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Fuzzy measures and discreteness☆

Congxin Wu*, Xuekun Ren

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China

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Abstract

The main result is to show that the space of nonmonotonic fuzzy measures on a measurable space $(X, \mathscr{X})$ with total variation norm is separable if and only if the σ-algebra $\mathscr{X}$ is a finite set. Our result is related to fuzzy analysis, functional spaces and discrete mathematics.

Keywords: Nonmonotonic fuzzy measure; Banach space; Separability; σ-Algebra

1. Introduction

It is well known that the main object in functional analysis is to discuss the infinite dimensional problem and continuity. Obviously, several conclusions of continuity in functional analysis which are characterized by the finiteness of the dimension in the universe space are closely related to discrete mathematics, and they are always remarkable. For example, the well known Dvoretsky–Rogers Theorem [1,8] states that for a Banach space X, every unconditionally convergent series in X is absolutely convergent if and only if X is finite dimensional.

On the other hand, since Zadeh introduced the notion of fuzzy subset [10] in 1965, the theory of fuzzy mathematics has been fast developed and can be applied to many branches in mathematical sciences, in particular, in information science.

It is clear that if we use the ordinary measures to describe the fuzzy structure, then it is not practical, because the additivity condition cannot be easily satisfied. In this aspect, Sugeno [9] firstly proposed the concept of fuzzy measure in 1974 which is a kind of non-additive measure. Later on, the fuzzy measure becomes a basic tool for multi-criteria decision making, image processing and recognition (see [2]).

The fuzzy measure and functional space were combined by Narukawa et al. [7] in 2003 to construct a Banach space, in fact, they introduced the concept $\mathscr{FM}(X, \mathscr{X}) = \{\mu : \mu$ *is the nonmonotonic fuzzy measure on a measurable space* $(X, \mathscr{X})$ *such that* $\|\mu\| = |\mu|(X) < \infty\}$, where X is a nonempty set, $\mathscr{X}$ is a σ-algebra on X, namely $\mathscr{X}$ is a family of some ordinary subsets of X and $\mathscr{X}$ has the following properties: (1) $X \in \mathscr{X}$; (2) If $A \in \mathscr{X}$, then $A^c \in \mathscr{X}$, where A^c is the complement of A relative to X; (3) If $A = \bigcup_{n=1}^{\infty} A_n$ and $A_n \in \mathscr{X}$ for $n = 1, 2, \ldots$, then $A \in \mathscr{X}$ (for example, see [3, p. 28]).

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* Corresponding author.
E-mail addresses: wucongxin@hit.edu.cn (C. Wu), renxuekun@hit.edu.cn (X. Ren).

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We note that a nonmonotonic fuzzy measure μ is a real-valued set function on $\mathscr{X}$ with $\mu(\phi)=0$ (ϕ is the empty set) and the total variation of μ on X is

$$|\mu|(X)=\sup\left\{\sum_{i=1}^{n}|\mu(A_i)-\mu(A_{i-1})|\right\},$$

where the sup is taken over all sequences $\phi=A_0\subset A_1\subset\cdots\subset A_n=X$, for $A_i\in\mathscr{X}, i=1,2,\ldots,n-1$.

In this paper, we use the following results and definitions.

(i) Cantor Theorem: The cardinal of a nonempty set X is smaller than the cardinal of the power set of X, where the power set of X is the family of all subsets of X (see [4, p. 276]).

(ii) Let (X,ρ) be a metric space. Then X is said to be separable if X contains a countable subset which is dense in X (see [6, p. 123]). Moreover the subset $A\subset X$ is dense in $X \Leftrightarrow$ for every $x\in X$ and any $\varepsilon>0$, there exists $y\in A$ such that $\rho(x,y)<\varepsilon \Leftrightarrow$ for every $\varepsilon>0$, $\bigcup_{y\in A}B(y,\varepsilon)=X$, where $B(y,\varepsilon)$ is an open ball with center y and radius ε (see [5, pp. 21–22]).

(iii) Let (X,ρ) be a metric space with $A\subset X$. Then A is said to be sequentially compact if any sequence in A has a subsequence which converges in X (see [6, pp. 123–128]).

(iv) Hausdorff Theorem: A subset M of a complete metric space (X,ρ) is sequentially compact if and only if M is totally bounded. Thus this theorem implies that any sequentially compact subspace M of X is separable (see [6, pp. 123–128]).

