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Stationary micropolar fluid with boundary data in L^2

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Abstract

We consider the Dirichlet boundary value problem for the equations of a stationary micropolar fluid in a bounded three-dimensional domain. We show the existence and uniqueness of a distributional solution with boundary values in L^2. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Micropolar fluid; Incompressible flow; Very weak solution

1. Introduction

The micropolar fluid model is an essential generalization of the well-established Navier–Stokes model in the sense that it takes into account the microstructure of the fluid. It may represent fluids consisting of randomly oriented (or spherical) particles suspended in a viscous medium, when the deformation of fluid parti-

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cles is ignored. Micropolar fluids were introduced in [1]. They are non-Newtonian fluids with nonsymmetric stress tensor.

The governing system of equations of micropolar fluids expresses the balance of momentum, mass, and moment of momentum [1,2], which in a stationary regime is

$$\begin{cases} -\mu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = a\,\mathrm{rot}\,\mathbf{w} + \mathbf{f}, \quad \mathrm{div}\,\mathbf{v} = 0, \\ -\alpha\Delta\mathbf{w} + (\mathbf{v}\cdot\nabla)\mathbf{w} - \beta\nabla\,\mathrm{div}\,\mathbf{w} + \gamma\mathbf{w} = a\,\mathrm{rot}\,\mathbf{v} + \mathbf{g}, \end{cases} \tag{1}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity field, p is the pressure and $\mathbf{w} = (w_1, w_2, w_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles. The fields $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ are given external forces and moments, respectively, and $\mu = \nu + \nu_r$, $a = 2\nu_r$, $\alpha = c_a + c_d$, $\beta = c_o + c_d - c_a$, $\gamma = 4\nu_r$, where $\nu, \nu_r, c_o, c_a, c_d$ are positive constants that represent viscosity coefficients, ν is the usual Newtonian viscosity and ν_r is called the microrotation viscosity. It is assumed that the density of the fluid is equal to one.

Observe that if the microrotation viscosity v_r equals zero then the first equations in system (1) reduce to the incompressible stationary Navier–Stokes system and the velocity field is independent of the microrotation field.

Several experiments show that solutions of the micropolar fluid model better describe behavior of numerous real fluids (e.g., blood [3]) than corresponding solutions of the Navier–Stokes model, especially when the characteristic dimensions of the flow (e.g., the diameter of a channel) become small.

In this paper we are interested in the boundary value problem for system (1) in a bounded domain Ω of $\mathbf{R}^3$ with a smooth boundary Γ and Dirichlet boundary data,

$$\mathbf{v}|_\Gamma = \mathbf{v}_0, \qquad w|_\Gamma = \mathbf{w}_0, \tag{2}$$

in $L^2(\Gamma)$. We assume that $\mathbf{f}, \mathbf{g} \in L^2(\Omega)$ and the compatibility condition $\int_\Gamma \mathbf{v}_0 \cdot \mathbf{n}\,ds = 0$, where we denote by $\mathbf{n}$ the unit outward normal of Γ. The case of null boundary data was studied by Łukaszewicz [4] (see also [2]), and in [5] in the case of exterior domain. The case where the boundary data are not null but sufficiently regular, such that they can be extended to the interior of the domain Ω accordingly with trace theorems, can be treated in a similar way as in [2]. (The case of stationary Navier–Stokes system with data in $H^{1/2}(\Gamma)$ goes back to the classical method of Leray—see, e.g., [6], and with data in $W^{1-1/q,q}(\Gamma)$, $3/2 < q < 2$, was solved in [7].) However, if they are not regular, for instance, if the boundary data are not the traces at the boundary of Ω of some functions in Sobolev spaces on Ω, then the problem is quite more difficult. This problem for the Stokes equations was treated by Conca [8], where the concept of *very weak solution* was introduced (see Appendix A in [8], or [9]). Then, more recently, Marusič-Paloka [10] proved the existence of a *very weak solution* for the stationary Navier–Stokes equations.

There are some physical motivations for considering fluid equations with irregular boundary data; e.g., in [8] it is considered the Stokes equations modeling a fluid in a domain containing a sieve and then it is shown that when the sieve becomes finer and finer the solution of the problem converges to a solution of a Stokes problem with boundary data only in L^2. Other examples, for the stationary Navier–Stokes equations with boundary data in some Sobolev space $W^{1-1/q,q}$, are pointed out in [7]; namely, the problem of a stationary fluid in a "domain with a cavity," i.e., the union of a semi-space with a bounded domain (the "cavity"), and the *Taylor problem*, i.e., the problem of equilibrium of a fluid between two co-centered cylinders with the external cylinder fixed and the internal one in a rotational motion about its axis.

The main idea used by Conca in [8] is the transposition method (see, e.g., [11]), which is very useful for linear equations. Marusič-Paloka [10] was able to extend Conca's result, first for small data by using a linearization of the Navier–Stokes equations and an iterative argument (in fact, the Banach's fixed point theorem) based on penalization method and an estimate on the Oseen's problem solution, and then for no small data assumption by splitting the data into a small irregular part and a large regular part.

We combine ideas from Conca [8], Marusič-Paloka [10], and Łukaszewicz [4], to obtain the existence of a *very weak solution* for the stationary micropolar fluid equations. That is, first we use the transposition method for obtaining a solution $\mathbf{w}$ to the microrotational field equation, which depends on the velocity field $\mathbf{v}$ that lives in $L^3(\Omega)$. This microrotational field solution $\mathbf{w}$ obeys a good estimate with respect to $\mathbf{v}$, as we prove below, provided $\mathbf{v}$ is split into a small irregular part in $L^3(\Omega)$ and a regular part $\mathbf{u}^\varepsilon$ in $H^1(\Omega)$ (see Lemma 3.1). To attain that, we needed to prove a regularity result for a second-order linear strongly elliptic system with an irregular coefficient (see the proof of Lemma 3.1). Then taking the small part of $\mathbf{v}$ as a solution for the Navier–Stokes equations, via Marusič-Paloka's theorem (Theorem 4 in [10]), we prove the existence of $\mathbf{u}^\varepsilon$ using an appropriate Leray–Hopf extension of a smooth approximation of the boundary value for $\mathbf{v}$, such that we may employ the Leray–Schauder fixed point theorem following [4].

Besides the existence of solutions, we obtain a result of continuous dependence on the boundary data for $\mathbf{w}$ and given external forces, which implies, in particular, uniqueness of solution.

