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Discrete Mathematics 308 (2008) 3048–3055
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Uniqueness of maximum planar five-distance sets

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Received 1 October 2005; accepted 9 August 2007
Available online 19 November 2007

Abstract

A subset X in the Euclidean plane is called a k-distance set if there are exactly k distances between two distinct points in X. We denote the largest possible cardinality of k-distance sets by $g(k)$. Erdős and Fishburn proved that $g(5)=12$ and also conjectured that 12-point five-distance sets are unique up to similar transformations. We classify 8-point four-distance sets and prove the uniqueness of the 12-point five-distance sets given in their paper. We also introduce diameter graphs of planar sets and characterize these graphs.

Keywords: Distance sets; Euclidean plane

1. Introduction

A subset X in Euclidean plane $\mathbb{R}^2$ is called a k-distance set if there are exactly k distances between two distinct points in X. For two subsets in $\mathbb{R}^2$, we say that they are isomorphic if there exists a similar transformation from one to the other. One of the many interesting problems on k-distance sets is to determine the largest possible cardinality of k-distance sets and to classify these k-distance sets. We denote this number by $g(k)$. A k-distance set X is said to be maximum if X has $g(k)$ points. Clearly $2k+1 \leqslant g(k)$ since the vertex set of a regular $(2k+1)$-gon is a k-distance set. On the other hand, Bannai–Bannai–Stanton [2] and Blokhuis [3] gave an upper bound $g(k) \leqslant \binom{k+2}{2}$. Let R_n denote the vertices of a regular n-gon, and let R_n^+ be R_n augmented by the point at the center of the regular n-gon. For $k=2$, $g(2)=5$ and every 5-point two-distance set is isomorphic to R_5 [4,5,9]. For $k=3$, the author classified three-distance sets with at least five points [10]. In particular, $g(3)=7$ and every 7-point three-distance set is isomorphic to R_7 or R_6^+. Erdős–Fishburn [7] determined $g(k)$ for $k \leqslant 5$ and classified maximum k-distance sets for $k \leqslant 4$. They proved that $g(5)=12$ and gave one example of 12-point five distance set. They also conjectured that every 12-point five-distance set is similar to this example.

The following theorem is proved in Erdős–Fishburn [7].

Theorem 1.1. (a) $g(4)=9$ *and every* 9-*point four-distance set in* $\mathbb{R}^2$ *is isomorphic to* R_9 *or one of the three configurations given in Figs.* 1(a)–(c).

(b) $g(5)=12$ *and the configuration given in Fig.* 1(d) *is an example of a* 12-*point five-distance set in* $\mathbb{R}^2$.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.08.028

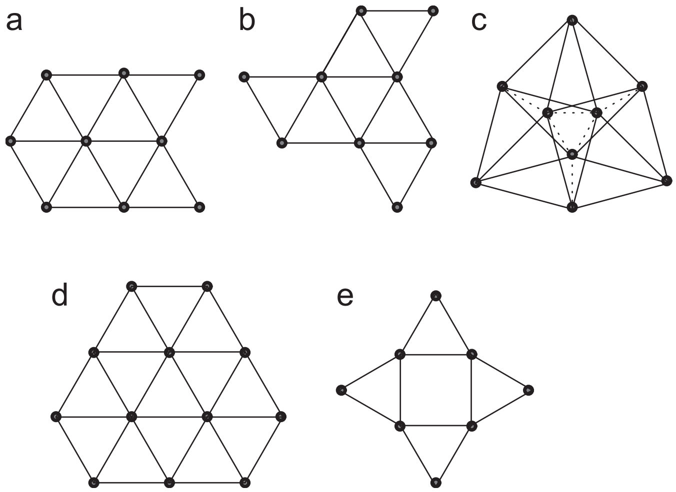


Fig. 1. (a)–(c) Three 9-point four-distance sets configurations. (d) A 12-point five-distance set configuration. (e) An 8-point four-distance set configuration.