Our result is that, by assuming that the number of the elements of the σ-algebra $\mathscr{X}$ is finite, we can characterize the separability of the space of nonmonotonic fuzzy measures $(\mathscr{F}\mathscr{M}(X,\mathscr{X}),\|\cdot\|)$. Our result shows that the separability, of the space of nonmonotonic fuzzy measures which is related to several topics in information science is equivalent to the finiteness of the elements of the σ-algebra. For notions and terminologies, the reader is referred to Narukawa et al. [7] if it is necessary.

2. The main results

Lemma. *Let X be a nonempty set, $\mathscr{X}$ be a σ-algebra of X. Then the σ-algebra $\mathscr{X}$ is a finite set or is an uncountable set.*

Proof. We divide the proof into several steps.

(1) Let $\mathscr{X}$ be a σ-algebra having an infinite chain which is ordered by the following sense: $A\subset B$ $(A,B\in\mathscr{X})$ if and only if A is properly contained in B. Then there exists a countable subchain $\{A_i\}_{i=1}^{\infty}$ and it can be represented by

$$\phi=A_0\subset A_1\subset\cdots\subset X\quad(A_i\in\mathscr{X}, i=1,2,\ldots).\tag{2.1}$$

Now we prove that $\mathscr{X}$ is uncountable. In fact, it follows from the definition of σ-algebra that

$$B_i=A_i\backslash A_{i-1}=(A_i^c\cup A_{i-1})^c\in\mathscr{X},\quad i=1,2,\ldots.$$

Now, formula (2.1) implies that

$$B_i\neq\phi\quad\text{and}\quad B_i\cap B_j=\phi\quad(i\neq j;\ \ i,j=1,2,\ldots).$$

We can assume that any subset of $\{B_i\}_{i=1}^{\infty}$ is of the form $\{B_{i_k}\}_{k=1}^{\infty}$ and we denote its union by

$$C_{\{i_k\}}=\bigcup_{k=1}^{\infty}B_{i_k}\in\mathscr{X}.$$

If $\{B_{i_k}\}_{k=1}^{\infty}\neq\{B_{i'_k}\}_{k=1}^{\infty}$, then there exists at least one element of $\{B_{i_k}\}_{k=1}^{\infty}$, denoted by B_{i_m}, such that $B_{i_m}\notin\{B_{i'_k}\}_{k=1}^{\infty}$; or at least one element of $\{B_{i'_k}\}_{k=1}^{\infty}$, denoted by $B_{i'_m}$, such that $B_{i'_m}\notin\{B_{i_k}\}_{k=1}^{\infty}$. By the property of pairwise disjointing of elements in $\{B_i\}_{i=1}^{\infty}$, we obtain

$$\bigcup_{k=1}^{\infty} B_{i_k} \neq \bigcup_{k=1}^{\infty} B_{i'_k}.$$

That is, $C_{\{i_k\}} \neq C_{\{i'_k\}}$ since $\{i_k\} \neq \{i'_k\}$. Therefore, we have

$$\{C_{\{i_k\}} : \{i_k\} \subseteq \mathbb{N}\} \subseteq \mathscr{X},$$

where $\mathbb{N}$ is the set of all positive integers.

We denote the cardinal number of set A by card$\{A\}$ and denote the power set of A by $P(A)$.

Notice that $\{B_i\}_{i=1}^{\infty}$ is countable, i.e., $\mathrm{card}\{\{B_i\}_{i=1}^{\infty}\} = \aleph_0$. By using the Cantor Theorem, we obtain

$$\mathrm{card}\{P(\{B_i\}_{i=1}^{\infty})\} > \mathrm{card}\{\{B_i\}_{i=1}^{\infty}\} = \aleph_0.$$

Obviously, the two sets $\{C_{\{i_k\}} : \{i_k\} \subseteq \mathbb{N}\}$ and $P(\{\{B_i\}_{i=1}^{\infty}\})$ preserve the one to one correspondence, namely

$$\mathrm{card}\{\{C_{\{i_k\}} : \{i_k\} \subseteq \mathbb{N}\}\} = \mathrm{card}\{P(\{B_i\}_{i=1}^{\infty}\})\}.$$

Hence, we infer that

$$\mathrm{card}\{\mathscr{X}\} \geqslant \mathrm{card}\{\{C_{\{i_k\}} : \{i_k\} \subseteq \mathbb{N}\}\} > \aleph_0.$$

This proves that $\mathscr{X}$ is an uncountable set.

