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Ramanujan's "Lost" Notebook
IV. Stacks and Alternating Parity in Partitions

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1. INTRODUCTION

This paper is centered around the partition function $OE(n)$, the number of partitions of n in which the parts (arranged in ascending order) alternate in parity starting with the smallest part odd. Briefly, $OE(n)$ counts the number of "odd-even" partitions of n. The first few examples for the computation of $OE(n)$ are

n	Relevant partitions of n	$OE(n)$
1	1	1
2	—	0
3	3, 1 + 2	2
4	—	0
5	5, 1 + 4	2
6	1 + 2 + 3	1
7	7, 1 + 6, 3 + 4	3

The generating function for $OE(n)$ turns out to be

$$1+\sum_{n=1}^{\infty} OE(n)\,q^n = 1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})},\tag{1.1}$$

a function of surprisingly familiar form. Indeed we should place (1.1) in the context of the identities

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)},$$
$$[3, \text{p. } 19],\tag{1.2}$$

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$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}(1+q^n),$$

[3, p. 19], (1.3)

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

[3, p. 104], (1.4)

$$1+\sum_{n=1}^{\infty}\frac{q^{n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{2n+1})},$$

[3, p. 5, Eq. (1.2.5) and p. 19], (1.5)

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=?, \tag{1.6}$$

$$1+\prod_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=\prod_{n=0}^{\infty}(1+q^{2n+1}),$$

[3, p. 19]. (1.7)

It is natural to ask for significant information concerning (1.6); however, there appears to be no mention of (1.6) in the literature. In Ramanujan's "Lost" Notebook (see [4] for the historical background of this document), there are several formulas involving (1.6), and while none is as simple as the five contrasting preceding identities, they nonetheless provide substantial combinational information on $OE(n)$ as we shall see in Section 4. The formulas of Ramanujan in question are

$$\begin{aligned}\prod_{n=1}^{\infty}(1-(-q)^n)&\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(1-q^2)(1-q^4)\cdots(1-q^{2m})}\\&=\sum_{n=0}^{\infty}\frac{q^{2n^2+n}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}\\&\quad+2\sum_{n=1}^{\infty}q^{n^2}(1+q^2)(1+q^4)\cdots(1+q^{2n-2}),\end{aligned} \tag*{$(1.8)_{\mathrm{R}}$}$$

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{4n})&\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(1+q^2)(1+q^4)\cdots(1+q^{2m})}\\&=\sum_{n=1}^{\infty}\frac{q^{2n^2-n}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})}\\&\quad+\sum_{n=0}^{\infty}q^{n^2+n}(1+q)(1+q^3)\cdots(1+q^{2n-1}),\end{aligned} \tag*{$(1.9)_{\mathrm{R}}$}$$

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{(1+q^n)} \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(1-q^2)(1-q^4)\cdots(1-q^{2m})} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} = 2\sum_{n=0}^{\infty} \frac{(-q)^{n(n+1)/2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}, \tag{1.10$_{\mathrm{R}}$}$$

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{(1+q^n)} \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(1-q^2)(1-q^4)\cdots(1-q^{2m})} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} = -4\sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n^2}}{(1+q)(1+q^3)\cdots(1+q^{4n-1})}. \tag{1.11$_{\mathrm{R}}$}$$

The subscript "R" designates that a formula is actually in Ramanujan's "Lost" Notebook. Actually one finds there "(1.10) + (1.11)" and "(1.10) − (1.11)" rather than the equivalent formulas given above.

There have been over the years numerous applications of partitions (in the additive number theory sense) and partition identities to statistical mechanics [6, 7, 12, 13]. More recently Baxter [9] has found an absolutely astounding application of the Rogers–Ramanujan identities in his solution of the hard-hexagon model. The earlier applications by Temperly, Auluck, *et al.* have led to purely combinatorial studies by Wright [15, 16] of objects called "stacks." Natural interpretations of Ramanujan's identities lead to results involving refinements of stacks which we shall call "stacks with summits." The basic facts about these new constructs will be developed in Section 3. Section 4 will consider the relationship of stacks with summits to Ramanujan's identities.

2. The Four Identities

There are two very different techniques required to handle these identities. Identities (1.8) and (1.9) are the least trouble and follow from methods developed to handle bilateral series [1]. Identities (1.10) and (1.11) require several devices including transformations of truncated series (see Lemmas 1 and 2 below) and results for very well-poised basic hypergeometric series including the q-analog of Whipple's theorem and its extensions [2, 8].

In the following, we shall utilize the standard notation related to basic hypergeometric functions [10, p. 89]:

$$(a)_\infty = (a; q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m), \tag{2.1}$$

$$\begin{aligned}(a)_n = (a; q)_n &= (a; q)_\infty/(aq^n; q)_\infty \\ &= (1-a)(1-aq) \cdots (1-aq^{n-1})\end{aligned}$$

wherever n is a nonnegative integer. (2.2)

The q-hypergeometric or basic hypergeometric series is defined by

$$\begin{aligned}&{}_r\phi_s\begin{pmatrix} a_1, a_2, \dots, a_r; q, t \\ b_1, b_2, \dots, b_s \end{pmatrix} \\ &\quad = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n t^n}{(q)_n (b_1)_n (b_2)_n \cdots (b_S)_n}.\end{aligned} \tag{2.3}$$

We shall say that this series is *well-poised* if $r = s + 1$, and

$$a_1 q = a_2 b_1 = a_3 b_2 = \cdots = a_r b_s. \tag{2.4}$$

We shall say it is *very well-poised* if in addition

$$b_1 = -b_2 = \sqrt{a_1}.$$

These seemingly artificial constraints have turned out to be extremely interesting in previous work, [10, pp. 100–106] and they turn up again here.