The plan of the paper is as follows. In Section 2 we give the definition of a *very weak solution* and state our main theorems. Section 3 deals with the system for the microrotational field $\mathbf{w}$ assuming that $\mathbf{v}$ is split into an appropriate sum, as explained above. In Section 4 we show a way of reducing the system for $\mathbf{v}$ to a new system for an unknown $\mathbf{u}$ in the space $\mathcal{V}$ of divergent free functions in $H_0^1(\Omega)$. That is, $\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$, where $\mathbf{v}^\varepsilon$ is the small part of $\mathbf{v}$ in $L^3(\Omega)$. This small part $\mathbf{v}^\varepsilon$ is a *very weak solution* of the stationary Navier–Stokes system, which exists due to the Marusič-Paloka's theorem [10], with null external force and with a boundary data very small in the norm of $L^2(\Gamma)$, depending on a smooth approximation $\mathbf{v}_0^\varepsilon$ of $\mathbf{v}_0$. The part $\mathbf{u}^\varepsilon$ is the "large" regular part of $\mathbf{v}$ in $H^1(\Omega)$. It is equal to $\mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon}$, where $\widetilde{\mathbf{v}_0^\varepsilon}$ is an appropriate Leray–Hopf extension of $\mathbf{v}_0^\varepsilon$ to Ω which is in $\mathcal{V}$, and $\mathbf{u}$ is the new unknown which satisfies its own system shown in Section 4. This system for $\mathbf{u}$ is a nonlinear one, where the nonlinearities come from the term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and from $\mathbf{w}$ that depends on $\mathbf{v}$. In Section 5 we prove the existence of a solution $\mathbf{u}$ in $\mathcal{V}$ for this system using the Leray–Schauder fixed point theorem, with the help of a good choice of $\mathbf{v}_0^\varepsilon$ and $\widetilde{\mathbf{v}_0^\varepsilon}$. Finally, in Section 6 we prove the continuous dependence of the *very weak solution* on the data $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{w}_0$.

Notations. Throughout this paper, besides standard or above stated notations, we fix the following one: $W^{k,p}$ is the Sobolev space of order k modelled in $L^p(\Omega;\mathbf{R}^3)$; $W_0^{k,p}$ is the closure in $W^{k,p}$ of the functions in C_0^∞; $H^k=W^{k,2}$; $H_0^k=W_0^{k,2}$; $((\ ,\ ))$ is the inner product in $\mathcal{V}$ ($\mathcal{V}$ is the closure in H_0^1 of the functions in C_0^∞ with null divergence); i.e.,

$$((\mathbf{u},\mathbf{v}))\stackrel{\text{def}}{=}\int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial u_i}{\partial x_j},\quad \mathbf{u}=(u_1,u_2,u_3),\quad \mathbf{v}=(v_1,v_2,v_3)\in\mathcal{V},$$

where repeated indices mean summation from 1 to 3; $\|\ \|$ is the norm associated with $((\ ))$; $\|\ \|_{k,p}$ is the norm of $W^{k,p}$; $\|\ \|_k$ is the norm of H^k; $(\ ,\ )$ is the inner product of L^2; $|\ ,\ |$ is the norm of L^2; $|\ ,\ |_p$ is the norm of L^p; $\mathcal{B}(\ ,\ ,\ )$ is the trilinear form given by $\mathcal{B}(\mathbf{u},\mathbf{v},\mathbf{w})\stackrel{\text{def}}{=}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})$; c is some positive constant that does not depend on the unknowns.

2. Very weak solution

In this section we give the definition of a *very weak solution* and state our main theorems.

Definition 2.1 (Very weak solution). A triple $(\mathbf{v},\mathbf{w},p)$ in $L^3\times L^2\times W^{-1,3}$ is a *very weak solution* of problem (1)–(2) if

$$(\mathbf{v},\nabla\theta)=\int_\Gamma(\mathbf{v}_0\cdot\mathbf{n})\theta\,ds,\quad \forall\theta\in W^{1,3/2},\tag{3}$$

$$-\mu(\mathbf{v},\Delta\varphi)-\mathcal{B}(\mathbf{v},\varphi,\mathbf{v})-(p,\operatorname{div}\varphi)$$
$$=a(\mathbf{w},\operatorname{rot}\varphi)+(\mathbf{f},\varphi)-\mu\int_\Gamma \mathbf{v}_0\cdot\frac{\partial\varphi}{\partial\mathbf{n}}\,ds,\quad \forall\varphi\in W^{2,3/2}\cap W_0^{1,3/2},\tag{4}$$

and

$$-\alpha(\mathbf{w},\Delta\psi)-\mathcal{B}(\mathbf{v},\psi,\mathbf{w})-\beta(\mathbf{w},\nabla\operatorname{div}\psi)+\gamma(\mathbf{w},\psi)$$
$$=a(\mathbf{v},\operatorname{rot}\psi)+(\mathbf{g},\psi)-\alpha\int_\Gamma \mathbf{w}_0\cdot\frac{\partial\psi}{\partial\mathbf{n}}\,ds-\beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\operatorname{div}\psi\,ds,$$
$$\forall\psi\in H^2\cap H_0^1.\tag{5}$$

The main goal of this paper is to prove the following theorems.

Theorem 2.1 (Existence). *There exists a very weak solution of problem* (1)–(2) *in the sense of the above definition, provided* $a \leqslant c^*\mu$, *where* c^* *is some positive constant depending only on* Ω *and on the parameters* α, β, *and* γ.

Theorem 2.2 (Continuous dependence on $\mathbf{f}$, $\mathbf{g}$, $\mathbf{w}_0$, and uniqueness). *Let* $(\mathbf{v}_i, \mathbf{w}_i)$, $i = 1, 2$, *be very weak solutions of problem* (1)–(2) *corresponding to the external fields* $\mathbf{f} = \mathbf{f}_i$, $\mathbf{g} = \mathbf{g}_i$, *and boundary data* $\mathbf{w}_{0,i}$, $i = 1, 2$, *respectively. Then there exists a constant* $\mu^* > 0$ *such that for all* $\mu \geqslant \mu^*$,

$$|\mathbf{v}_1 - \mathbf{v}_2|_3 + |\mathbf{w}_1 - \mathbf{w}_2| \leqslant c\big(|\mathbf{f}_1 - \mathbf{f}_2| + |\mathbf{g}_1 - \mathbf{g}_2| + |\mathbf{w}_{01} - \mathbf{w}_{02}|\big), \tag{6}$$

where the constant c *depends only on the data of the problem and on* Ω. *In particular, for* $\mu \geqslant \mu^*$ *the problem is uniquely solvable.*

Remark 2.1. Observe that the solution exists if $a \leqslant c^*\mu$. In particular, with $a = 0$, $\mathbf{g} = 0$, and $\mathbf{w}_0 = 0$ our existence theorem reduces to that in [10]. On the other hand, the solution is unique provided the viscosity μ is large enough, exactly as in the case of more regular solutions [4].