(2) If there does not exist any infinite chain which is ordered by '$\subset$' in $\mathscr{X}$, then there exists a maximal finite chain in $\mathscr{X}$, denoted by the chain

$$\phi = A_0 \subset A_1 \subset \cdots \subset A_n = X. \tag{2.2}$$

Now, by the maximality of the finite chain, we see that in the finite chain (2.2), we cannot insert any set $A \in \mathscr{X}$ between A_{i-1} and A_i for $i = 1, 2, \ldots, n$, i.e., there does not exist a set $A \in \mathscr{X}$ such that $A_{i-1} \subset A \subset A_i$. We now consider the following cases:

(a) We first denote the set of all maximal finite chains in $\mathscr{X}$ by the following form:

$$\phi = A_0^{(\tau)} \subset A_1^{(\tau)} \subset A_2^{(\tau)} \subset \cdots \subset A_{k_\tau}^{(\tau)} = X, \quad \tau \in \Omega. \tag{2.3}$$

Now we prove that the set $\{A_1^{(\tau)}\}_{\tau\in\Omega}$ in (2.3) only appears in the following cases: there exists an infinite subset $\{A_1^{(\tau_j)}\}_{j=1}^{\infty}$ of $\{A_1^{(\tau)}\}_{\tau\in\Omega}$ and $A_1^{(\tau_j)}$ $(j = 1, 2, \ldots)$ are pairwise disjoint, or there exists a finite subset $\{A_1^{(\tau'_j)}\}_{j=1}^{m}$ of $\{A_1^{(\tau)}\}_{\tau\in\Omega}$ and $A_1^{(\tau'_j)}$ $(j = 1, 2, \ldots, m)$ are pairwise disjoint, and the others in $\{A_1^{(\tau)}\}_{\tau\in\Omega}$ are coincident with some one of $\{A_1^{(\tau'_j)}\}_{j=1}^{m}$.

We have the following situations:

(i) If $A_1^{(\tau_1)} \cap A_1^{(\tau_2)} \neq \phi$, then $A_1^{(\tau_1)} = A_1^{(\tau_2)}$, for any $\tau_1, \tau_2 \in \Omega$. In fact, we suppose that $A_1^{(\tau_1)} \cap A_1^{(\tau_2)} \neq \phi$ and $A_1^{(\tau_1)} \neq A_1^{(\tau_2)}$. Then, it is obvious that

$$\phi \subset A_1^{(\tau_1)} \cap A_1^{(\tau_2)} \subset A_1^{(\tau_1)} \quad \text{or} \quad \phi \subset A_1^{(\tau_1)} \cap A_1^{(\tau_2)} \subset A_1^{(\tau_2)}.$$

This shows that at least one of the finite chains $\{A_i^{(\tau_1)}\}_{i=0}^{k_{\tau_1}}$ and $\{A_i^{(\tau_2)}\}_{i=0}^{k_{\tau_2}}$ in (2.3) is not maximal. This is clearly a contradiction.

(ii) For $\tau_i \in \Omega$, where $i = 1, 2, 3$. If $A_1^{(\tau_1)} \cap A_1^{(\tau_2)} = \phi$, then $A_1^{(\tau_3)} \cap A_1^{(\tau_1)} = \phi$ and $A_1^{(\tau_3)} \cap A_1^{(\tau_2)} = \phi$, or $A_1^{(\tau_3)} = A_1^{(\tau_1)}$, or $A_1^{(\tau_3)} = A_1^{(\tau_2)}$.