Proof of (1.8). Using the above definitions we see that the right-hand side of (1.8) may be written as

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q^2; q^2)_n} + 2 \sum_{n=1}^{\infty} q^{n^2}(-q^2; q^2)_{n-1} \\ &= \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}(-q^{2n+2}; q^2)_\infty}{(-q^2; q^2)_\infty} \\ &= \frac{1}{(-q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \sum_{m=0}^{\infty} \frac{q^{m^2+2nm+m}}{(q^2; q^2)_m}\end{aligned}$$

(by one of Euler's summations [3, p. 19])

$$\begin{aligned}&= \frac{1}{(-q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \left(\sum_{m=0}^{\infty} \frac{q^{4m^2+2m+4nm}}{(q^2; q^2)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{4m^2+6m+2+4nm+2n}}{(q^2; q^2)_{2m+1}} \right) \\ &= \frac{1}{(-q^2; q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{2m^2+m}}{(q^2; q^2)_{2m}} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \\ &\quad + \frac{1}{(-q^2; q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{2m^2+3m+2}}{(q^2; q^2)_{2m+1}} \sum_{n=-\infty}^{\infty} q^{2n^2+3n}\end{aligned}$$

(where after interchange of summation n has been replaced by $n - m$)

$$= \frac{(q^4;q^4)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty}{(-q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+m}}{(q^2;q^2)_{2m}}$$

$$+\frac{(q^4;q^4)_\infty(-q^{-1};q^4)_\infty(-q^5;q^4)_\infty}{(-q^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+2}}{(q^2;q^2)_{2m+1}}$$

(by Jacobi's Triple product identity [3, p. 21])

$$=(q^2;q^2)_\infty(-q;q^2)_\infty\left(\sum_{m=0}^{\infty}\frac{q^{(2m+1)(2m)/2}}{(q^2;q^2)_{2m}}+\sum_{m=0}^{\infty}\frac{q^{(2m+2)(2m+1)/2}}{(q^2;q^2)_{2m+1}}\right)$$

$$=(-q;-q)_\infty\sum_{m=0}^{\infty}\frac{q^{(m+1)m/2}}{(q^2;q^2)_m},$$

and thus (1.8) is established.

Proof of (1.9). In exactly the same way we see that the right-hand side of (1.9) is

$$\sum_{n=1}^{\infty}\frac{q^{2n^2-n}}{(-q;q^2)_n}+\sum_{n=0}^{\infty}q^{n^2+n}(-q;q^2)_n$$

$$=\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+3n+1}}{(-q;q^2)_{n+1}}$$

$$=\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+3n+1}}{(-q;q^2)_\infty}(-q^{2n+3};q^2)_\infty$$

$$=\frac{1}{(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}q^{2n^2+3n+1}\sum_{m=0}^{\infty}\frac{q^{m^2-m+(2n+3)m}}{(q^2;q^2)_m}$$

(by one of Euler's summation [3, p. 19])

$$=\frac{1}{(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}q^{2n^2+3n+1}\left(\sum_{m=0}^{\infty}\frac{q^{4m^2+4m+4nm}}{(q^2;q^2)_{2m}}\right.$$

$$\left.+\sum_{m=0}^{\infty}\frac{q^{4m^2+8m+3+4nm+2n}}{(q^2;q^2)_{2m+1}}\right)$$

$$=\frac{1}{(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+m+1}}{(q^2;q^2)_{2m}}\sum_{n=-\infty}^{\infty}q^{2n^2+3n}$$

$$+\frac{1}{(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+4}}{(q^2;q^2)_{2m+1}}\sum_{n=-\infty}^{\infty}q^{2n^2+5n}$$

(where after interchange of summation n has been replaced by $n-m$)

$$=\frac{1}{(-q;q^2)_\infty}(q^4;q^4)_\infty(-q^{-1};q^4)_\infty(-q^5;q^4)_\infty\sum_{m=0}^{\infty}\frac{q^{2m^2+m+1}}{(q^2;q^2)_{2m}}$$
$$+\frac{1}{(-q;q^2)_\infty}(q^4;q^4)_\infty(-q^7;q^4)_\infty(-q^{-3};q^4)_\infty\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+4}}{(q^2;q^2)_{2m+1}}$$

(by Jacobi's triple product identity [3, p. 21])

$$=(q^4;q^4)_\infty\left(\sum_{m=0}^{\infty}\frac{q^{2m^2+m}}{(q^2;q^2)_{2m}}+\sum_{m=0}^{\infty}\frac{q^{2m^2+3m+1}}{(q^2;q^2)_{2m+1}}\right)$$
$$=(q^4;q^4)_\infty\sum_{m=0}^{\infty}\frac{q^{(m+1)m/2}}{(q^2;q^2)_m},$$

and thus (1.9) is established.

Identities (1.10) and (1.11) require (among other things) the following propositions:

LEMMA 1. *For each nonnegative integer m,*

$$(q;q^2)_m\sum_{n=0}^{2m}\frac{(-1)^n}{(q)_n(\alpha)_{2m-n}}=\sum_{n=0}^{m}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_n}. \tag{2.5}$$

Remark. The case $\alpha=0$ appears as identity (4.3) in [5]. The case $\alpha=q$ may be seen to reduce easily to a famous formula of Gauss for the Gaussian polynomials.