Our main result is:

Theorem 1.2. (a) *Every* 8-*point four-distance set in* $\mathbb{R}^2$ *is isomorphic to* R_8, R_7^+, *Fig.* 1(e) *or an* 8-*point subsets of a* 9-*point four-distance set*.

(b) *The configuration given in Fig.* 1(d) *is the only* 12-*point five-distance set in* $\mathbb{R}^2$.

The proof of Theorem 1.2(b) is based on the classification of 8-point four-distance sets which in turn uses the classification of 5-point three-distance sets. We give an outline of the proof of Theorem 1.2 in Section 2. In Section 3, we introduce diameter graphs of planar sets and give their fundamental properties. In particular, Propositions 3.1–3.3 are of independent interest. It seems that the independence numbers of diameter graphs and the concept of replaceable vertices in independent sets of diameter graphs defined below play an important role in the classification of k-distance sets. In Section 4, we give a complete proof of Theorem 1.2.

2. Preliminaries

Let $D = D(X)$ be the diameter of a finite set $X \subset \mathbb{R}^2$, and let

$$X_D = \{x \in X : d(x, y) = D \text{ for some } y \in X\} \quad \text{and} \quad m = m(X) = |X_D|.$$

The following lemmas are known.

Lemma 2.1. *Let D be the diameter of an n-point planar set X with* $n \geqslant 3$. *Then*

(a) *if* $m \geqslant 3$, *the points in* X_D *are the vertices of a convex m-gon*;
(b) *D can be eliminated as an interpoint distance by removing at most* $\lceil m/2 \rceil$ *points from X*, *where* $\lceil m/2 \rceil$ *is the smallest integer at least* $m/2$.

Proof. See Lemma 1 in [7]. □

We denote the set of all n-point k-distance sets and the set of all n-point convex k-distance sets by $E_n(k)$ and $M_n(k)$, respectively. A k-distance set is said to be maximal if it is not contained in other k-distance sets. We denote the set of all non-maximal n-point k-distance sets by $E_n^*(k)$. For $i \leqslant n-3$, we let $R_n - i$ denote a set of $n-i$ vertices of a regular n-gon. (When $i \geqslant 2$, $R_n - i$ depends on the choice of the i vertices removed from R_n.)

