{N+1} \) with half-spaces \( H = \{ y \in \mathbb{R}^{N+1}; \; y \cdot \xi \leq 0 \} \) such that the hyperplane \( T := \partial H \) contains the \( y_{N+1} \)-axis. We shall denote by \( \mathcal{H} \) the set of such half-spaces and by \( \mathcal{H}_P \) the set of such half-spaces containing \( P \) in their interior, where \( P \in S^{N-1} \times \{0\} \).

**Definition 2.3.** The symmetrization \( A^P \) of a measurable set \( A \subset S^{N-1} \) with respect to \( P \in S^{N-1} \) is the closed geodesic ball in \( S^{N-1} \) centered on \( P \) which satisfies \( m(A^P) = m(A) \) where \( m \) is the superficial measure on \( S^{N-1} \). The symmetrization \( A^P \) of a measurable set \( A \subset C \) is the unique subset of \( C \) such that, for almost every \( t \in \mathbb{R} \):

\[
Q[A^P \cap (\mathbb{R}^N \times \{t\})] = \left( Q[A \cap (\mathbb{R}^N \times \{t\})] \right)^P,
\]

where \( Q \) is the orthogonal projection of \( \mathbb{R}^{N+1} \) onto the hyperplane \( y_{N+1} = 0 \). If \( \Omega \) is an open subset of \( C \), \( v: \Omega \to \mathbb{R} \) a continuous function and \( P \in S^{N-1} \times \{0\} \), the foliated Schwarz symmetrization \( v^P: \Omega^P \to \mathbb{R} \) of \( v \) with respect to \( P \) is defined by the condition

\[
\forall r \in \mathbb{R}: \quad \{ v^P \geq r \} = \{ v \geq r \}^P.
\]

If \( \Omega = \Omega^P \) and \( v = v^P \), then \( v \) is said to be foliated Schwarz symmetric with respect to \( P \).

The interpretation of this symmetry property is given by the following lemma.

**Lemma 2.4.** Let \( \Omega = S^{N-1} \times (a,b) \) and \( v: \Omega \to \mathbb{R} \) a continuous function. For \( \theta \in S^{N-1} \) let us define \( \varphi = \varphi(\theta) \) the geodesic distance between \( \theta \) and \( P = (1,0,\ldots,0) \). \( v \) is foliated Schwarz symmetric following \( P \) if and only if there exists a continuous function \( f: [0, \pi] \times \mathbb{R} \to \mathbb{R} \) such that for all \( t \in \mathbb{R} \), \( f(\cdot, t) \) is non-increasing and

\[
\forall \sigma = (\theta, t) \in \Omega: \quad v(\theta, t) = f(\varphi(\theta), t).
\]

We can also describe this symmetry property with half-spaces and polarizations. Let us recall that if \( S_H \) is the orthogonal symmetry associated to the boundary of a half-space \( H \), and if \( v: \Omega \to \mathbb{R} \) is a function defined on an open set \( \Omega \) invariant with respect to \( S_H \), then the polarization \( v_H \) of \( v \) is defined by

\[
v_H(x) := \begin{cases} 
\max\{v(x), v(S_H(x))\} & \text{if } x \in \Omega \cap H, \\
\min\{v(x), v(S_H(x))\} & \text{if } x \notin \Omega \cap H.
\end{cases}
\]

The following lemma can be found in [1] but the proof given here is simpler.

**Lemma 2.5.** Let \( \Omega = S^{N-1} \times (a, b) \), \( v: \Omega \to \mathbb{R} \) a continuous function, and \( P \in S^{N-1} \times \{0\} \). Then, \( v = v^P \iff \forall H \in \mathcal{H}_P: v = v_H \).
Proof. In this proof, we shall assume that \( P = (1, 0, \ldots, 0) \).

(⇒) Let \( H \in \mathcal{H}^P \) and \( \sigma \in \Omega \cap H \). As \( \varphi(\sigma) \leq \varphi(S_H(\sigma)) \) because \( P \in H \), by Lemma 2.4 we have \( v(\sigma) \geq v(S_H(\sigma)) \). The thesis follows.