3. Problem in w

In this section we study the following problem in $\mathbf{w}$:

Problem 3.1. Given $\mathbf{w}_0 \in L^2(\Gamma)$ and $\mathbf{v} \in L^3$ with $\operatorname{div} \mathbf{v} = 0$ (see Remark 3.1 below) and such that $\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$, where $\mathbf{u}^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{u}$, $\mathbf{u} \in \mathcal{V}$, $\mathbf{v}_0^\varepsilon \in H^2$, with $\operatorname{div} \widetilde{\mathbf{v}_0^\varepsilon} = 0$, and $\mathbf{v}^\varepsilon \in L^3$ with $|\mathbf{v}^\varepsilon|_3$ sufficiently small; find $\mathbf{w} \in L^2$ such that (5) is satisfied.

Remark 3.1. Above, the condition $\operatorname{div} \mathbf{v} = 0$ is understood in the weak sense; i.e., $(\mathbf{v}, \nabla\theta) = 0$ for all $\theta \in W_0^{1,3/2}$. As a consequence, we have that the bilinear form

$$B(\phi, \psi) \stackrel{\text{def}}{=} \alpha(\nabla\phi, \nabla\psi) - \mathcal{B}(\mathbf{v}, \psi, \phi) + \beta(\operatorname{div}\phi, \operatorname{div}\psi) + \gamma(\phi, \psi) \tag{7}$$

is strongly elliptic; i.e., it is bilinear continuous and coercive. Indeed, $\mathcal{B}(\mathbf{v}, \phi, \psi) = -(1/2)(\mathbf{v}, \nabla(|\phi|^2)) = 0$, for all $\phi \in H_0^1$, since $\operatorname{div} \mathbf{v} = 0$ and $H_0^1 \subset W_0^{1,3/2}$.

Lemma 3.1. *There exists a unique solution* $\mathbf{w}$ *of Problem* 3.1. *Moreover,*

$$|\mathbf{w}| \leqslant c\big(1 + \|\mathbf{u}^\varepsilon\|_1\big), \tag{8}$$

where c *is independent of* $\mathbf{v}$.

Proof. We use the transposition method [11]. Let

$$L(\psi) \stackrel{\text{def}}{=} -\alpha\Delta\psi - (\mathbf{v}\cdot\nabla)\psi - \beta\nabla\operatorname{div}\psi + \gamma\psi. \tag{9}$$

Given $h \in L^2$, let ψ be the unique weak solution in H_0^1 of the equation $L(\psi) = h$; i.e.,

$$B(\phi,\psi) = (h,\phi) \tag{10}$$

for all $\phi \in H_0^1$. Existence and uniqueness of such solution ψ in H_0^1 easily follows from Lax–Milgram's lemma, since $\operatorname{div}\mathbf{v} = 0$ (cf. Remark 3.1 above). Besides, we can easily get the estimates

$$\|\psi\| \leqslant \alpha^{-1}|h|, \qquad |\psi| \leqslant \gamma^{-1}|h| \tag{11}$$

by taking $\phi = \psi$ in (10).

Next we prove higher regularity of the solution of (10); i.e., we show that $\psi \in H^2$. Moreover, we obtain the following estimate:

$$\|\psi\|_2 \leqslant c\bigl(1 + \|\mathbf{u}^\varepsilon\|_1^2\bigr)|h|, \tag{12}$$

where c is independent of $\mathbf{v}$. To attain that we first regularize $\mathbf{v}$ by making use of the convolution operator with a smooth family of mollifiers $\{\rho_\eta\}$, $\eta > 0$. For $\mathbf{v}_\eta \stackrel{\text{def}}{=} \mathbf{u}_\eta^\varepsilon + \mathbf{v}_\eta^\varepsilon$, where $\mathbf{u}_\eta^\varepsilon = \mathbf{u} * \rho_\eta + \widetilde{\mathbf{v}_0^\varepsilon}$, $\mathbf{v}_0^\varepsilon = \mathbf{v}^\varepsilon * \rho_\eta$, we let ψ_η be the solution in H_0^1 of the following regularization of system $L(\psi) = h$:

$$-\alpha\Delta\psi_\eta - \beta\nabla\operatorname{div}\psi_\eta + \gamma\psi_\eta - F_\eta, \tag{13}$$

where $F_\eta \stackrel{\text{def}}{=} h + (\mathbf{v}_\eta\cdot\nabla)\psi_\eta = h + (\mathbf{u}_\eta^\varepsilon\cdot\nabla)\psi_\eta + (\mathbf{v}_\eta^\varepsilon\cdot\nabla)\psi_\eta$. Since $\mathbf{u}_\eta^\varepsilon, \mathbf{v}_\eta^\varepsilon \in C(\bar{\Omega})$ and $\nabla\psi_\eta \in L^2$, we have that $F_\eta \in L^2$; thus by Nečas result on strongly elliptic systems (Theorem 5 in [12]) we obtain

$$\|\psi_\eta\|_2 \leqslant c|F_\eta|, \tag{14}$$

where c is independent of $\mathbf{v}_\eta$. But

$$\begin{aligned} \bigl|(\mathbf{u}_\eta^\varepsilon\cdot\nabla)\psi_\eta\bigr| &\leqslant c\|\mathbf{u}_\eta^\varepsilon\|_1|\nabla\psi_\eta|_3 \leqslant c\|\mathbf{u}_\eta^\varepsilon\|_1\|\psi_\eta\|^{1/2}\|\psi_\eta\|_2^{1/2} \\ &\leqslant \frac{c^2}{4\sigma}\|\mathbf{u}_\eta^\varepsilon\|_1^2\|\psi_\eta\| + \sigma\|\psi_\eta\|_2 \end{aligned} \tag{15}$$

for any $\sigma > 0$. (In the second inequality above we used the Gagliardo–Nirenberg (see, e.g., [13]) inequality $\|u\|_{W^{k,p}} \leqslant c\|u\|_{W^{m,q}}^\theta|u|_r^{1-\theta}$ with $k = 1$, $p = n = 3$, $m = q = 2$, $\theta = 1/2$, and $r = 6$.) Besides,

$$\bigl|(\mathbf{v}_\eta^\varepsilon\cdot\nabla)\psi_\eta\bigr| \leqslant |\mathbf{v}_\eta^\varepsilon|_3|\nabla\psi_\eta|_6 \leqslant |\mathbf{v}^\varepsilon|_3|\nabla\psi_\eta|_6 \leqslant \sigma c\|\psi_\eta\|_2, \tag{16}$$

if $|\mathbf{v}^\varepsilon|_3 \leqslant \sigma$. Then, using (15) and (16) in (14) with an appropriate σ, we obtain $\|\psi_\eta\|_2 \leqslant c(|h| + \|\mathbf{u}_\eta^\varepsilon\|_1^2\|\psi_\eta\|)$. As $\|\psi_\eta\| \leqslant c|h|$, we arrive at (12), with ψ_η in place of ψ and $\mathbf{u}_\eta^\varepsilon$ in place of $\mathbf{u}^\varepsilon$. Then we pass to the limit for a subsequence of $\{\eta\}$

and get (12). Here we used Banach–Alaoglu's theorem in H^2 and the uniqueness of solution of (10) in H_0^1.