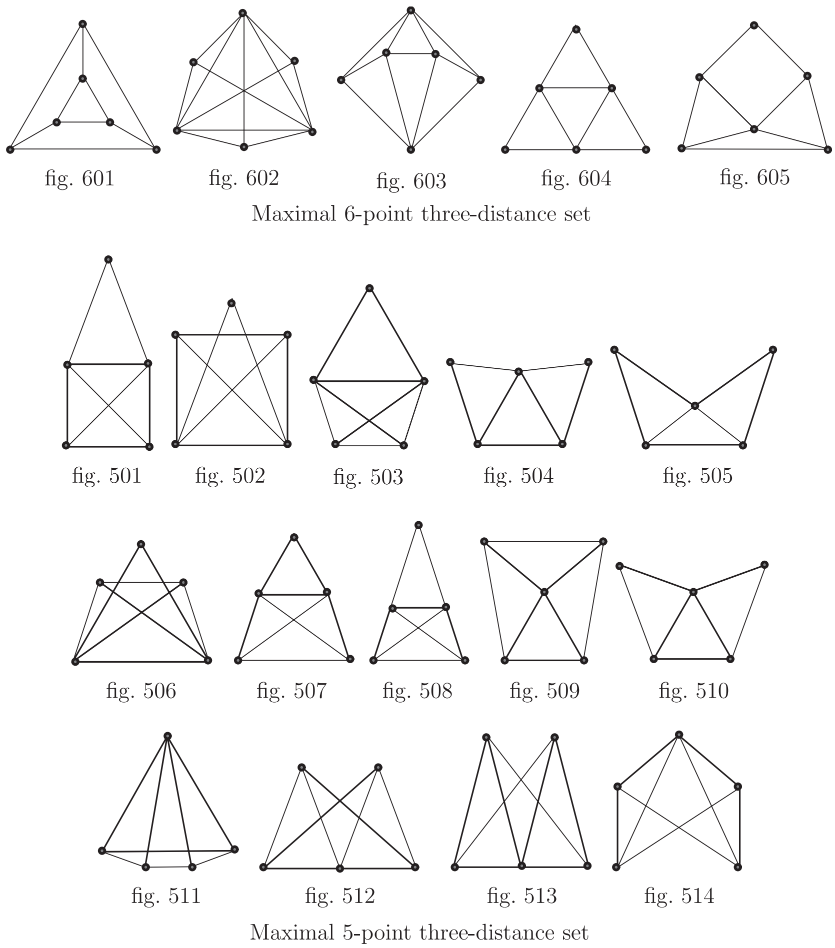


Fig. 2.

Lemma 2.2 (*Altman [1], Erdős and Fishburn [6], Fishburn [8], Shinohara [10]*). (a) *Let X be an n-point convex k-distance set, where* $n \geqslant 3$. *Then* $k \geqslant \lfloor n/2 \rfloor$, *where* $\lfloor n/2 \rfloor$ *is the greatest integer at most* $n/2$. *Moreover*,

(i) $M_{2k+1}(k) = \{R_{2k+1}\}$;
(ii) $M_{2k}(k) = \{R_{2k}, R_{2k+1} - 1\}$ *for* $k \geqslant 4$;
(iii) $M_7(4) = \{R_8 - 1, \textit{every } R_9 - 2\}$;
(iv) $M_9(5) = \{R_{10} - 1, \textit{every } R_{11} - 2\}$.

(b) $g(3) = 7$. *Moreover*,

(i) $E_7(3) = \{R_7, R_6^+\}$;
(ii) *every maximal* 6-*point three-distance set in* $\mathbb{R}^2$ *is isomorphic to* R_5^+ *or one of the configurations* (*Fig.* 2);
(iii) *every maximal* 5-*point three-distance set in* $\mathbb{R}^2$ *is isomorphic to* R_4^+ *or one of the configurations* (*Fig.* 2).

See [1] for the inequality $k \geqslant \lfloor n/2 \rfloor$ and (a)(i), [8] for (a)(ii)–(iii), [6] for (a)(iv) and [10] for (b).

For $E_8(4)$ and $E_{12}(5)$, we have the following proposition.

Proposition 2.1. (a) *If* $X \in E_8(4)$, *then X contains a subset* $Y \in \{R_5, R_7, R_8 - 1, \textit{every } R_9 - 2\} \cup E_5(3)$.

(b) *If* $X \in E_{12}(5)$, *then X contains a subset* $Y \in \{R_9, R_{10} - 1, \textit{every } R_{11} - 2\} \cup E_8(4)$.

Proof. (a) Let X be an 8-point four-distance set. If $m \geqslant 7$, then the subset X_D of X is a convex m-gon by Lemma 2.1(a). Therefore X contains a subset $Y \in \bigcup_{k \leqslant 4} M_7(k) = \{R_7, R_8 - 1, \text{every } R_9 - 2\}$. If $m \leqslant 6$, then $8 - \lceil m/2 \rceil \geqslant 5$. By Lemma 2.1(b), X contains a subset $Y \in \bigcup_{k \leqslant 3} E_5(k) = \{R_5\} \cup E_5(3)$.

(b) Let X be a 12-point five-distance set. If $m \geqslant 9$, then the subset X_D of X is a convex m-gon by Lemma 2.1(a). Therefore X contains a subset $Y \in \bigcup_{k \leqslant 5} M_{12}(k) = \{R_{10} - 1, \text{every } R_{11} - 2\}$. If $m \leqslant 8$, then $12 - \lceil m/2 \rceil \geqslant 8$. By Lemma 2.1(b), X contains a subset $Y \in \bigcup_{k \leqslant 4} E_8(k) = E_8(4)$. □

In Section 3, we slightly improve Proposition 2.1.