In fact, it follows from (i) that

$$A_1^{(\tau_3)} \cap A_1^{(\tau_1)} = \phi \quad \text{or} \quad A_1^{(\tau_3)} = A_1^{(\tau_1)},$$

$$A_1^{(\tau_3)} \cap A_1^{(\tau_2)} = \phi \quad \text{or} \quad A_1^{(\tau_3)} = A_1^{(\tau_2)}.$$

Since $A_1^{(\tau_1)} \cap A_1^{(\tau_2)} = \phi$, we see that $A_1^{(\tau_3)} = A_1^{(\tau_1)}$ and $A_1^{(\tau_3)} = A_1^{(\tau_2)}$ cannot hold simultaneously. Hence, only the following three cases may be true: $A_1^{(\tau_3)} \cap A_1^{(\tau_1)} = \phi$ and $A_1^{(\tau_3)} \cap A_1^{(\tau_2)} = \phi$, or $A_1^{(\tau_3)} \cap A_1^{(\tau_1)} = \phi$ and $A_1^{(\tau_3)} = A_1^{(\tau_2)}$, or $A_1^{(\tau_3)} \cap A_1^{(\tau_2)} = \phi$ and $A_1^{(\tau_3)} = A_1^{(\tau_1)}$. The proof is now completed.

(iii) Similar to (i) and (ii), we can easily see that if $A_1^{(\tau_1)}, A_1^{(\tau_2)}$ and $A_1^{(\tau_3)}$ are pairwise disjoint, then for any $\tau_4 \in \Omega$, we know that: $A_1^{(\tau_1)}, A_1^{(\tau_2)}, A_1^{(\tau_3)}$ and $A_1^{(\tau_4)}$ are pairwise disjoint; or $A_1^{(\tau_4)}$ coincides with some of the $\{A_1^{(\tau_i)}\}_{i=1}^3$. Thus, by induction, the above process has to stop after a finite number of steps, i.e., there exists a finite number of $A_1^{(\tau'_j)}$, for $j = 1, 2, \ldots, m$ which are pairwise disjoint and the others in $\{A_1^{(\tau)}\}_{\tau\in\Omega}$ coincide with some one of $\{A_1^{(\tau'_j)}\}_{i=1}^m$, or there exists an infinite number of $A_1^{(\tau_j)}$, for $j = 1, 2, \ldots$ which are pairwise disjoint in $\{A_1^{(\tau)}\}_{\tau\in\Omega}$.

(b) If there exists an infinite number of $A_1^{(\tau_j)}$, for $j = 1, 2, \ldots$ which are pairwise disjoint in $\{A_1^{(\tau)}\}_{\tau\in\Omega}$, then $\mathscr{X}$ is an uncountable set.

In fact, since $A_1^{(\tau)} \neq \phi$ for every $\tau \in \Omega$ and $A_1^{(\tau_j)} \cap A_1^{(\tau_i)} = \phi$ as $j \neq i$, we have

$$\mathrm{card}\{\{A_1^{(\tau_j)}\}_{j=1}^\infty\} = \aleph_0.$$

By using the Cantor Theorem, we obtain that

$$\mathrm{card}\{P(\{A_1^{(\tau_j)}\}_{j=1}^\infty)\} > \mathrm{card}\{\{A_1^{(\tau_j)}\}_{j=1}^\infty\} = \aleph_0.$$

Thus, by the fact that $P(\{A_1^{(\tau_j)}\}_{j=1}^\infty) \subseteq \mathscr{X}$, we infer that $\mathscr{X}$ is uncountable.

(c) If there exists a finite number of $A_1^{(\tau'_j)}$, for $j = 1, 2, \ldots, m$ which are pairwise disjoint and the others in $\{A_1^{(\tau)}\}_{\tau\in\Omega}$ coincide with some one of $\{A_1^{(\tau'_j)}\}_{i=1}^m$, then $\mathscr{X}$ is a finite set. We proceed to the proof as follows:

(A) We first prove that $\bigcup_{j=1}^m A_1^{(\tau'_j)} = X$.

For if otherwise, then we have $\bigcup_{j=1}^m A_1^{(\tau'_j)} \subset X$. Thereby, we see that the following set:

$$X \Big\backslash \left(\bigcup_{j=1}^m A_1^{(\tau'_j)}\right) \in \mathscr{X}$$

is nonempty and is disjoint with $A_1^{(\tau'_j)}$ for $j = 1, 2, \ldots, m$. Therefore, we have

$$\phi \subset X \Big\backslash \left(\bigcup_{j=1}^m A_1^{(\tau'_j)}\right) \subset X,$$

which can be extended to a different maximal finite chain in (2.3). This is a contradiction.