Now we consider the map that takes h in L^2 into the unique solution ψ of (10) which is in H^2. Since we have (12) and Eq. (10) is linear, this is a continuous linear map from L^2 into H^2. Then the expression

$$l(h) \stackrel{\text{def}}{=} a(\mathbf{v}, \operatorname{rot}\psi) + (\mathbf{g}, \psi) - \alpha \int_\Gamma \mathbf{w}_0 \cdot \frac{\partial \psi}{\partial \mathbf{n}}\, ds - \beta \int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \operatorname{div}\psi\, ds$$

defines a continuous linear functional in h acting on L^2. Writing the equation for $\mathbf{w}$ in the form

$$(\mathbf{w}, h) = l(h) \tag{17}$$

for all $h \in L^2$, we conclude directly from the Riesz representation theorem that it has a unique solution $\mathbf{w}$ in L^2. This prove the existence and uniqueness part of the lemma.

Next we proceed to get the estimate (8). Setting $h = \mathbf{w}$ in the equation $(\mathbf{w}, h) = l(h)$ we get

$$|\mathbf{w}|^2 = a(\mathbf{v}, \operatorname{rot}\psi) + (\mathbf{g}, \psi) - \alpha \int_\Gamma \mathbf{w}_0 \frac{\partial \psi}{\partial \mathbf{n}}\, ds - \beta \int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \operatorname{div}\psi\, ds, \tag{18}$$

where $L(\psi) = \mathbf{w}$, $\psi \in H_0^1 \cap H^2$. We shall show that the right-hand side of (18) can be estimated by $c(1 + \|\mathbf{u}^\varepsilon\|_1)|\mathbf{w}|$, where c is independent of $\mathbf{v}$. From the estimate $\alpha\|\psi\|^2 + \gamma|\psi|^2 \leqslant (\mathbf{w}, \psi) \leqslant (1/2\gamma)|\mathbf{w}|^2 + (\gamma/2)|\psi|^2$ we have

$$\|\psi\| \leqslant \frac{1}{\sqrt{2\alpha\gamma}}|\mathbf{w}| \quad \text{and} \quad |\psi| \leqslant \frac{1}{\gamma}|\mathbf{w}|. \tag{19}$$

The difficult term in (18) is $\int_\Gamma \mathbf{w}_0(\partial\psi/\partial\mathbf{n})\, ds$. To estimate it we need to use the fact that

$$|z|_{L^2(\Gamma)} \leqslant c\big(|\nabla z|^{1/2}|z|^{1/2} + |z|\big) \tag{20}$$

for any z in $H^1(\Omega)$. This estimate can be inferred from $|z|_{L^2(\Gamma)} \leqslant c|\nabla z|^{1/2}|z|^{1/2}$ for all $z \in H^1(\Omega)$ with null average in Ω (see, e.g., [14, p. 50]) by applying it to z minus its average in Ω. Using (19), (20) with $z = \nabla\psi$ and (12), we have

$$a(\mathbf{v}, \operatorname{rot}\psi) \leqslant a|\mathbf{v}|\|\psi\| \leqslant \frac{a}{\sqrt{2\alpha\gamma}}|\mathbf{v}||\mathbf{w}| \leqslant ac\big(1 + \|\mathbf{u}^\varepsilon\|_1\big)|\mathbf{w}|,$$

$$(\mathbf{g}, \psi) \leqslant |\mathbf{g}||\psi| \leqslant \frac{1}{\gamma}|\mathbf{g}||\mathbf{w}| \leqslant c|\mathbf{w}|,$$

$$\begin{aligned}\alpha \int_\Gamma \mathbf{w}_0 \frac{\partial\psi}{\partial\mathbf{n}}\, ds &\leqslant \alpha|\mathbf{w}_0|_{L^2(\Gamma)}|\nabla\psi|_{L^2(\Gamma)} \\ &\leqslant \alpha|\mathbf{w}_0|_{L^2(\Gamma)} c\big(\|\psi\|_2^{1/2}\|\psi\|^{1/2} + \|\psi\|\big)\end{aligned}$$

$$\leqslant \alpha c|\mathbf{w}_0|_{L^2(\Gamma)}\big(\big(1+\|\mathbf{u}^\varepsilon\|_1^2\big)^{1/2}|\mathbf{w}|^{1/2}|\mathbf{w}|^{1/2}+|\mathbf{w}|\big)$$
$$\leqslant c\big(1+\|\mathbf{u}^\varepsilon\|_1\big)|\mathbf{w}|,$$

and

$$\beta\int_\Gamma (\mathbf{w}_0\cdot\mathbf{n})\,\mathrm{div}\,\psi\,ds \leqslant c\beta|\mathbf{w}_0|_{L^2(\Gamma)}|\mathbf{w}| \leqslant c\big(1+\|\mathbf{u}^\varepsilon\|_1\big)|\mathbf{w}|.$$

In conclusion, (18) together with the above estimates gives (8). □

We finish this section with the following lemma which will be used in the end of the proof of Lemma 5.2.

Lemma 3.2. *Let* $(\mathbf{u}_n^\varepsilon)$ *be a bounded sequence in* H^1, $\mathbf{v}_n \overset{\text{def}}{=} \mathbf{u}_n^\varepsilon + \mathbf{v}^\varepsilon$, *and* $\mathbf{w}_n$ *the unique solution of Problem* 3.1 *with* $\mathbf{v} = \mathbf{v}_n$. *Then there exists a subsequence* $(\mathbf{w}_{n_k})$ *that is strongly convergent in* L^2.