(⇐) Let us prove that if \( H \) is a half-space such that \( \partial H \) contains the \( y_{N+1} \)-axis and \( P \), then \( v = v^H \). Let us note that proving this fact for all such \( H \) is equivalent to prove that

\[
\forall \sigma \in \Omega: \quad v(\sigma) = v(S_H(\sigma)).
\]

We can then conclude by continuity of \( v \). Hence by (2) there exists a function \( f \) such that

\[
f : [0, \pi] \times \mathbb{R} \to \mathbb{R}, \quad v(\sigma) = v(\theta, t) = f(\varphi(\theta), t).
\]

If we prove that for all \( t \in \mathbb{R} \), \( f(\cdot, t) \) is non-increasing, by Lemma 2.4 we have the thesis. If \( \varphi, \varphi' \in (0, \pi), \quad \varphi < \varphi' \), then

\[
\exists H \in \mathcal{H}^P, \exists \sigma \in \Omega \cap H: \quad \varphi(\sigma) = \varphi, \quad \varphi(S_H(\sigma)) = \varphi',
\]

and hence, because \( v = v^H \), we have

\[
f(\varphi, t) = v(\theta, t) = v(\sigma) \geq v(S_H(\sigma)) = f(\varphi', t).
\]

We conclude then for \( \varphi, \varphi' \in [0, \pi] \) by continuity of \( f \) (which follows from continuity of \( v \)). \( \Box \)

Finally, let us give the following lemma, which is completely similar to its analog on an open subset of \( \mathbb{R}^N \).

**Lemma 2.6.** Let us consider \( \Omega \) an open subset of \( \mathcal{C} \) invariant with respect to \( S_H \) for some half-space \( H \in \mathcal{H} \). If \( v \in H^1(\Omega) \) then \( v^H \in H^1(\Omega) \) and

\[
\int_\Omega v^2 = \int_\Omega \nu^2_H, \quad \int_\Omega |\nabla v|^2 = \int_\Omega |\nabla v_H|^2.
\]

3. The main theorem

In this section, we shall work with \( \Omega \) an open subset of \( \mathcal{C} \), \( \lambda \) a non-negative real and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) a continuous function, and consider problem \((\mathcal{P})\), functional \( \phi \) and Nehari manifold \( \mathcal{N} \) defined in the introduction. We shall often use the following hypotheses on \( f \) and \( \Omega \):

\( \mathbf{(H_1)} \) (a) \( f \) is continuous,

(b) \( f \) is Hölder continuous on \( \Omega \times [-R, R] \) for every \( R > 0 \),
Theorem 3.4. Under the hypotheses (H1)(a)–(H2), the functional φ is well defined, it is of class $C^1$ and its critical points are weak solutions of (P).

Theorem 3.2. Under the hypotheses (H1)(b)–(H2), a function v is a classical solution of (P) if and only if v is a weak solution of (P).

Theorem 3.3. Under the hypotheses (H1)(a)–(H2)(b), the functional φ is bounded from below on $\mathcal{N}$ by 0 and every minimizer $v \in \mathcal{N}$ of $\phi|_{\mathcal{N}}$ is a critical point of $\phi$.

We shall now prove the main result of this paper by proceeding as in [1].

Theorem 3.4. Under the hypotheses (H1)(b)–(H2)(H3)(a), (b)–(H4)(b), every minimizer $v \in \mathcal{N}$ of $\phi|_{\mathcal{N}}$ is foliated Schwarz symmetric.

To do that, let us begin by proving the two following lemmas.

Lemma 3.5. Under the hypotheses (H1)(a)–(H2)–(H4)(a), if $v \in \mathcal{N}$ is such that $\phi(v) = \inf_{\mathcal{N}} \phi$ and if $H$ is a half-space, then $v_H$ is also such that $\phi(v_H) = \inf_{\mathcal{N}} \phi$ and $v_H \in \mathcal{N}$.