(B) We next show that each $A \in \mathscr{X}\backslash\{\phi, X, A_1^{(\tau'_1)}, A_1^{(\tau'_2)}, \ldots, A_1^{(\tau'_m)}\}$ can be represented by

$$A = \bigcup_{k=1}^l A_1^{(\tau'_{j_k})},$$

where $1 < l < m$, and $\mathscr{X}$ is a finite set.

In fact, because we may assume that $\phi \subset A \subset X$ is extendable to a maximal finite chain as (2.3), namely, there exists a positive integer j_1 such that $A_1^{(\tau'_{j_1})} \subset A$ where $1 \leqslant j_1 \leqslant m$. By the conclusion in (A), there exists a positive integer j_2 such that $1 \leqslant j_2 \leqslant m$ and

$$(A\backslash A_1^{(\tau'_{j_1})}) \cap A_1^{(\tau'_{j_2})} \neq \phi.$$

Thus we prove that $A_1^{(\tau'_{j_2})} \subseteq A\backslash A_1^{(\tau'_{j_1})}$. For if otherwise, then we will have

$$\phi \subset (A\backslash A_1^{(\tau'_{j_1})}) \cap A_1^{(\tau'_{j_2})} \subset A_1^{(\tau'_{j_2})}. \tag{2.4}$$

By the assumption of (a) in (2), we can extend the chain (2.4) to a maximal finite chain as the form given in (2.3). Denote the first term which is nonempty in the chain by $A_1^{(\tau_0)}$ ($\tau_0 \in \Omega$). Then,we obtain that

$$\phi \subset A_1^{(\tau_0)} \subseteq (A\backslash A_1^{(\tau'_{j_1})}) \cap A_1^{(\tau'_{j_2})} \subset A_1^{(\tau'_{j_2})}.$$

This implies that $A_1^{(\tau_0)} \subset A_1^{(\tau'_{j_2})}$. On the other hand, by the hypothesis of (2) (c), we get $A_1^{(\tau_0)}$ which coincides with some of $\{A_1^{(\tau'_j)}\}_{j=1}^m$. This is clearly a contradiction.

If $A_1^{(\tau'_{j_2})} = A\backslash A_1^{(\tau'_{j_1})}$, then we have

$$A = \bigcup_{k=1}^{2} A_1^{(\tau'_{j_k})}.$$

If $A_1^{(\tau'_{j_2})} \subset A\backslash A_1^{(\tau'_{j_1})}$, then by repeating the above discussion to the case of $\bigcup_{k=1}^2 A_1^{(\tau'_{j_k})} \subset A$, we finally obtain

$$A = \bigcup_{k=1}^{l} A_1^{(\tau'_{j_k})} \in P(\{A_1^{(\tau'_j)}\}_{j=1}^m),$$

where $1 < l < m$. Hence $\mathscr{X} \subseteq P(\{A_1^{(\tau_j)}\}_{j=1}^m)$. Since $\{A_1^{(\tau_j)}\}_{j=1}^m$ is finite, $\mathscr{X}$ is also finite.

From (1) and (2) above, we conclude that the σ-algebra $\mathscr{X}$ is a finite set or an uncountable set. □

Main Theorem. *($\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|$) is separable if and only if the σ-algebra $\mathscr{X}$ is a finite set.*

Proof. (1) (*Necessity*): This part follows directly from our Lemma. We only need to prove that: If the σ-algebra $\mathscr{X}$ is uncountable, then ($\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|$) is nonseparable.

Now, we let $\mathscr{X}$ be an uncountable set and $\mathscr{X}\backslash\{\phi\} = \{A_\tau : \tau \in \Omega\}$, where Ω is an uncountable index set. Then, we define a family of set functions on $\mathscr{X}$ by

$$\mu_\tau(A) = \begin{cases} 1 & \text{if } A = A_\tau, \\ 0 & \text{if } A \in \mathscr{X}\backslash\{A_\tau\}, \end{cases}$$

where $\tau \in \Omega$. It is obvious that $\{\mu_\tau : \tau \in \Omega\}$ is an uncountable set. Hence, we have

$$\|\mu_\tau\| = \sup\left\{\sum_{i=1}^{k} |\mu_\tau(A_i) - \mu_\tau(A_{i-1})|\right\} \leqslant 2,$$

where $\tau \in \Omega$ and the sup is taken over all sequences, $\phi = A_0 \subset A_1 \subset \cdots \subset A_k = X$ for $A_i \in \mathscr{X}$, $i = 1, 2, \ldots, k-1$, and so we obtain

$$\{\mu_\tau : \tau \in \Omega\} \subseteq \mathscr{F}\mathscr{M}(X, \mathscr{X}).$$