Proof. From inequality (8) we conclude that the sequence $(\mathbf{w}_n)$ is bounded in L^2. Thus, there exists a subsequence $(\mathbf{w}_{n_k})$ that is weakly convergent in L^2. From (18) written for $\mathbf{w}_{n_k}$ and $\mathbf{w}_{n_l}$, we get

$$\begin{aligned} &|\mathbf{w}_{n_k}|^2 - |\mathbf{w}_{n_l}|^2 \\ &\quad = a(\mathbf{v}_{n_k} - \mathbf{v}_{n_l}, \mathrm{rot}\,\psi_{n_k}) + a\big(\mathbf{v}_{n_k}, \mathrm{rot}(\psi_{n_k} - \psi_{n_l})\big) + (\mathbf{g}, \psi_{n_k} - \psi_{k_l}) \\ &\qquad - \alpha\int_\Gamma \mathbf{w}_0 \frac{\partial}{\partial\mathbf{n}}(\psi_{n_l} - \psi_{n_k})\,ds - \beta\int_\Gamma (\mathbf{w}_0\cdot\mathbf{n})\,\mathrm{div}(\psi_{n_l} - \psi_{n_k})\,ds, \end{aligned} \tag{21}$$

where $L(\psi_{n_k}) = \mathbf{w}_{n_k}$ and $L(\psi_{n_l}) = \mathbf{w}_{n_l}$.

From the boundedness of $(\mathbf{w}_n)$ in L^2 and inequality (12) it follows that the sequence (ψ_{n_k}) is bounded in H^2. From the compact embedding $H^1 \hookrightarrow\hookrightarrow L^{3/2}$ we conclude the existence of a subsequence $(\psi_{n_{k_m}}$, $m = 1, 2, \ldots,$ such that $(\nabla\psi_{n_{k_m}})$ converges strongly in $L^{3/2}$. Since $H^2 \hookrightarrow H^{3/2}(\Gamma) \hookrightarrow\hookrightarrow H^1(\Gamma)$, we can assume also that $|\nabla(\psi_{n_{k_m}} - \psi_{n_{k_i}})|_{L^2(\Gamma)}$ converges to zero as m, i go to infinity. Taking that into account, we can see easily from (21) that $|\mathbf{w}_{n_{k_m}}|^2 - |\mathbf{w}_{n_{k_i}}|^2 \to 0$, as m, i go to infinity. This, together with the weak convergence of $(\mathbf{w}_{n_{k_m}})$ in L^2, gives the strong convergence of $(\mathbf{w}_{n_{k_m}})$ in L^2. □

4. Problem in v and a related problem

Assume that $\mathbf{w} \in L^2$ is given and consider the problem (3), (4) in $\mathbf{v}$. We want to get rid of the pressure (it can be recovered when needed from De Rham's lemma) and to this end we take test functions that are divergent free. Then the problem (3), (4) reduces to the following one.

Problem 4.1. Given $\mathbf{w} \in L^2$, $\mathbf{v}_0 \in L^2(\Gamma)$ and $\mathbf{f} \in L^2$; find $\mathbf{v} \in L^3$ such that

$$(\mathbf{v}, \nabla\theta) = \int_\Gamma (\mathbf{v}_0 \cdot \mathbf{n})\theta \, ds, \quad \forall \theta \in W^{1,3/2}, \tag{22}$$

and

$$-\mu(\mathbf{v}, \Delta\varphi) - \mathcal{B}(\mathbf{v}, \varphi, \mathbf{v}) = a(\mathbf{w}, \operatorname{rot}\varphi) + (\mathbf{f}, \varphi) - \mu \int_\Gamma \mathbf{v}_0 \cdot \frac{\partial \varphi}{\partial \mathbf{n}} \, ds, \tag{23}$$

for all φ in $W^{2,3/2} \cap W_0^{1,3/2}$ such that $\operatorname{div}\varphi = 0$.

Now, we introduce a problem that is related to Problem 4.1. Assume that $\mathbf{v}$ is a solution of Problem 4.1 and that we can write $\mathbf{v}$ in the form

$$\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon \quad (\varepsilon > 0), \tag{24}$$

where $\mathbf{u}^\varepsilon$ is a "large regular part": $\mathbf{u}^\varepsilon \in H^1$, $\operatorname{div}\mathbf{u}^\varepsilon = 0$, $\mathbf{u}^\varepsilon|_\Gamma = \mathbf{v}_0^\varepsilon$ ($\mathbf{v}_0^\varepsilon$ is a smooth approximation of $\mathbf{v}_0$ in $L^2(\Gamma)$ such that $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_{L^2(\Gamma)} \ll 1$), and $\mathbf{v}^\varepsilon$ is a "small irregular part": $\mathbf{v}^\varepsilon \in L^3$ and is very weak solution of the problem (cf. Lemma 4.2 below)

$$\begin{cases} -\mu\Delta\mathbf{v}^\varepsilon + (\mathbf{v}^\varepsilon \cdot \nabla)\mathbf{v}^\varepsilon + \nabla p^\varepsilon = 0 \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v}^\varepsilon = 0 \quad \text{in } \Omega, \\ \mathbf{v}^\varepsilon|_\Gamma = \mathbf{v}_0 - \mathbf{v}_0^\varepsilon. \end{cases} \tag{25}$$

According to the definition of a very weak solution, we have, in particular,

$$-\mu(\mathbf{v}^\varepsilon, \Delta\varphi) - \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \mathbf{v}^\varepsilon) = -\mu \int_\Gamma (\mathbf{v}_0 - \mathbf{v}_0^\varepsilon) \frac{\partial \varphi}{\partial \mathbf{n}} \, ds \tag{26}$$

for all $\varphi \in W^{2,3/2} \cap W_0^{1,3/2}$ with $\operatorname{div}\varphi = 0$. From (23), (24) and (26) it follows that

$$\begin{aligned} -\mu(\mathbf{u}^\varepsilon, \Delta\varphi) = {} & \mathcal{B}(\mathbf{u}^\varepsilon, \varphi, \mathbf{v}) + \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \mathbf{u}^\varepsilon) + a(\mathbf{w}, \operatorname{rot}\varphi) + (\mathbf{f}, \varphi) \\ & - \mu \int_\Gamma \mathbf{v}_0^\varepsilon \frac{\partial \varphi}{\partial \mathbf{n}} \, ds. \end{aligned}$$

Observe that $\mathbf{v}_0^\varepsilon$ is smooth and that $\mathbf{u}^\varepsilon$ belongs to H^1. We can integrate by parts on the left-hand side of this equation to get

$$\mu((\mathbf{u}^\varepsilon, \varphi)) = \mathcal{B}(\mathbf{u}^\varepsilon, \varphi, \mathbf{v}) + \mathcal{B}(\mathbf{v}^\varepsilon, \varphi, \mathbf{u}^\varepsilon) + a(\mathbf{w}, \operatorname{rot}\varphi) + (\mathbf{f}, \varphi). \tag{27}$$