3. Diameter graphs

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of G, respectively. We denote a path and a cycle with n vertices by P_n and C_n, respectively. The diameter graph $DG(X)$ of $X \subset \mathbb{R}^2$ is the graph with X as its vertices and where two vertices $x, y \in X$ are adjacent if $d(x, y) = D$. Clearly $DG(R_{2n+1}) = C_{2n+1}$ and $DG(R_{2n}) = n \cdot P_2$. The diameter graph $G = DG(X)$ of $X \subset \mathbb{R}^2$ does not contain C_4 and if G contains C_3, then any two vertices in $V(G) \backslash V(C_3)$ are not adjacent. We generalize these properties as follows.

Proposition 3.1. *Let $G = DG(X)$ for $X \subset \mathbb{R}^2$. Then*

(a) *G contains no C_{2k} for any $k \geqslant 2$;*
(b) *if G contains C_{2k+1}, then any two vertices in $V(G) \backslash V(C_{2k+1})$ are not adjacent and every vertex not in the cycle is adjacent to at most one vertex of the cycle.*

In particular, G contains at most one cycle.

Proof. We can prove this easily by using the fact that two length-D segments in X must cross if they do not share an end point. □

A subset H of $V(G)$ is an *independent set* of $V(G)$ if no two vertices in H are adjacent, and H is said to be *maximal* if no other independent sets contain H. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality among the independent sets of G. An independent set H of G is said to be *maximum* if $|H| = \alpha(G)$.

Proposition 3.2. *Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^2$ with $|X| = n$. If $G \neq C_n$, then we have $\alpha(G) \geqslant \lceil n/2 \rceil$.*

Proof. We may assume G has no isolated vertex. Moreover, we may assume G contains a cycle C_k with $k < n$, otherwise $\alpha(G) \geqslant \lceil n/2 \rceil$, since G is a bipartite graph. Then there exist two vertices $v \in V(G) \backslash V(C_k)$ and $w \in V(C_k)$ such that v and w are adjacent. Since $G - \{v, w\}$ is a tree, so we have $\alpha(G - \{v, w\}) \geqslant \lceil (n-2)/2 \rceil$. Therefore $\alpha(G) \geqslant \alpha(G - \{v, w\}) + |\{v\}| = \lceil (n-2)/2 \rceil + 1 = \lceil n/2 \rceil$. □

Let H be an independent set of $V(G)$. We say that a vertex v in H is a *replaceable vertex* in H if there exists a vertex w in $V(G) \backslash H$ such that $H \backslash \{v\} \cup \{w\}$ is also an independent set of $V(G)$, and w is called a *replacement vertex* of v in H. We denote a replacement vertex of v by v^+ if the replacement is uniquely determined. Let H be a maximum independent set of $V(G)$ and v be a replaceable vertex in H. If $w_1, w_2, \ldots, w_k$ are replacement vertices of v, then vw_i and $w_i w_j$ must be edges of G for $1 \leqslant i < j \leqslant k$. Thus for any diameter graph, except for C_3, if v is replaceable then there is a unique replacement if the independent set is maximum. Clearly all vertices of a non-maximal independent set are replaceable.