It is easy to see that for every $A_\tau, A_\gamma \in \mathscr{X}$ as $\tau \neq \gamma$, we have

$$(\mu_\tau - \mu_\gamma)(A) = \begin{cases} 1 & \text{if } A = A_\tau, \\ -1 & \text{if } A = A_\gamma, \\ 0 & \text{if } A \in \mathscr{X}\backslash\{A_\tau, A_\gamma\}, \end{cases}$$

and hence, we have

$$\|\mu_\tau - \mu_\gamma\| \geqslant 2, \tag{2.5}$$

where $\tau, \gamma \in \Omega$ and $\tau \neq \gamma$.

Suppose that $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ is separable. Then there exists a countable dense subset $D = \{\mu^{(n)}\}$. This leads to

$$\bigcup_{n=1}^{\infty} B(\mu^{(n)}, 1) = \mathscr{F}\mathscr{M}(X, \mathscr{X}) \supseteq \{\mu_\tau : \tau \in \Omega\},$$

where

$$B(\mu^{(n)}, 1) = \{\mu \in \mathscr{F}\mathscr{M}(X, \mathscr{X}) : \|\mu - \mu^{(n)}\| < 1\}.$$

Therefore, there exists a positive integer n_0 such that $B(\mu^{(n_0)}, 1)$ which contains at least two elements $\mu_{\tau_1}, \mu_{\tau_2}$ of $\{\mu_\tau : \tau \in \Omega\}$ for $\tau_1, \tau_2 \in \Omega$ and $\tau_1 \neq \tau_2$. Therefore, we further infer that

$$\|\mu_{\tau_1} - \mu_{\tau_2}\| \leqslant \|\mu_{\tau_1} - \mu^{(n_0)}\| + \|\mu^{(n_0)} - \mu_{\tau_2}\| < 1 + 1 = 2.$$

This contradicts to (2.5). That is, $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ is nonseparable.

(2) (*Sufficiency*): Let $\mathscr{X} = \{\phi, A_1, A_2, \ldots, A_m = X\}$. Define a family of set functions on $\mathscr{X}$ by

$$\mu_{r_1, r_2, \ldots, r_m}(A) = \begin{cases} 0 & \text{if } A = \phi, \\ r_k & \text{if } A = A_k, \quad k = 1, 2, \ldots, m, \end{cases}$$

where $r_k \in \mathbb{Q}$ for $k = 1, 2, \ldots, m$, and $\mathbb{Q}$ is the set of all rational numbers. Then we have

$$M = \{\mu_{r_1, r_2, \ldots, r_m} : \ r_k \in \mathbb{Q}, k = 1, 2, \ldots, m\} \subseteq \mathscr{F}\mathscr{M}(X, \mathscr{X}),$$

and M is countable. Now we prove that M is a dense set of $\mathscr{F}\mathscr{M}(X, \mathscr{X})$, i.e., $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ is separable.

In fact, any $\mu \in \mathscr{F}\mathscr{M}(X, \mathscr{X})$ can be represented by

$$\mu(A) = \begin{cases} 0 & \text{if } A = \phi, \\ a_k & \text{if } A = A_k, \quad k = 1, 2, \ldots, m, \end{cases}$$

where $a_k \in \mathbb{R}^1$ for $k = 1, 2, \ldots, m$. For every $\varepsilon > 0$, we can find $r_k^{(0)} \in \mathbb{Q}$ such that

$$\max_{1 \leqslant k \leqslant m} |r_k^{(0)} - a_k| < \varepsilon/(2m - 1).$$

The following shows that $\mu_{r_1^{(0)}, r_2^{(0)}, \ldots, r_m^{(0)}} \in M$ satisfies