Proof. Let us call the left-hand side of (2.5) $L_m(\alpha)$ and denote the right-hand side by $R_m(\alpha)$. Mathematical induction proves our lemma once we show that both of these functions satisfy

$$f_0(\alpha)=1, \tag{2.6}$$

$$(1-\alpha)f_{m+1}(\alpha)-\frac{(1-q^{2m+1})}{(1-\alpha q)}f_m(\alpha q^2)=\frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}}. \tag{2.7}$$

Both functions clearly satisfy (2.6). Let us turn to $L_m(\alpha)$ for (2.7):

$$L_{m+1}(\alpha)-\frac{(1-q^{2m+1})}{(1-\alpha)(1-\alpha q)}L_m(\alpha q^2)$$
$$=(q;q^2)_{m+1}\left(\sum_{n=0}^{2m+2}\frac{(-1)^n}{(\alpha)_n(q)_{2m-n+2}}-\sum_{n=0}^{2m}\frac{(-1)^n}{(\alpha)_{n+2}(q)_{2m-n}}\right)$$
$$=(q;q^2)_{m+1}\left(\sum_{n=-2}^{2m}\frac{(-1)^n}{(\alpha)_{n+2}(q)_{2m-n}}-\sum_{n=0}^{2m}\frac{(-1)^n}{(\alpha)_{n+2}(q)_{2m-n}}\right)$$

$$= (q;q^2)_{m+1}\left(\frac{1}{(q)_{2m+2}} - \frac{1}{(1-\alpha)(q)_{2m+1}}\right)$$

$$= \frac{(q;q^2)_{m+1}}{(q)_{2m+2}(1-\alpha)}((1-\alpha)-(1-q^{2m+2}))$$

$$= \frac{q^{2m+2}-\alpha}{(1-\alpha)(q^2;q^2)_{m+1}},$$

and this result is clearly equivalent to (2.7) for $L_m(\alpha)$.

As for $R_m(\alpha)$ in (2.7), we see that

$$(1-\alpha)R_{m+1}(\alpha) - \frac{R_m(\alpha q^2)}{(1-\alpha q)}$$

$$= \sum_{n=0}^{m+1} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_n}\left(1-\alpha-\frac{(1-q^{2m+2-2n})}{(1-\alpha q^{2n+1})}\right)$$

$$= \sum_{n=0}^{m+1} \frac{(-1)^n q^{n^2}(q^{2m+2-2n}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}$$

$$= \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}} - \frac{\alpha q}{(q^2;q^2)_m(1-\alpha q)}$$
$$+ \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2}(q^{2m+2-2n}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}$$

$$= \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}} - \frac{\alpha q}{(q^2;q^2)_m(1-\alpha q)}$$
$$+ \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2}(q^{2m+2-2n}(1-\alpha q^{2n+1}))}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}$$
$$+ \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2}(\alpha q^{2m+3}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}$$

$$= \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}} - \frac{q^{2m+1}R_m(\alpha q^2)}{(1-\alpha q)}$$
$$+ \left\{-\frac{\alpha q}{(q^2;q^2)_m(1-\alpha q)}\right.$$
$$\left.+ \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2}(\alpha q^{2m+3}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}\right\}.$$

Thus to show that $R_m(\alpha)$ satisfies (2.7) it suffices to show that the expression inside the curly brackets is identically zero. This follows easily since

$$-\frac{\alpha q}{(q^2;q^2)_m\,(1-\alpha q)}+\sum_{n=1}^{m+1}\frac{(-1)^n\,q^{n^2}(\alpha q^{2m+3}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}\,(\alpha q;q^2)_{n+1}}$$

$$=-\frac{\alpha q}{(q^2;q^2)_m\,(1-\alpha q)}$$

$$+\sum_{n=1}^{m+1}\frac{(-1)^n\,q^{n^2}(-\alpha q^{2n+1}(1-q^{2m-2n+2})-\alpha(1-\alpha q^{2n+1}))}{(q^2;q^2)_{m+1-n}\,(\alpha q;q^2)_{n+1}}$$

$$=-\alpha\sum_{n=0}^{m}\frac{(-1)^n\,q^{(n+1)^2}}{(q^2;q^2)_{m-n}\,(\alpha q;q^2)_{n+1}}-\alpha\sum_{n=1}^{m+1}\frac{(-1)^n\,q^{n^2}}{(q^2;q^2)_{m+1-n}\,(\alpha q;q^2)_n}$$

$$=-\alpha\sum_{n=0}^{m}\frac{(-1)^n\,q^{(n+1)^2}}{(q^2;q^2)_{m-n}\,(\alpha q;q^2)_{n+1}}+\alpha\sum_{n=0}^{m}\frac{(-1)^n\,q^{(n+1)^2}}{(q^2;q^2)_{m-n}\,(\alpha q;q^2)_{n+1}}$$

$$=0.$$

Hence Lemma 1 is valid. ∎

LEMMA 2. *For each positive integer m,*

$$(q;q^2)_m\sum_{n=0}^{2m-1}\frac{(-1)^n}{(q)_n\,(\alpha)_{2m-1-n}}=\left(1-\frac{\alpha}{q}\right)\sum_{n=1}^{m}\frac{(-1)^n\,q^{n^2}}{(q^2;q^2)_{m-n}\,(\alpha;q^2)_n}. \tag{2.8}$$