Lemma 3.1. *Every tree T with $|V(T)| = n \geqslant 4$ and $\alpha(T) = \lfloor n/2 \rfloor$ has a maximum independent set with at least two replaceable vertices.*

Proof. Clearly n is even, thus $\alpha(T) = n/2$. We may assume that T contains a vertex v with degree $k \geqslant 3$ by the fact that the assertion holds for a path with at least four vertices. Let T_i be a connected component of $T - \{v\}$ and $n_i = |V(T_i)|$

for $1 \leqslant i \leqslant k$. Since $\sum_{i=1}^{k} n_i$ is odd, we may assume that n_1 is odd. Then we have

$$\frac{n}{2} = \alpha(T) \geqslant \sum_{i=1}^{k} \alpha(T_i) \geqslant \sum_{i=1}^{k} \left\lceil \frac{n_i}{2} \right\rceil = \frac{n_1+1}{2} + \sum_{i=2}^{k} \left\lceil \frac{n_i}{2} \right\rceil .$$

Therefore exactly one connected component has odd order and $\alpha(T_i) = \lfloor n_i/2 \rfloor$ for $2 \leqslant i \leqslant k$. We may assume $n_i = 2$ for $2 \leqslant i \leqslant k$, since if $n_j \neq 2$, we apply the same argument for T_j instead of T. Then T has a maximum independent set with at least $k-1 \geqslant 2$ replaceable vertices. □

Proposition 3.3. *Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^2$. Suppose $|V(G)| = n \geqslant 6$ and $\alpha(G) = \lfloor n/2 \rfloor$. Then there exists a maximum independent set which has at least two replaceable vertices.*

Proof. This proposition is proved easily by using Proposition 3.1 and Lemma 3.1. □

We say that a subset Y of $X \subset \mathbb{R}^2$ is an independent set if the corresponding vertex set of Y in the diameter graph of X is an independent set of $V(DG(X))$. We also apply other definitions of graphs given above to subsets or points of the Euclidean plane in the same manner.

Remark 3.1. Let X be a k-distance set and Y an independent set of X with two replaceable points p_1, p_2. Let $Y_i = Y \backslash \{p_i\} \cup \{p_i^+\}$ for $i = 1, 2$. Then

(a) Y and Y_i are at most $(k-1)$-distance sets;
(b) $Y \cup \{p_1^+, p_2^+\}$ is a k'-distance set for some $k' \leqslant k$.

Moreover, if Y is a maximal independent set of X, then

(c) $d(p_i, p_i^+) = D(X)$ for $i = 1, 2$;
(d) $p_1^+ \neq p_2^+$.

Lemma 3.2. (a) *Let X be an 8-point four-distance set such that $m = |X_D| \leqslant 6$. Then X contains a 5-point independent set with at least two replaceable points.*

(b) *Let X be a 12-point five-distance set such that $m = |X_D| \leqslant 8$. Then X contains an 8-point independent set with at least two replaceable points.*

Proof. (a) Let X be an 8-point four-distance set. If $\alpha(DG(X)) \geqslant 6$, then clearly the assertion holds. Therefore we may assume $\alpha(DG(X)) < 6$. Since $|X \backslash X_D| = 8 - m$ and $X \backslash X_D$ consists of isolated points, the assertion holds for $m \leqslant 3$. If $m = 4$, then there exists $v, w \in X_D$ satisfying $d(v, w) \neq D$. Then $(X \backslash X_D) \cup \{w\}$ satisfies the condition we want. Hence $m = 5$ or 6. Since $\alpha(DG(X)) = \alpha(DG(X_D)) + 8 - m < 6$, we obtain $\alpha(DG(X_D)) \leqslant m - 3$. If $m = 5$ and $DG(X_D) \neq C_5$, then Proposition 3.2 implies $\alpha(DG(X_D)) \geqslant \lceil \frac{5}{2} \rceil = 3$. This is a contradiction. Hence $DG(X_D) = C_5$. C_5 has a 2-point independent set H'. Then $H = H' \cup (X \backslash X_D)$ is a 5-point independent set. Clearly H has two replaceable vertices. If $m = 6$, Proposition 3.1 implies that $DG(X_D) \neq C_6$. Then Proposition 3.2 implies $\alpha(DG(X_D)) \geqslant 3$. Therefore $\alpha(DG(X_D)) = 3$ holds. Then Proposition 3.3 implies that there exists a 3-point independent set $H' \subset X_D$ having at least two replaceable points. Then $H = H' \cup \{v, w\}$ is a 5-point independent set for any distinct two points in $X \backslash X_D$. Since v, w are isolated points in X, any replaceable point in H' with respect to X_D is also replaceable with respect to H and X.