$$\begin{aligned} \|\mu - \mu_{r_1^{(0)}, r_2^{(0)}, \ldots, r_m^{(0)}}\| = \ & \sup \left\{ \sum_{i=1}^{l} |(\mu(A_{k_i}) - \mu_{r_1^{(0)}, r_2^{(0)}, \ldots, r_m^{(0)}}(A_{k_i})) \right. \\ & \left. - (\mu(A_{k_{i-1}}) - \mu_{r_1^{(0)}, r_2^{(0)}, \ldots, r_m^{(0)}}(A_{k_{i-1}}))| \right\} \\ \leqslant \ & \sup \left\{ \sum_{i=1}^{l} (|a_{k_i} - r_{k_i}^{(0)}| + |a_{k_{i-1}} - r_{k_{i-1}}^{(0)}|) \right\} \\ = \ & \sup \left\{ \sum_{i=1}^{l} |a_{k_i} - r_{k_i}^{(0)}| + \sum_{i=1}^{l} |a_{k_{i-1}} - r_{k_{i-1}}^{(0)}| \right\} \\ \leqslant \ & (2l - 1) \max_{1 \leqslant i \leqslant l} |a_{k_i} - r_{k_i}^{(0)}| < \varepsilon, \end{aligned}$$

where $\phi = A_0 = A_{k_0} \subset A_{k_1} \subset A_{k_2} \subset \cdots \subset A_{k_l} = A_m = X$ for $1 \leqslant l \leqslant m$, and $a_{k_0} = r_{k_0}^{(0)} = 0$. This shows that M is dense in $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ and our proof is complete. □

Corollary. *Any subset which is bounded with respect to the norm in the space* $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ *is sequentially compact if and only if the σ-algebra* $\mathscr{X}$ *is a finite set*.

Proof. (1) (*Necessity*). Define a sequence of subsets in $\mathscr{F}\mathscr{M}(X, \mathscr{X})$ by

$$M_i = \{\mu \in \mathscr{F}\mathscr{M}(X, \mathscr{X}) : \|\mu\| \leqslant i\},$$

where $i = 1, 2, \ldots$. Obviously, M_i is bounded with respect to the norm in space $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$. By assumption, M_i is sequentially compact. Hence, by Hausdorff Theorem, M_i is separable, and so, there exists countable sets $A_i \subset M_i$ for $i = 1, 2, \ldots$ which is dense in M_i. Let $A = \bigcup_{i=1}^{\infty} A_i$. Then A is countable and we can easily see that A is dense in $\mathscr{F}\mathscr{M}(X, \mathscr{X})$ and $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ is separable. Thus, by our theorem, we know that $\mathscr{X}$ is a finite set.

(2) (*Sufficiency*): Let M be a bounded subset with respect to the norm in $(\mathscr{F}\mathscr{M}(X, \mathscr{X}), \|\cdot\|)$ and $\{\mu_n\}$ be a sequence in M. Then there exists a constant $K > 0$ such that $\|\mu_n\| \leqslant K$ for $n = 1, 2, \ldots$. The definition of the norm $\|\cdot\|$ shows that for any $n \in \mathbb{N}$ and $A \in \mathscr{X} \backslash \{\phi\}$, we have

$$K \geqslant \|\mu_n\| \geqslant |\mu_n(A) - \mu_n(\phi)| + |\mu_n(X) - \mu_n(A)|$$

$$= |\mu_n(A)| + |\mu_n(X) - \mu_n(A)|,$$

and thereby, $|\mu_n(A)| \leqslant K$ $(n = 1, 2, \ldots, A \in \mathscr{X})$. Now, by our assumption, $\mathscr{X}$ is finite. Thus, we obtain a subsequence $\{\mu^{(m)}\} \subset \{\mu_n\}$ such that $\{\mu^{(m)}(A)\}$ is convergent for every $A \in \mathscr{X}$. Define $\mu(A) = \lim_{m \to \infty} \mu^{(m)}(A)$ for each $A \in \mathscr{X}$. Obviously $\mu \in \mathscr{F}\mathscr{M}(X, \mathscr{X})$.

Clearly, there exists a real number $r > 0$ such that $r > \text{card}\{P(\mathscr{X})\}$. Moreover, for any $A \in \mathscr{X}$ and $\varepsilon > 0$, there exists a positive integer m_0 such that

$$|\mu^{(m)}(A) - \mu(A)| < \varepsilon/2r \quad (m \geqslant m_0).$$

Thus it is easy to explain that

$$\|\mu^{(m)} - \mu\| \leqslant \varepsilon \quad (m \geqslant m_0).$$

Hence, $\mu^{(m)} \to \mu$. This proves that M is sequentially compact. □

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