Proof. Let us note that by Lemma 2.6 we know that $v_H \in H^1_0(\Omega)$ and that $\int_{\Omega} v^2 = \int_{\Omega} v_H^2$ and $\int_{\Omega} |\nabla v|^2 = \int_{\Omega} |\nabla v_H|^2$ hold. Hence, it suffices to prove that

$$v_H \in \mathcal{N}, \quad \phi(v_H) = \phi(v),$$

i.e.,

$$\int_{\Omega} F(\sigma, v_H) = \int_{\Omega} F(\sigma, v), \quad \int_{\Omega} f(\sigma, v_H)v = \int_{\Omega} f(\sigma, v)v.$$

The proofs are exactly the same as $\int_{\Omega} v^2 = \int_{\Omega} v_H^2$. \[\square\]

Lemma 3.6. Under the hypotheses (H1)(a)–(H2)–(H3)(a)–(H4)(a), if $v \in C^2(\Omega) \cap C_0(\overline{\Omega})$ is a solution of problem (P) such that

$$v_H \in C^2(\Omega) \cap C_0(\overline{\Omega}) \quad \text{and} \quad v_H \text{ is a solution of (P)},$$

then we have one of the three following possibilities: $\forall \sigma \in \text{int} \Omega \cap H$,

(a) $v(\sigma) > v(S_H(\sigma))$,  \hspace{1cm} (b) $v(\sigma) < v(S_H(\sigma))$,  \hspace{1cm} (c) $v(\sigma) = v(S_H(\sigma))$.

Proof. Let us consider the function $w \in C^2(\Omega) \cap C_0(\overline{\Omega})$ defined by the restriction of $v - \nu$ to $V := \text{int}(\Omega \cap H)$, where $\nu := v \circ S_H$. We have
\[-\Delta_\sigma |w| + \lambda |w| = -\Delta \left(2v_H - (v + \bar{v})\right) + \lambda \left(2v_H - (v + \bar{v})\right) = 2f(\sigma, v_H) - f(\sigma, v) - f(\sigma, \bar{v}) = \left[f(\sigma, v_H) - f(\sigma, v)\right] + \left[f(\sigma, v_H) - f(\sigma, \bar{v})\right] \geq 0,
\]

where the inequality is due to (H3)(a). Hence, by the Strong Maximum Principle on the cylinder applied to the function \(-|w|\), we have \(-|w| \equiv 0\) or \(-|w| < 0\) on \(V\). The thesis follows. \(\square\)

We can then mention the following theorem.

**Theorem 3.7.** Under the hypotheses (H1)(b)–(H2)–(H3)(a)–(H4)(b), if \(v \in C^2(\Omega) \cap C_0(\overline{\Omega})\) is a solution of problem \((P)\) such that

\[\forall \mathcal{H} \in \mathcal{H}: \quad v_H \in C^2(\Omega) \cap C_0(\overline{\Omega}) \quad \text{and} \quad v_H \text{ is a solution of } (P),\]

then there exists \(P\in S^{N-1} \times \{0\}\) such that \(v = v^P\).

**Proof.** This theorem is a consequence of the second lemma. \(\square\)

**Proof of Theorem 3.4.** Theorem 3.7 joint to the first lemma lead to Theorem 3.4. \(\square\)

### 4. Application to the Caffarelli–Kohn–Nirenberg inequalities

Particular cases of the Caffarelli–Kohn–Nirenberg inequalities are the following:

\[\exists C > 0, \quad \forall u \in D(\mathbb{R}^N): \quad \left(\int_{\mathbb{R}^N} |x|^{\alpha} |u|^p \right)^\frac{1}{p} \leq C \left(\int_{\mathbb{R}^N} |x|^{\beta} |\nabla u|^2 \right)^\frac{1}{2}, \quad (\text{CKN})\]

where \(p \in [1, \infty)\) and \(\alpha, \beta \in \mathbb{R}\) are related by

\[
\begin{align*}
N = 1, & \quad \frac{1}{2} < \beta < +\infty, \quad \beta - 1 \leq \alpha \leq \beta - \frac{1}{2}, \quad p = \frac{2}{\beta - \alpha}; \\
N = 2, & \quad 0 < \beta < +\infty, \quad \beta - 1 \leq \alpha \leq \beta, \quad p = \frac{2}{\beta - \alpha}; \\
N \geq 3, & \quad \frac{2 - N}{2} < \beta < +\infty, \quad \beta - 1 \leq \alpha \leq \beta, \quad p = \frac{2N}{N - 2 + 2(\beta - \alpha)}.
\end{align*}
\]