(b) By a similar argument as given in the proof of (a), we can prove (b). □

For a maximal 5-point three-distance set (resp. maximal 8-point four-distance set), it is hard to have replaceable points satisfying the property of Remark 3.1. Therefore Lemma 3.2 is a useful tool to classify 8-point four-distance sets (resp. 12-point five-distance set). Using Lemma 3.2 and Remark 3.1, we can improve Proposition 2.1 as follows.

Proposition 3.4. (a) *If $X \in E_8(4)$, then X contains a subset $Y \in \{R_5, R_7, R_8 - 1$, every $R_9 - 2\} \cup E_5^*(3)$.*

(b) *If $X \in E_{12}(5)$, then X contains a subset $Y \in \{R_9, R_{10} - 1$, every $R_{11} - 2\} \cup E_8^*(4)$.*

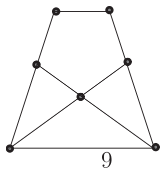


Fig. 3.

Proof. (a) Let $X \in E_8(4)$ and assume that X does not contain any subset contained in $\{R_5, R_7, R_8 - 1, \text{every } R_9 - 2\}$. As we mentioned at the beginning of Proposition 2.1 we may assume $m \leqslant 6$. Then Lemma 3.3 implies that X contains a 5-point independent set Y having at least two replaceable points. Then Y is at most a three-distance set. Since R_5 is the unique 5-point two-distance set in $\mathbb{R}^2$, we may assume $Y \in E_5(3)$. If $Y \in E_5^*(3)$, then the proof is done. Hence we may assume that Y is a maximal three-distance set. Then again Remark 3.1(b) and (d) imply that there exist distinct two replacement points $v, w \in X \backslash Y$ and $Y \cup \{v, w\}$ is a 7-point four-distance set. Thus Y is a maximal 5-point three-distance set which is contained in a 7-point four-distance set X'. Moreover Y is an independent set in X having at least two replaceable points. We use the classification of $E_5(3)$ given in [10] and show that no such Y exists. Only 5-point three-distance sets of fig. 505 and fig. 508 in Fig. 2 have such two points. Then X' must be the seven points in Fig. 3. However, we cannot put another point q such that $X' \cup \{q\}$ is a four-distance set. This means there exists no 8-point four-distance set which contains maximal 5-point three-distance sets.

(b) By Theorem 1.2(i), every maximal 8-point four-distance set is isomorphic to R_8, R_7^+ or the configuration of Fig. 1(e). By a similar argument as given in (a), we conclude that 12-point five-distance set cannot contain a maximal 8-point four-distance set. □

Proposition 3.4 means that when $m = |X_D| \leqslant 6$ (resp. $m \leqslant 8$) the classification of 8-point four-distance sets (resp. 12-point five-distance sets) results in that of 6-point three-distance sets (resp. 9-point four-distance sets). However, the classification of 5-point three-distance sets (resp. 8-point four-distance sets) is still essential to classify 8-point four-distance sets (resp. 12-point five-distance sets), as it is used in the proof of Proposition 3.4. It would be interesting if one can prove these facts without reference to the classification of 5-point three-distance sets (resp. 8-point four-distance sets).

4. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2.