Now we write $\mathbf{u}^\varepsilon$ in the form

$$\mathbf{u}^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{u}, \tag{28}$$

where $\widetilde{\mathbf{v}_0^\varepsilon}$ is a suitable Leray–Hopf extension of $\mathbf{v}_0^\varepsilon$ to Ω (cf. Lemma 4.1 below), and $\mathbf{u} \in \mathcal{V}$. From (27) and (28) we can derive the equation for $\mathbf{u}$. We also write

$$\mathbf{v} = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{u} + \mathbf{v}^\varepsilon = \mathbf{u} + V^\varepsilon, \tag{29}$$

where $V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$. We observe that V^ε belongs to L^3 and $V^\varepsilon|_\Gamma = \mathbf{v}_0$. Applying (28) and (29) to (27) we obtain

$$\begin{aligned}\mu((\mathbf{u}, \varphi)) &= \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) + \mathcal{B}(V^\varepsilon, \varphi, \mathbf{u}) + \mathcal{B}(\mathbf{u}, \varphi, V^\varepsilon) + a(\mathbf{w}, \operatorname{rot}\varphi)\\ &\quad + (\mathbf{f}, \varphi) - \mu\big(\big(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi\big)\big) + \mathcal{B}\big(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi, V^\varepsilon\big) + \mathcal{B}\big(\mathbf{v}^\varepsilon, \varphi, \widetilde{\mathbf{v}_0^\varepsilon}\big).\end{aligned}$$

Denote

$$\mathcal{L}(\mathbf{u}, \varphi) \stackrel{\text{def}}{=} \mathcal{B}(V^\varepsilon, \varphi, \mathbf{u}) + \mathcal{B}(\mathbf{u}, \varphi, V^\varepsilon), \quad V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon, \tag{30}$$

and

$$\begin{aligned}\langle \mathcal{F}, \varphi \rangle &\stackrel{\text{def}}{=} (\mathbf{f}, \varphi) - \mu\big(\big(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi\big)\big) + \mathcal{B}\big(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi\big) + \mathcal{B}\big(\widetilde{\mathbf{v}_0^\varepsilon}, \varphi, V^\varepsilon\big)\\ &\quad + \mathcal{B}\big(\mathbf{v}^\varepsilon, \varphi, \widetilde{\mathbf{v}_0^\varepsilon}\big).\end{aligned} \tag{31}$$

Then

$$\mu((\mathbf{u}, \varphi)) = \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) + \mathcal{L}(\mathbf{u}, \varphi) + a(\mathbf{w}, \operatorname{rot}\varphi) + \langle \mathcal{F}, \varphi \rangle \tag{32}$$

for all $\varphi \in W^{2,3/2} \cap W_0^{1,3/2}$ with $\operatorname{div}\varphi = 0$. If $\mathbf{u}$ is a solution of problem (32), then it is also a variational solution; that is,

$$\mu((\mathbf{u}, \varphi)) = \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) + \mathcal{L}(\mathbf{u}, \varphi) + a(\mathbf{w}, \operatorname{rot}\varphi) + \langle \mathcal{F}, \varphi \rangle \tag{33}$$

for all $\varphi \in \mathcal{V}$, as from (30), (31) we can see that $\mathcal{L}(\mathbf{u}, \varphi)$, $\langle \mathcal{F}, \varphi \rangle$, and $\mathcal{B}(\mathbf{u}, \varphi, \mathbf{u})$ are continuous in φ with respect to the H^1 topology.

Let us assume now that $\mathbf{u} \in \mathcal{V}$ is a solution of (33). From the above considerations it follows then that $\mathbf{v} = \mathbf{u} + V^\varepsilon$, $V^\varepsilon = \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, is a very weak solution of Problem 4.1.

In the next section we prove existence of a very weak solution of problem (1)–(2), where the velocity field is of the form $\mathbf{v} = \mathbf{u} + V^\varepsilon = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$, with $\mathbf{u} \in \mathcal{V}$, and with $V^\varepsilon \stackrel{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, $\mathbf{u}^\varepsilon = \mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon}$, suitably constructed on the basis of the boundary data $\mathbf{v}_0 \in L^2$. We will use the following lemmas.

Lemma 4.1 (Leray–Hopf extension). *Let Ω be an open connected and bounded set in $\mathbf{R}^3$ of class C^2 and $\mathbf{z}_0 \in H^{3/2}(\Gamma)$ with $\int_\Gamma \mathbf{z}_0 \cdot \mathbf{n}\, ds = 0$. Then for every $\sigma > 0$ there exists a function $\tilde{\mathbf{z}}_0$ such that $\tilde{\mathbf{z}}_0 \in H^2(\Omega)$, $\operatorname{div}\tilde{\mathbf{z}}_0 = 0$ in Ω, $\tilde{\mathbf{z}}_0 = \mathbf{z}_0$ on Γ and $|\mathcal{B}(\mathbf{u}, \tilde{\mathbf{z}}_0, \mathbf{u})| \leqslant \sigma \|\mathbf{u}\|^2$ for all $\mathbf{u} \in \mathcal{V}$.*

Proof. See [6, Chapter II, §1.4 and Appendix 1]. □

Lemma 4.2 (Marusič-Paloka). *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain in $\mathbf{R}^3$ with a boundary Γ of class C^2. Consider the following boundary value problem for the Navier–Stokes equations with data $\mathbf{g}$ in $L^2(\Gamma)$ satisfying $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,ds = 0$:*

$$\begin{cases} -\mu\Delta\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{z} = 0 & \text{in } \Omega, \\ \mathbf{z} = \mathbf{g} & \text{on } \Gamma. \end{cases}$$

If $|\mathbf{g}|_{L^2(\Gamma)}$ is sufficiently small, then there exists a unique very weak solution $\mathbf{z}$ in L^3 of the above problem. Furthermore, there is a constant c_1 depending only on μ such that

$$|\mathbf{z}|_3 < \frac{c_1\mu|\mathbf{g}|_{L^2(\Gamma)}}{\mu - c_1|\mathbf{g}|_{L^2(\Gamma)}}. \tag{34}$$

Proof. See Theorem 4 in [10]. □

5. Existence theorem

At the beginning of this section we shall show how to construct a map $\mathcal{A}:\mathcal{V}\to\mathcal{V}$ whose fixed point gives a very weak solution of (1)–(2) in the sense of Definition 2.1. Then we prove two lemmas which yield the proof of Theorem 2.1.