Proof. Utilizing the notation of Lemma 1, we see that

$$\left(1-\frac{\alpha}{q}\right)\sum_{n=0}^{2m-1}\frac{(-1)^n}{(q)_n\,(\alpha/q)_{2m-n}}$$

$$=\left(1-\frac{\alpha}{q}\right)\left(\frac{L_m(\alpha/q)}{(q;q^2)_m}-\frac{1}{(q)_{2m}}\right)$$

$$=\left(1-\frac{\alpha}{q}\right)\left(\frac{R_m(\alpha/q)}{(q;q^2)_m}-\frac{1}{(q)_{2m}}\right)$$

$$=\frac{(1-\alpha/q)}{(q;q^2)_m}\sum_{n=1}^{m}\frac{(-1)^n\,q^{n^2}}{(q^2;q^2)_{m-n}\,(\alpha;q^2)_n},$$

as desired. ∎

The proofs of identities (1.10) and (1.11) require (besides Lemma 1) three identities for the very well-poised ${}_{10}\phi_9$. The first two were given by Bailey in his seminal paper on the "Bailey transform" [8, Eq. (6.1) and Eq. (6.3)]:

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, r_1, -r_1, r_2, -r_2, q^{-N}, -q^N; q, \dfrac{-a^3q^{3+2N}}{br_1^2r_2^2} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \dfrac{aq}{r_1}, -\dfrac{aq}{r_1}, \dfrac{aq}{r_2}, -\dfrac{aq}{r_2}, aq^{N+1}, -aq^{N+1} \end{matrix}\right)$$

$$= \frac{(a^2q^2;q^2)_\infty\,(a^2q^2/r_1^2r_2^2;q^2)_\infty}{(a^2q^2/r_1^2;q^2)_\infty\,(a^2q^2/r_2^2;q^2)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(r_1^2;q^2)_n\,(r_2^2;q^2)_n\,(-aq/b)_{2n}}{(q^2;q^2)_n\,(a^2q^2/b^2;q^2)_n\,(-aq)_{2n}} \left(\frac{a^2q^2}{r_1^2r_2^2}\right)^n. \tag{2.9}$$

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, \sqrt{a}q^2, -\sqrt{a}q^2, p_1, p_1q, p_2, p_2q, f, q^{-2N}, q^{-2N+1}; q^2, \dfrac{a^3q^{4N+3}}{p_1^2p_2^2f} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq^2}{p_1}, \dfrac{aq}{p_1}, \dfrac{aq^2}{p_2}, \dfrac{aq}{p_2}, \dfrac{aq^2}{f}, aq^{2N+2}, aq^{2N+1} \end{matrix}\right)$$

$$= \frac{(aq)_\infty\,(aq/p_1p_2)_\infty}{(aq/p_1)_\infty\,(aq/p_2)_\infty} \sum_{n=0}^{\infty} \frac{(p_1)_n\,(p_2)_n\,(aq/f;q^2)_n}{(q)_n\,(aq;q^2)_n\,(aq/f)_n} (aq/p_1p_2)^n \tag{2.10}$$

The third result is the special case $k=3$ of a generalization of Watson's q-analog of Whipple's theorem [10, p. 100] to the very well-poised ${}_{2k+4}\phi_{2k+3}$ [2, Theorem 4]:

$${}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, b_3, c_3, q^{-N}; q, \dfrac{a^3q^{3+N}}{b_1b_2b_3c_1c_2c_3} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b_1}, \dfrac{aq}{c_1}, \dfrac{aq}{b_2}, \dfrac{aq}{c_2}, \dfrac{aq}{b_3}, \dfrac{aq}{c_3}, aq^{N+1} \end{matrix}\right)$$

$$= \frac{(aq)_N\,(aq/b_3c_3)_N}{(aq/b_3)_N\,(aq/c_3)_N} \sum_{m_1,m_2\geqslant 0} \frac{(aq/b_1c_1)_{m_1}\,(aq/b_2c_2)_{m_2}\,(b_2)_{m_1}\,(c_2)_{m_1}}{(q)_{m_1}\,(q)_{m_2}\,(aq/b_1)_{m_1}\,(aq/c_1)_{m_1}}$$

$$\times \frac{(b_3)_{m_1+m_2}\,(c_3)_{m_1+m_2}\,(q^{-N})_{m_1+m_2}\,(aq)^{m_1}\,q^{m_1+m_2}}{(aq/b_2)_{m_1+m_2}\,(aq/c_2)_{m_1+m_2}\,(b_3c_3q^{-N}/a)_{m_1+m_2}\,(b_2c_2)^{m_1}}. \tag{2.11}$$

With these exceedingly unlikely tools in hand, we are prepared for the next pair of identities:

Proof of (1.10). We may write the left-hand side of (1.10) as

$$\frac{1}{(-q)_\infty}\left\{\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(-q)_m}(q^{m+1})_\infty+\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}(-q^{m+1})_\infty}{(-q)_m}\right\}$$

$$=\frac{1}{(-q)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{m(m+1)/2+n(n+1)/2+mn}((-1)^n+(-1)^m)}{(-q)_m\,(q)_n}\qquad\text{(by [3, p. 19])}$$

$$=\frac{1}{(-q)_\infty}\sum_{N=0}^{\infty}q^{N(N+1)/2}\sum_{n=0}^{N}\frac{((-1)^n+(-1)^{N-n})}{(-q)_{N-n}\,(q)_n}$$

$$=\frac{2}{(-q)_\infty}\sum_{N=0}^{\infty}q^{N(2N+1)}\sum_{n=0}^{2N}\frac{(-1)^n}{(-q)_{2N-n}\,(q)_n}$$

$$=\frac{2}{(-q)_\infty}\sum_{N=0}^{\infty}\frac{q^{N(2N+1)}}{(q;q^2)_N}\sum_{j=0}^{N}\frac{(-1)^j q^{j^2}}{(q^2;q^2)_{N-j}\,(-q^2;q^2)_j}\qquad\text{(by Lemma 1)}$$