Let

$$A(X) = \{d(x, y) : x, y \in X, x \neq y\},$$

$$N_p(X) = \{x \in X : d(x, p) \notin A(X)\} \quad \text{for any } p \in \mathbb{R}^2$$

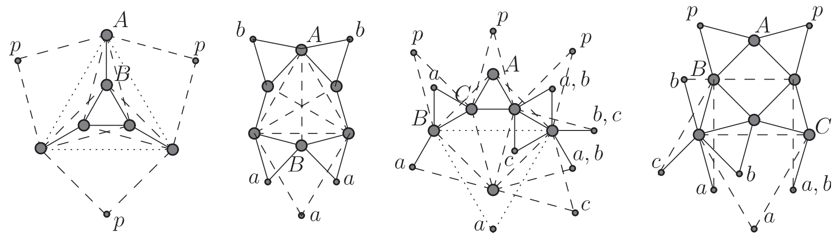
and

$$L_\Delta = \left\{ a(1, 0) + b\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) : a, b \in \mathbb{Z} \right\}.$$

(I) Proof of Theorem 1.2(a): Let $X \in E_8(4)$ and D the diameter of X.

(i) $m \geqslant 7$.

As we have seen at the beginning of the proof of Proposition 2.1, X contains a subset $Y \in \{R_7, R_8 - 1, \text{every } R_9 - 2\}$. First, we assume $X = R_7 \cup \{p\}$. If $|N_p(R_7)| \geqslant 3$, then p must be the center of R_7, so $X = R_7^+$. If $|N_p(R_7)| = k \leqslant 2$, then $R_7 - k \cup \{p\} \in E_{8-k}(3)$. Using elementary plane geometry we can show that this is impossible. Next we assume $X = R_8 - 1 \cup \{p\}$. Suppose $p \notin X_D$. Since X is a four-distance set, $|\{d(p, r) | r \in R_8 - 1\}| \leqslant 3$. Then there exist three points $r_1, r_2, r_3 \in R_8 - 1$ such that $d(p, r_1) = d(p, r_2) = d(p, r_3)$. This means that p is the center of R_8 and $X = R_8^+ - 1$, but this is a five-distance set. Hence $p \in X_D$. Thus X is 8-point convex four-distance set and we have $X = R_8$. Finally, we assume that X contains an $R_9 - 2$. We can prove $X = R_9 - 1$ using an argument similar to the preceding case.



Y_1 : fig. 601 Y_2 : fig. 602 Y_3 : fig. 603 Y_4 : fig. 605

Fig. 4.

(ii) $m \leqslant 6$.

By Proposition 3.4, X contains a subset $Y \in \{R_5\} \cup E_5^*(3)$. We divide the proof for $m \leqslant 6$ into two cases:

Case A: $R_5 \subset X$.

Case B: $Y \subset X$ with $Y \in E_5^*(3)$.

Case A: Let $R_5 \cup \{p\} \subset X$ with $p \notin R_5$. Then p is on a perpendicular bisector l of two points in R_5. It is easy to check that any other point q which is on l cannot be contained in X. This forces X to be equal to $R_5 \cup \{p, \sigma_i(p), \sigma_j(p)\}$ where $\sigma_j(p)$ is the point rotated p by $2\pi j/5$ about the center of R_5. Then there exists a 10-point four-distance set Y such that $|m(Y)| = 5$. This contradicts the fact that there exists no 7-point three-distance set which contains R_5.

Case B. First, we assume X contains an $R_7 - 2$. By Proposition 3.3, $R_7 - 2$ has a point satisfying the condition in Remark 3.1, so X contains $R_7 - 1$. Then we can show that this case does not occur by the argument similar to that in (i). Next we assume that X contains a 5-point subset of a three-distance set Y of R_6^+ or fig. 604 which is a subset of L_Δ. We may assume $A(Y) = \{1, \sqrt{3}, 2\}$. Suppose $D > 2$ holds and let $p \in X \backslash Y$. Every subset Y' of $\mathbb{R}^2$ with at least four points, $A(Y') \subset \{1, \sqrt{3}, 2\}$ and $1 \in A(Y')$, is a subset of L_Δ. Therefore we can find all possible points p easily and we conclude that in this case X is an 8-point subset of Fig. 1(a) or (b) which is a subset of L_Δ. Finally, we assume that X contains a 5-point non-maximal three-distance subset Y in one of the configurations: fig. 601, fig. 602, fig. 603, fig. 605 in Fig. 2. In Fig. 4, Y_i means a 6-point three-distance set whose points are denoted by big black dots. Then Y is among $\{Y_i \backslash \{v\} | 1 \leqslant i \leqslant 4, v = A, B, \text{or } C\}$. For each 5-point three-distance set above, we list possible candidates which can be the sixth point of a four-distance set such that the fourth distance is the diameter of the four-distance set, and we denote the point by small letter. The point with small letter a (or b, c) means the points such that $Y_i \backslash \{A\} \cup \{a\}$ is at most a four-distance set, and the point with small letter p means the points such that $Y_i \cup \{p\}$ is a four-distance set. In this case it is easy to see that X is either Fig. 1(e) or an 8-point subset of Fig. 1(c).