We start with $\mathbf{v}_0 \in L^2(\Gamma)$—the irregular boundary condition. We take a smooth approximation $\mathbf{v}_0^\varepsilon$ of $\mathbf{v}_0$ in $L^2(\Gamma)$ such that $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_{L^2(\Gamma)}$ is small enough with respect to μ, and let $\mathbf{v}^\varepsilon$ to be a very weak solution of (25) (cf. Lemma 4.2); we take $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_{L^2(\Gamma)}$ so small that the Problem 3.1 has a solution for each $\mathbf{u}^\varepsilon$ in H^1 and that the last inequality in (38) below holds true. Then we construct the Leray–Hopf extension $\widetilde{\mathbf{v}_0^\varepsilon}$ of $\mathbf{v}_0^\varepsilon$ satisfying

$$\mathcal{B}\big(\mathbf{u}, \widetilde{\mathbf{v}_0^\varepsilon}, \mathbf{u}\big) \leqslant \frac{\mu}{8}\|\mathbf{u}\|^2 \tag{35}$$

for all $\mathbf{u} \in \mathcal{V}$ (cf. Lemma 4.1).

Now, for $\mathbf{u} \in \mathcal{V}$, we define $\mathbf{v} = \mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon = \mathbf{u}^\varepsilon + \mathbf{v}^\varepsilon$, $\mathbf{u}^\varepsilon \overset{\text{def}}{=} \mathbf{u} + \widetilde{\mathbf{v}_0^\varepsilon}$, and for this $\mathbf{v}$ we solve Problem 3.1 in $\mathbf{w}$. Having $\mathbf{w}$—the unique solution of Problem 3.1—we can define $\mathcal{A}(\mathbf{u}) \in \mathcal{V}$ by the relation

$$E\big(\mathcal{A}(\mathbf{u}), \varphi\big) = a(\mathbf{w}, \operatorname{rot}\varphi) + \langle\mathcal{F}, \varphi\rangle + \mathcal{B}(\mathbf{u}, \varphi, \mathbf{u}) \tag{36}$$

for all $\varphi \in \mathcal{V}$, where $E(\mathbf{u}, \varphi) \overset{\text{def}}{=} \mu((\mathbf{u}, \varphi)) - \mathcal{L}(\mathbf{u}, \varphi)$ ($\mathcal{L}$ defined in (30)) is continuous and coercive under our assumptions. For each $\mathbf{w} \in L^2$ and $\mathbf{u} \in \mathcal{V}$ the right-hand side of (36) defines a linear and bounded functional in φ on $\mathcal{V}$. Thus, by the Lax–Milgram lemma, the map $\mathcal{A}$ is well defined.

Observe that each fixed point $\mathbf{u}$ of the map $\mathcal{A}$ defines a pair $(\mathbf{v}, \mathbf{w}) = (\mathbf{u} + V^\varepsilon, \mathbf{w})$, $V^\varepsilon \overset{\text{def}}{=} \widetilde{\mathbf{v}_0^\varepsilon} + \mathbf{v}^\varepsilon$, which is a very weak solution of (1)–(2). Using the De

Rham lemma we show then that there exists a $p \in W^{-1,3}$ such that the triple $(\mathbf{v}, \mathbf{w}, p)$ satisfies all conditions in Definition 2.1.

We can prove that the operator $\mathcal{A}$ is completely continuous and that for $a \leqslant c^*\mu$, with some constant c^*, all $\mathbf{u} \in \mathcal{V}$ such that for some $\lambda \in [0,1]$ it is $\mathbf{u} = \lambda\mathcal{A}\mathbf{u}$ are contained in a ball $\|\mathbf{u}\| \leqslant M$. The existence of a fixed point of $\mathcal{A}$ follows then from the Leray–Schauder fixed point theorem.

Lemma 5.1. *If a is small enough, $a \leqslant c^*\mu$ with some constant c^*, then there exists a constant $M > 0$ such that for all $\mathbf{u} \in \mathcal{V}$ satisfying the equation $\mathbf{u} = \lambda\mathcal{A}\mathbf{u}$ for some $\lambda \in [0,1]$ we have $\|\mathbf{u}\| \leqslant M$.*

Proof. If $\lambda = 0$, then $\mathbf{u} = 0$. Now, if $0 < \lambda \leqslant 1$, then setting $\mathcal{A}\mathbf{u} = (1/\lambda)\mathbf{u}$ in (36) with $\varphi = \mathbf{u}$, we obtain

$$\mu\|\mathbf{u}\|^2 - \mathcal{L}(\mathbf{u},\mathbf{u}) = \lambda\big\{a(\mathbf{w}, \operatorname{rot}\mathbf{u}) + \langle\mathcal{F}, \mathbf{u}\rangle\big\}. \tag{37}$$

By the definition of $\mathcal{L}$ (see (30)) together with the fact that div $V^\varepsilon = 0$ and $V^\varepsilon = \widetilde{\mathbf{v}}_0^\varepsilon + \mathbf{v}^\varepsilon$, and by the estimates (35) and (34) in Lemma 4.2 with $\mathbf{g} = \mathbf{v}_0 - \mathbf{v}_0^\varepsilon$ (cf. problem (25)), we have

$$\begin{aligned}\big|\mathcal{L}(\mathbf{u},\mathbf{u})\big| &= \big|\mathcal{B}(\mathbf{u},\mathbf{u},V^\varepsilon)\big| = \big|\mathcal{B}\big(\mathbf{u},\mathbf{u},\widetilde{\mathbf{v}}_0^\varepsilon\big) + \mathcal{B}(\mathbf{u},\mathbf{u},\mathbf{v}^\varepsilon)\big| \\ &= \big|-\mathcal{B}\big(\mathbf{u},\widetilde{\mathbf{v}}_0^\varepsilon,\mathbf{u}\big) + \mathcal{B}(\mathbf{u},\mathbf{u},\mathbf{v}^\varepsilon)\big| \leqslant \frac{\mu}{8}\|\mathbf{u}\|^2 + c|\mathbf{v}^\varepsilon|_3\|\mathbf{u}\|^2 \\ &\leqslant \left(\frac{\mu}{8} + c\frac{c_1\mu|\mathbf{v}_0^\varepsilon - \mathbf{v}_0|_{L^2(\Gamma)}}{\mu - c_1|\mathbf{v}_0^\varepsilon - \mathbf{v}_0|_{L^2(\Gamma)}}\right)\|\mathbf{u}\|^2 \leqslant \frac{\mu}{4}\|\mathbf{u}\|^2\end{aligned} \tag{38}$$

for $|\mathbf{v}_0^\varepsilon - \mathbf{v}_0|_{L^2(\Gamma)}$ sufficiently small with respect to μ. Also, by (8),