$$=\frac{2}{(-q)_\infty}\sum_{j=0}^{\infty}\sum_{N=0}^{\infty}\frac{q^{(N+j)(2N+2j+1)}(-1)^j q^{j^2}}{(-q^2;q^2)_j\,(q;q^2)_{N+j}\,(q^2;q^2)_N}$$

$$=\frac{2}{(-q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{3j^2+j}}{(-q^2;q^2)_j\,(q;q^2)_j}\sum_{N=0}^{\infty}\frac{q^{2N^2+N+4Nj}}{(q^2;q^2)_N\,(q^{2j+1};q^2)_N}$$

$$=\frac{2}{(-q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{3j^2+j}}{(-q^2;q^2)_j\,(q;q^2)_j}\,\frac{1}{(q^{2j+1};q^2)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2jn}(q^{2j+2};q^2)_n}{(q^2;q^2)_n}\qquad\text{(by [3, last equation on p. 38])}$$

$$=2\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n+j}q^{(n+j)^2+2j^2+j}(q^2;q^2)_{n+j}}{(q^2;q^2)_j\,(q^2;q^2)_n\,(-q^2;q^2)_j}$$

$$=(1-q)\lim_{c_1,b_2,c_2,b_3,N\to\infty}$$

$${}_{10}\phi_9\left(\begin{matrix}q,q^{5/2},-q^{5/2},-q,c_1,b_2,c_2,b_3,q^2,q^{-2N};q^2,\dfrac{-q^{2N+6}}{c_1b_2c_2b_3}\\[2ex] q^{1/2},-q^{1/2},-q^2,\dfrac{q^3}{c_1},\dfrac{q^3}{b_2},\dfrac{q^3}{c_2},\dfrac{q^3}{b_3},q,q^{3+2N}\end{matrix}\right)$$

(by (2.11))

$$=(1-q)\lim_{f,r_2,N\to\infty}$$

$$
{}_{10}\phi_9\left(\begin{matrix} q, q^{5/2}, -q, q^2, f, r_2, -r_2 q, q^{-2N}, -q^{-2N+1}; q^2, \dfrac{q^{4N+4}}{fr_2^2} \\ q^{1/2}, -q^{1/2}, -q^2, q, \dfrac{q^3}{f}, \dfrac{q^3}{r_2}, -\dfrac{q^2}{r_2}, q^{3+2N}, -q^{2+2N} \end{matrix}\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n} \qquad \text{(by (2.10) with a replaced by } -q).
$$

Thus (1.10) is established.

We now proceed to the final member of this quartet.

Proof of (1.11). We may write the left-hand side of (1.11) as

$$
\frac{1}{(-q)_\infty}\left\{\sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(-q)_m}(q^{m+1})_\infty - \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2}(-q^{m+1})_\infty}{(-q)_m}\right\}
$$

$$
= \frac{1}{(-q)_\infty} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{m(m+1)/2+n(n+1)/2+mn}((-1)^n-(-1)^m)}{(-q)_m (q)_n} \qquad \text{(by [3, p. 19])}
$$

$$
= \frac{1}{(-q)_\infty} \sum_{N=0}^{\infty} q^{N(N+1)/2} \sum_{n=0}^{N} \frac{((-1)^n-(-1)^{N-n})}{(-q)_{N-n}(q)_n}
$$

$$
= \frac{2}{(-q)_\infty} \sum_{N=0}^{\infty} q^{(N+1)(2N+1)} \sum_{n=0}^{2N+1} \frac{(-1)^n}{(-q)_{2N+1-n}(q)_n}
$$

$$
= \frac{4}{(-q)_\infty} \sum_{N=0}^{\infty} \frac{q^{(N+1)(2N+1)}}{(q;q^2)_{N+1}} \sum_{n=0}^{N} \frac{(-1)^{n+1} q^{(n+1)^2}}{(q^2;q^2)_{N-n}(-q;q^2)_{n+1}} \qquad \text{(by Lemma 2)}
$$

$$
= \frac{4}{(-q)_\infty} \sum_{n=0}^{\infty}\sum_{N=0}^{\infty} \frac{q^{(N+n+1)(2N+2n+1)}(-1)^{n+1} q^{(n+1)^2}}{(q^2;q^2)_N (q;q^2)_{N+n+1}(-q;q^2)_{n+1}}
$$

$$
= \frac{4}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{3n^2+5n+2}}{(q;q^2)_{n+1}(-q;q^2)_{n+1}} \sum_{N=0}^{\infty} \frac{q^{2N^2+3N+4Nn}}{(q^2;q^2)_N (q^{2n+3};q^2)_N}
$$

$$
= \frac{4}{(-q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{3n^2+5n+2}}{(q;q^2)_{n+1}(-q;q^2)_{n+1}} \frac{1}{(q^{2n+3};q^2)_\infty}
$$

$$
\times \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+2m+2mn}(q^{2n+2};q^2)_m}{(q^2;q^2)_m}
$$

$$
= 4 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n+m+1} q^{(m+n)^2+2n^2+3n+2+2(m+n)}(q^2;q^2)_{m+n}}{(-q;q^2)_{n+1}(q^2;q^2)_n (q^2;q^2)_m}
$$

$$
= \frac{-4q^2}{(1+q)}(1-q^3) \lim_{c_1,b_2,c_2,b_3,N\to\infty}
$$

$$\times {}_{10}\phi_9 \left(\begin{matrix} q^3, q^{7/2}, -q^{7/2}, -q^2, c_1, b_2, c_2, b_3, q^2, q^{-2N}; q^2, \dfrac{-q^{2N+8}}{c_1 b_2 c_2 b_3} \\ q^{3/2}, -q^{3/2}, -q^3, \dfrac{q^5}{c_1}, \dfrac{q^5}{b_2}, \dfrac{q^5}{c_2}, \dfrac{q^5}{b_3}, q^3, q^{2N+5} \end{matrix} \right)$$

$$= \frac{-4q^2(1-q^3)}{(1+q)} \frac{(q^{10}; q^4)_\infty}{(q^6; q^4)_\infty} \sum_{n=0}^{\infty} \frac{(q^4; q^4)_n (-1)^n q^{2n^2+4n}}{(q^4; q^4)_n (-q^5; q^2)_{2n}}$$