The proof is contained in [5]. Let us denote by \(S_{\alpha, \beta}\) the optimal constant of (CKN), i.e.,

\[S_{\alpha, \beta}^{-1} = \inf_{u \in D(\mathbb{R}^N) \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |x|^{2\beta} |\nabla u|^2 \ dx}{\left(\int_{\mathbb{R}^N} |x|^{\alpha p} |u|^p \ dx\right)^{1/p}}\right)^{1/2} > 0.
\]

Our purpose is to give a new proof of the result obtained in [6] by Lin and Wang insuring that, when \(N \geq 2\), if \(S_{\alpha, \beta}\) is reached by a \(D_{\beta}^{1,2}(\mathbb{R}^N)\)-function \(u\) (that we shall call extremal function), then \(u\) is foliated Schwarz symmetric, i.e., \(u\) can be written as \(u(x) = \tilde{u}(x \cdot \xi, |x|)\) where \(\xi \in \mathbb{R}^N\) and \(\tilde{u}(. , r)\) is non-increasing for every \(r > 0\). Let us note that existence results concerning extremal functions can be found in [7,8], and that those functions are in the space \(D_{\beta}^{1,2}(\mathbb{R}^N) := \overline{D(\mathbb{R}^N)}\) where the closure is taken with respect to the norm \(\| \cdot \|_{\beta}\) induced by the inner product \(D(\mathbb{R}^N)\) by

\[\langle u_1, u_2 \rangle_{\beta} = \int_{\mathbb{R}^N} |x|^{2\beta} \nabla u_1 \cdot \nabla u_2.
\]

Let us note also that the following injections are continuous:
\[ \forall (\alpha, p) \text{ as in (3), (4), (5):} \quad \mathcal{D}_{\beta}^{1,2}(\mathbb{R}^N) \subset L^p_\alpha(\mathbb{R}^N), \]

\[ L^p_\alpha(\mathbb{R}^N) := \left\{ u \in L^1_{\text{Loc}}(\mathbb{R}^N): \int_{\mathbb{R}^N} |x|^{\alpha p}|u|^p < \infty \right\} \]

(it is direct a consequence of (CKN)) and that \( \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \) is dense in \( \mathcal{D}_{\beta}^{1,2}(\mathbb{R}^N) \) for the norm \( \| \cdot \|_\beta \) (see for example [8]). By homogeneity, we can formulate the result of Lin and Wang as

**Theorem 4.1.** Let us consider

\[ \psi_{\alpha,\beta}: \mathcal{D}_{\beta}^{1,2}(\mathbb{R}^N) \to \mathbb{R}: u \to \int_{\mathbb{R}^N} |x|^\alpha |u|^p \, dx. \]

If \( N \geq 2 \), every \( \mathcal{D}_{\beta}^{1,2}(\mathbb{R}^N) \)-solution \( u \) of the minimization problem \( \inf_M \| \cdot \|_\beta^2 \) where \( M = \psi_{\alpha,\beta}^{-1}(1) \), is foliated Schwarz symmetric.

As in [6–8], let us consider the map

\[ T: \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \to \mathcal{D}(\mathcal{C}): u \to v, \]

\[ u(x) = |x|^{-\frac{N-1+2\beta}{2}} \cdot v \left( \frac{x}{|x|}, -\ln |x| \right), \quad v(\theta, t) = e^{-\frac{N-1+2\beta}{2} \, t} \cdot u(\theta \cdot e^{-t}). \]  

(6)

A direct computation shows that \( T \) satisfies

\[ \int_{\mathbb{R}^N} |x|^{2\beta} \nabla u_1 \cdot \nabla u_2 \, dx = \int_{\mathcal{C}} \left\{ \nabla_\alpha v_1 \cdot \nabla_\alpha v_2 + \left( \frac{N-2+2\beta}{2} \right)^2 v_1 v_2 \right\}. \]  

\[ \int_{\mathbb{R}^N} |x|^\alpha |u|^p \, dx = \int_{\mathcal{C}} |v|^p. \]  

(8)