(II) Proof of Theorem 1.2(b): Let $X \in E_{12}(5)$.

(i) $m \geqslant 9$.

Similar as in I(i) one can show that X contains a subset $Y \in \{R_9, R_{10} - 1, \text{every } R_{11} - 2\}$. It is clear that they cannot be extended to a five-distance set on 12 points.

(ii) $m \leqslant 8$.

Proposition 3.4, X contains a subset $Y \in E_8^*(4)$. First, we assume X contains a subset $Y \in \{R_9 - 1$, an 8-point subset of Fig. 1(c)$\}$. By an argument similar to the preceding case, X must contain R_9 or 9-point four-distance set of Fig. 1(c). However, X does not contain R_9 by (i). Therefore X contains the 9-point four-distance set of Fig. 1(c). Let Y' be the 9-point four-distance set of Fig. 1(c). We cannot take a new point p such that $|N_p(Y')| \geqslant 3$ since $\angle r_1 r_2 r_3 \leqslant 5\pi/6$ for every convex vertices r_1, r_2, r_3 in Y', so $|N_p(Y')| \leqslant 2$. Then $X \backslash N_p(Y') \cup \{p\}$ is also a four-distance set with at least eight points, so we have $p \in Y'$. This is a contradiction. Next we assume X contains an 8-point subset of a four-distance set of Fig. 1(a), (b). Clearly $X \subset L_\Delta$ and X is a 12-point five-distance set of Fig. 1(d). □

References

[1] E. Altman, On a problem of P. Erdős, Amer. Math. Monthly (1963) 148–157.

[2] E. Bannai, E. Bannai, D. Stanton, An upper bound for the cardinality of an *s*-distance subset in real Eucldiean space, II, Combinatorica 3 (1983) 147–152.
[3] A. Blokhuis, Few-distance sets, Ph.D. Thesis, Eindhoven University of Technology, 1983 (CWI Tract (7) 1984).
[4] S.J. Einhorn, I.J. Schoenberg, On Euclidean sets having only two distances between points II, Nederl Akad. Wetensch. Proc. Ser. A69—Indag. Math. 28 (1966) 489–504.
[5] S.J. Einhorn, I.J. Schoenberg, On Euclidean sets having only two distances between points I, Nederl Akad. Wetensch. Proc. Ser. A69—Indag. Math. 28 (1966) 479–488.
[6] P. Erdős, P. Fishburn, Convex nonagons with five intervertex distance, Geom. Dedicata 60 (1996) 317–332.
[7] P. Erdős, P. Fishburn, Maximum planar sets that determine *k* distances, Discrete Math. 160 (1996) 115–125.
[8] P. Fishburn, Convex polygons with few intervertex distance, Comput. Geom. Theory Appl. 5 (1995) 65–93.
[9] L.M. Kelly, Elementary problems, Solutions. Isosceles *n*-points, Amer. Math. Monthly 54 (1947) 227–229.
[10] M. Shinohara, Classification of three-distance sets in two dimensional Euclidean space, European J. Combin. 25 (2004) 1039–1058.