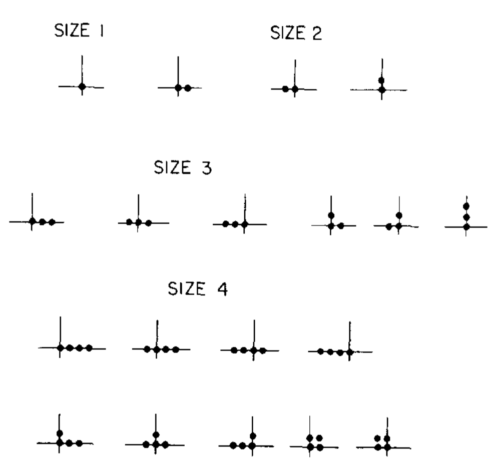
(by (2.9) with q replaced by q^2, then $a = q^3, r_1 = q^2, b, r_2, N \to \infty$)

$$= -4 \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+4n+2}}{(-q; q^2)_{2n+2}}$$

$$= 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n^2}}{(-q; q^2)_{2n}}.$$

3. Stacks with Summits

A *stack with summit* is a subset S of the set I of integer points in $\mathbb{R}^2$ with nonnegative second coordinate such that: (1) all the elements of I lying on a vertical or horizontal segment connecting two elements of S are also in S; (2) if $(x, y) \in S$ and $(x, z) \in I$ then $(x, z) \in S$ for each such point with $0 \leqq z \leqq y$; (3) if y_0 is the largest y-coordinate of any element of S, then $(0, y_0) \in S$. The point $(0, y_0)$ is called the *summit* of S. The number of points in S is called the *size* of the stack. Since stacks are much more easily understood by geometric examples, we list all the stacks with summits that have size $\leqq 4$:

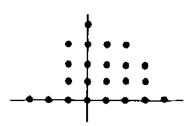


Thus if we denote by $\sigma\sigma(n)$ the number of stacks with summits of size n, we see that

$$\sum_{n=0}^{\infty} \sigma\sigma(n)\, q^n = 1 + q + 3q^2 + 6q^3 + 12q^4 + \cdots. \tag{3.1}$$

At this point, we mention that Wright [15, 16] considers two of our "stacks with summits" identical if one can be transformed into the other by a horizontal translation (i.e., he does not take account of "summits").

If we utilize the idea of Ferrers graphs [3, Section 1.3] we see immediately that each stack with summit can be decomposed into a vertical column of say j points plus to the right a Ferrers graph of a partition with at most j parts and also one to the left. As an example, the stack with summit



decomposes into the vertical column of 5 points plus to the right the Ferrers graph of the partition $2 + 3 + 3 + 4$ plus to the left the Ferrers graph of the partition $1 + 1 + 1 + 3$. Thus since $1/(q)_j$ is the generating function for partitions with at most j parts, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma\sigma(n)\, q^n &= \sum_{j=0}^{\infty} q^j \times \frac{1}{(q)_j} \times \frac{1}{(q)_j} \\ &= \sum_{j=0}^{\infty} \frac{q^j}{(q)_j^2}.\end{aligned} \tag{3.2}$$

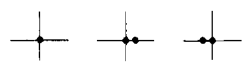
We note in passing that from Heine's fundamental transformation for the ordinary basic hypergeometric function [3, p. 19], we may directly infer that

$$\sum_{n=0}^{\infty} \sigma\sigma(n)\, q^n = \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{3.3}$$

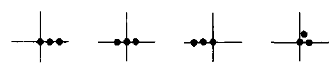
This last formula provides a convenient way of calculating $\sigma\sigma(n)$ since the power series expansion of $(q)_\infty^{-2}$ has been extensively computed (cf. [11, seq. 536, p. 68]).

Next we consider a *gradual stack with summit*. Intuitively this is the same idea as before except now we consider a "diamond" lattice instead of the standard lattice of integer points. Namely, a *gradual stack with summit* is a subset S of the set D of integer points with nonnegative y-coordinate in $\mathbb{R}^2$ whose coordinates have indentical parity such that: (1) all the elements of D lying on a segment with slope 0, 1 or -1 connecting two elements of S are also in S; (2) if $(x, y) \in S$ then so is every point of D of the form $(x-t, y-t)$, $(x+t, y-t)$ with $t \geqq 0$; (3) if y_0 is the largest y-coordinate of any element of S, then one of $(1, y_0)$ and $(0, y_0)$ is in S. This last point is called the summit of S. The number of points in S is called the size of the stack. To make this definition more transparent, we list all the gradual stacks with summits that have size $\leqq 6$:

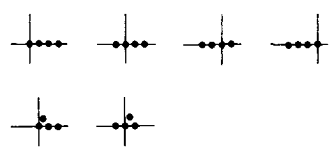
SIZE 1 SIZE 2



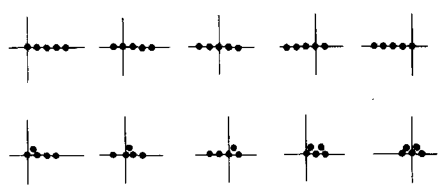
SIZE 3



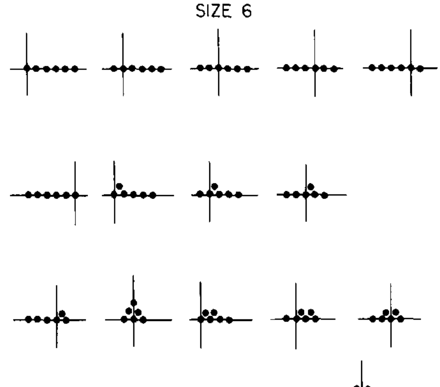
SIZE 4



SIZE 5



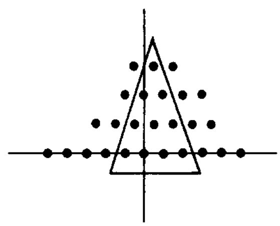
SIZE 6



Thus if we denote by $g\sigma(n)$ the number of gradual stacks with summits of size n, we set that

$$\sum_{n=0}^{\infty} g\sigma(n)^n = 1 + q + 2q^2 + 4q^3 + 6q^4 + 10q^5 + 15q^6 + \cdots. \tag{3.4}$$

There is also a nice formula for this generating function that may be derived with the use of Ferrers graphs. Indeed, we see that each gradual stack with summit can be decomposed into a central triangle of say $j(j+1)/2$ points plus to the right a Ferrers graph (skewed at a 45° angle) of a partition with at most j parts and also one to the left. As an example, the gradual stack with summit



decomposes into the indicated triangle of $(4 \times 5)/2 = 10$ points plus to the right the Ferrers graph of the partition $1 + 2 + 2 + 3$ plus to the left the Ferrers graph of the partition $1 + 1 + 2 + 4$. Thus instead of (3.2) we find that

$$\begin{aligned} \sum_{n=0}^{\infty} g\sigma(n)\, q^n &= \sum_{j=0}^{\infty} q^{j(j+1)/2} \times \frac{1}{(q)_j} \times \frac{1}{(q)_j} \\ &= \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2}}{(q)_j^2}. \end{aligned} \tag{3.5}$$

This function is also amenable to transformation. Indeed Ramanujan asserted (and Watson proved [14, pp. 59–60]) that

$$\sum_{n=0}^{\infty} g\sigma(n)\, q^n = \frac{1}{(q)_\infty} \sum_{m=0}^{\infty} \frac{q^{m(2m+1)}}{(q^2; q^2)_m}. \tag{3.6}$$

This last result is useful for computations of $g\sigma(n)$ since $(q)_\infty^{-1}$ is the generating function for ordinary partitions, and the series in (3.6) converges much faster than the one in (3.5).

4. An Application of Ramanujan's Identities

If we subtract identity (1.11) from (1.10) and divide the result by 2, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1+q)^2 (1+q^2)^2 \cdots (1+q^n)^2} = \sum_{n=0}^{\infty} \frac{(-q)^{n(n+1)/2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n^2}}{(1+q)(1+q^3)\cdots(1+q^{4n-1})}. \qquad (4.1)_R$$

Now we may readily interpret this result in terms of partition functions and functions related to stacks.

Definition 1. Let $g\sigma_+(n)$ (resp. $g\sigma_-(n)$) denote the number of gradual stacks with summit with an even number (resp. odd number) points of the stack lying on the x-axis. For brevity we say that $g\sigma_+(n)$ (resp. $g\sigma_-(n)$) enumerates the number of gradual stacks with summit and even (resp. odd) *base.*

If we return to the argument we used to establish (3.5) we may directly establish that

$$\sum_{n=0}^{\infty} (g\sigma_+(n) - g\sigma_-(n))\, q^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n^2}. \qquad (4.2)$$

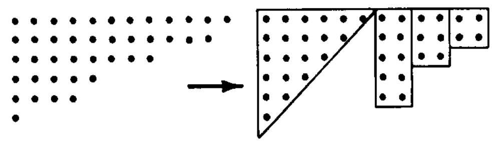
Before we define our next partition functions, we note that for any partition enumerated by $OE(n)$, the rank (i.e., largest part minus number of parts) is always even. Consequently one half the rank (or the half-rank) of such partitions is an integral parameter.

Definition 2. Let $OE_+(n)$ (resp. $OE_-(n)$) denote the number of partitions of n of the type enumerated by $OE(n)$ with even (resp. odd) half-rank.

As we remarked in the Introduction,

$$\sum_{n=0}^{\infty} OE(n)\, q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n}. \qquad (4.3)$$

This assertition easily follows once we observe that any odd-even partition (say $1+4+5+8+11+12$) may have its Ferrers graph decomposed as



This last object may be viewed as representing a triangular number $1 + 2 + 3 + 4 + 5 + 6 = 21$ plus (reading boxed columns) a partition with even parts $(10 + 6 + 4)$ each $\leqq$ twice the number of parts of the original partition. Since $(q^2; q^2)_n^{-1}$ is the generating function for partitions with even parts each $\leqq 2n$, we see that the above argument provides for the construction of the representation of the generating function for $OE(n)$ given in (4.3).

If we also keep count of the half-rank, we are led to

$$\sum_{n=0}^{\infty} (OE_+(n) - OE_-(n))\, q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q^2; q^2)_n}. \tag{4.4}$$

DEFINITION 3. Let $P_+(n)$ (resp. $P_-(n)$) denote the number of partitions of n into an even (resp. odd) number of parts in which: (i) no part is divisible by 4; (ii) at least one part is even; (iii) each integer congruent to 2 modulo 4 and $\leqq$ largest part in the partition appears exactly once as a part.