Let us then endow \( \mathcal{D}(\mathcal{C}) \) with the inner product

\[ (v_1|v_2)_{\beta,\mathcal{C}} = \int_{\mathcal{C}} \left\{ \nabla_\alpha v_1 \nabla_\alpha v_2 + \left( \frac{N-2+2\beta}{2} \right)^2 v_1 v_2 \right\}, \]

and with the induced norm \( \| \cdot \|_{\beta,\mathcal{C}} \). By the equivalence of this norm with \( \| \cdot \| \) defined in Section 2, by the isometry of the bijection \( T \), and the density of \( \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \) in \( \mathcal{D}_{\beta}^{1,2}(\mathbb{R}^N) \), Theorem 4.1 is equivalent to:

**Theorem 4.2.** Let us consider

\[ \psi_{\alpha,\beta,\mathcal{C}}: H^1_0(\mathcal{C}) \to \mathbb{R}: v \to \int_{\mathcal{C}} |v|^p. \]

If \( N \geq 2 \), every \( C^2 \)-solution \( v \) of the minimization problem \( \inf_{M_\mathcal{C}} \| \cdot \|_{\beta,\mathcal{C}}^2 \) where \( M_\mathcal{C} = \psi_{\alpha,\beta,\mathcal{C}}^{-1}(1) \), is foliated Schwarz symmetric.

It is clear that functions \( v \) (respectively \( u \)) mentioned in Theorem 4.2 (respectively Theorem 4.1) verify the equation \( \varphi_{\beta,\mathcal{C}}(v) = \lambda \psi_{\alpha,\beta,\mathcal{C}}(v) \) (respectively \( \varphi_{\beta}(u) = \lambda \psi_{\alpha,\beta}(u) \)) where the Lagrange multiplier \( \lambda \) equals \( \frac{2}{p} S_{\alpha,\beta}^{-2} \). Hence the equation for \( v \) is

\[ -\Delta_\sigma v + \left( \frac{N-2+2\beta}{2} \right)^2 v = S_{\alpha,\beta}^{-2}|v|^{p-2}v, \]  

(9)

and for \( u \) is

\[ -\text{div}(|x|^{2\beta} \nabla u) = S_{\alpha,\beta}^{-2}|x|^\alpha |u|^{p-2}u. \]
Remark 4.3. Let us note that

\[
S^{-1}_{\alpha,\beta} = \inf_{v \in D(C) \setminus \{0\}} \left( \frac{\int_C (|\nabla \sigma v|^2 + (\frac{N-2+2\beta}{2})v^2))^{\frac{1}{2}}}{(\int_C |v|^p)^{1/p}} \right) > 0.
\]

This is equivalent to Theorem 2.2, i.e., the continuous injection of \( H^1_0(C) \) in \( L^p(C) \) for all \( p \) in \([2, \infty)\) or \([2, 2^*]\). This can be regarded in parallel to the injections of \( D^{1,2}_\beta(\mathbb{R}^N) \) in \( L^p_\alpha(\mathbb{R}^N) \) mentioned above, with \( \beta > \frac{2-N}{2} \) fixed and \( \alpha \) (and so \( p \)) taking all its values obtained by (3), (4), (5).

We can conclude by the proof of Theorem 4.2 in the case \( p > 2 \). Because of (9), it is clear that we shall work with

\[
\lambda = \left( \frac{N - 2 + 2\beta}{2} \right)^2, \quad f(\sigma, v) = S^{-2}_{\alpha,\beta}|v|^{p-2}v.
\]

The homogeneity of the problem ensures that

\[
v \in M_C, \quad \|v\|^2_{\beta,C} = \inf_{M_C} \|\cdot\|^2_{\beta,C} \iff \|v\|^2_{\beta,C} \cdot v \in N, \quad \phi\left(\|v\|^2_{\beta,C} \cdot v\right) = \inf_N \phi,
\]

which, together with Theorem 3.4, permits to conclude that if \( v \) is a minimizer of \( \|\cdot\|^2_{\beta,C} \) on \( M_C \), then \( \|v\|^2_{\beta,C} \cdot v \) is foliated Schwarz symmetric. Hence the same holds for \( v \).

Remark 4.4. Let us notice that this proof does not work if \( N = 1 \) because in this case, \( C \) is not connected.

References