Clearly the standard construction of generating series for partition functions [3, Chapter 1] shows that

$$\sum_{n=0}^{\infty} (P_+(n) - P_-(n))\, q^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{2+6+10+\cdots+(4n-2)}}{(1+q)(1+q^3)\cdots(1+q^{4n-1})}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n^2}}{(-q; q^2)_{2n}}. \tag{4.5}$$

We may now use (4.2), (4.4) and (4.5) to interpret (4.1) purely combinatorially:

THEOREM 1. *For each* $n \geqq 0$,

$$g\sigma_+(n) + (-1)^n OE_-(n) + 2P_+(n) = g\sigma_-(n) + (-1)^n OE_+(n) + 2P_-(n). \tag{4.6}$$

The following table provides the first few values of the various expressions in (4.6)

n	$g\sigma_+(n)$	$(-1)^n OE_-(n)$	$2P_n(n)$	$g\sigma_-(n)$	$(-1)^n OE_+(n)$	$2P_-(n)$
0	1	0	0	0	1	0
1	0	0	0	1	−1	0
2	2	0	0	0	0	2
3	1	−1	2	3	−1	0
4	4	0	0	2	0	2
5	3	−1	4	7	−1	0
6	10	0	0	5	1	4

Consequently, we see that the assertion in Theorem 1 is confirmed for the first seven values of n in the table:

n	$g\sigma_+(n)+(-1)^n OE_-(n)+2P_+(n)$	$g\sigma_-(n)+(-1)^n OE_+(n)+2P_-(n)$
0	1	1
1	0	0
2	2	2
3	2	2
4	4	4
5	6	6
6	10	10

Three results similar to but more complex then Theorem 1 are easily deduced from (1.8)–(1.11). We state the results without proof since their derivation is much the same as that in Theorem 1.

DEFINITION 4. A partition without gaps is one in which each positive integer not exceeding the largest part appears at least once.

DEFINITION 5. Let $\gamma e_+(n)$ (resp. $\gamma e_-(n)$) denote the number of partitions of n with largest part even, without gaps, no repeated odd parts and in which an even (resp. odd) number of different summands each occurs an even number of times.

DEFINITION 6. Let $\gamma O_+(n)$ (resp. $\gamma O_-(n)$) denote the number of partitions of n with largest part odd, without gaps, no repeated even parts and in which an even (resp. odd) number of different summands each occurs an even number of times.

DEFINITION 7. Let $D_e(n)$ (resp. $D_0(n)$) denote the number of partitions of n into distinct parts with largest part even (resp. odd) and with all even (resp. odd) positive integers not exceeding the largest part actually appearing as parts.

From (1.8) we find

THEOREM 2.

$$\begin{aligned}
&OE(n) + OE(n-1) - OE(n-2) - OE(n-5) - \cdots \\
&\quad + (-1)^{j+j(3j-1)/2}\, OE(n - j(3j-1)/2) \\
&\quad + (-1)^{j+j(3j+1)/2}\, OE(n - j(3j+1)/2) + \cdots \\
&= \gamma e_+(n) - \gamma e_-(n) + 2D_0(n).
\end{aligned}$$

Equation (1.9) yields

THEOREM 3.

$$\begin{aligned}
&OE(n) - OE(n-4) - OE(n-8) + \cdots \\
&\quad + (-1)^j\, OE(n - 2j(3j-1) + (-1)^j\, OE(n - 2j(3j+1)) + \cdots \\
&= \gamma O_+(n) - \gamma O_-(n) + D_e(n).
\end{aligned}$$

Finally the addition of (1.10) and (1.11) yields after division by 2:

THEOREM 4.

$$\begin{aligned}
&OE(n) - 2OE(n-1) - 2OE(n-4) + \cdots \\
&\quad + 2(-1)^j\, OE(n - j^2) + \cdots \\
&= (OE_+(n) - OE_-(n))(-1)^n - 2(P_+(n) - P_-(n)).
\end{aligned}$$

Theorems 2, 3 and 4 can be viewed as results which provide reasonably efficient recurrences for the computation of $OE(n)$. This is because the partition functions $\gamma e_+(n)$, $\gamma e_-(n)$, $\gamma O_+(n)$, $\gamma O_-(n)$, $D_e(n)$ and $D_0(n)$ are all relatively small compared to $OE(n)$.

5. CONCLUSION

We have chosen in this paper to illustrate combinatorial applications of four apparently innocent yet surprisingly intricate identities from the "Lost" Notebook related to the seemingly elementary series

$$1 + \frac{q}{1-q^2} + \frac{q^3}{(1-q^2)(1-q^4)} + \frac{q^6}{(1-q^2)(1-q^4)(1-q^6)} + \cdots.$$

As usual our treatment of Ramanujan's results raises more questions than it answers.

Examination of the four essential separate transformations that are required in each of the proofs of (1.10) and (1.11) does not immediately yield any reasonable straightforward generalizations. Professor R. Askey observes that Lemma 1 (and consequently also Lemma 2) undoubtedly follows from some quadratic transformation for basic hypergeometric series. Thus presumably this result may be generalized substantially. Also the two results ((2.9) and (2.10)) on ${}_{10}\phi_9$'s can be invoked with the three variables r_2, b and a in tact.

Finally we note that the analytic complexity lying behind a result like Theorem 1 suggests that a purely combinatorial explanation of this result would be illuminating. For practice one might try to prove (3.3) or (3.6) purely combinatorially.

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