$$\begin{aligned}a(\mathbf{w},\operatorname{rot}\mathbf{u}) &\leqslant a|\mathbf{w}|\|\mathbf{u}\| \leqslant ac\big(1 + \|\mathbf{u}^\varepsilon\|_1\big)\|\mathbf{u}\| \\ &\leqslant ac\big(1 + \|\mathbf{u}\|_1 + \big\|\widetilde{\mathbf{v}}_0^\varepsilon\big\|_1\big)\|\mathbf{u}\| \\ &\leqslant ac\big(1 + \|\mathbf{u}\| + \big\|\widetilde{\mathbf{v}}_0^\varepsilon\big\|_1\big)\|\mathbf{u}\| \\ &\leqslant ac\|\mathbf{u}\|^2 + ac'\|\mathbf{u}\| \leqslant \frac{\mu}{4}\|\mathbf{u}\|^2 + ac'\|\mathbf{u}\|\end{aligned} \tag{39}$$

for $ac \leqslant \mu/4$ (we can set $c^* = 1/(4c)$) and, by the definition of $\mathcal{F}$ (see (31)),

$$\langle\mathcal{F},\mathbf{u}\rangle = (\mathbf{f},\mathbf{u}) - \mu\big(\big(\widetilde{\mathbf{v}}_0^\varepsilon,\mathbf{u}\big)\big) + \mathcal{B}\big(\widetilde{\mathbf{v}}_0^\varepsilon,\mathbf{u},V^\varepsilon\big) + \mathcal{B}\big(\mathbf{v}^\varepsilon,\mathbf{u},\widetilde{\mathbf{v}}_0^\varepsilon\big) \leqslant c\|\mathbf{u}\|. \tag{40}$$

From (37), together with (38)–(40), we obtain the desired result. □

Lemma 5.2. *The operator $\mathcal{A}$ is completely continuous.*

Proof. Let $(\mathbf{u}_n)$ be a bounded sequence in $\mathcal{V}$. We shall show that then $(\mathcal{A}\mathbf{u}_{n_k})$ is a Cauchy sequence in $\mathcal{V}$ (for a subsequence (n_k)). Let

$$E(\mathcal{A}\mathbf{u}_m, \varphi) = a(\mathbf{w}_m, \operatorname{rot}\varphi) + \langle \mathcal{F}, \varphi\rangle + \mathcal{B}(\mathbf{u}_m, \varphi, \mathbf{u}_m), \tag{41}$$

$$E(\mathcal{A}\mathbf{u}_n, \varphi) = a(\mathbf{w}_n, \operatorname{rot}\varphi) + \langle \mathcal{F}, \varphi\rangle + \mathcal{B}(\mathbf{u}_n, \varphi, \mathbf{u}_n) \tag{42}$$

for all $\varphi \in \mathcal{V}$, where

$$\begin{aligned}&\big(\mathbf{w}_m, -\alpha\Delta\psi + (\mathbf{v}_m\cdot\nabla)\psi - \beta\nabla\operatorname{div}\psi + \gamma\psi\big)\\ &\quad = a(\mathbf{v}_m, \operatorname{rot}\psi) + (\mathbf{g}, \psi) - \alpha\int_\Gamma \mathbf{w}_0\frac{\partial\psi}{\partial\mathbf{n}}\,ds - \beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\operatorname{div}\psi\,ds,\end{aligned} \tag{43}$$

$$\begin{aligned}&\big(\mathbf{w}_n, -\alpha\Delta\psi + (\mathbf{v}_n\cdot\nabla)\psi - \beta\nabla\operatorname{div}\psi + \gamma\psi\big)\\ &\quad = a(\mathbf{v}_n, \operatorname{rot}\psi) + (\mathbf{g}, \psi) - \alpha\int_\Gamma \mathbf{w}_0\frac{\partial\psi}{\partial\mathbf{n}}\,ds - \beta\int_\Gamma(\mathbf{w}_0\cdot\mathbf{n})\operatorname{div}\psi\,ds,\end{aligned} \tag{44}$$

$\mathbf{v}_m = \mathbf{u}_m + V^\varepsilon$, $\mathbf{v}_n = \mathbf{u}_n + V^\varepsilon$, $V^\varepsilon \overset{\text{def}}{=} \tilde{\mathbf{v}}_0^\varepsilon + \mathbf{v}^\varepsilon$. Taking the difference of (41) and (42) we obtain

$$\begin{aligned}E(\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n, \varphi) &= a(\mathbf{w}_m - \mathbf{w}_n, \operatorname{rot}\varphi)\\ &\quad + \mathcal{B}(\mathbf{u}_m - \mathbf{u}_n, \varphi, \mathbf{u}_n) + \mathcal{B}(\mathbf{u}_m, \varphi, \mathbf{u}_m - \mathbf{u}_n).\end{aligned} \tag{45}$$

Set $\varphi = \mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n$ and we have

$$\begin{aligned}\frac{3}{4}\mu\|\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n\|^2 &\leqslant a|\mathbf{w}_m - \mathbf{w}_n|\|\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n\|\\ &\quad + c\big(\|\mathbf{u}_m\| + \|\mathbf{u}_n\|\big)\|\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n\||\mathbf{u}_m - \mathbf{u}_n|_3,\end{aligned}$$

where for obtaining the left-hand side we used $E(\mathbf{u}, \varphi) \overset{\text{def}}{=} \mu((\mathbf{u}, \varphi)) - \mathcal{L}(\mathbf{u}, \varphi)$ and the estimate for $\mathcal{L}(\mathbf{u}, \mathbf{u})$ in (38). Thus

$$\frac{3}{4}\mu\|\mathcal{A}\mathbf{u}_m - \mathcal{A}\mathbf{u}_n\| \leqslant a|\mathbf{w}_m - \mathbf{w}_n| + c\big(\|\mathbf{u}_m\| + \|\mathbf{u}_n\|\big)|\mathbf{u}_m - \mathbf{u}_n|_3. \tag{46}$$

Now, as $(\mathbf{u}_m)$ is a bounded sequence in $\mathcal{V}$, there exists a subsequence (we denote it also by $(\mathbf{u}_n)$) that is convergent is L^3. Moreover, in view of Lemma 3.1, $(\mathbf{w}_m)$ converges in L^2. Thus, by (46), $(\mathcal{A}\mathbf{u}_n)$ is a Cauchy sequence in $\mathcal{V}$. In consequence, the operator $\mathcal{A}$ is compact. Observe that from inequality (46) the continuity of $\mathcal{A}$ in $\mathcal{V}$ immediately follows. □

6. Continuous dependence

In this section we prove Theorem 2.2. Let

$$\mu((\mathbf{u}_i, \phi)) = \mathcal{B}(\mathbf{u}_i, \phi, \mathbf{u}) + \mathcal{L}(\mathbf{u}_i, \phi) + a(\mathbf{w}_i, \operatorname{rot}\phi) + \langle \mathcal{F}_i, \phi\rangle, \tag{47}$$

where

$$\mathcal{L}(\mathbf{u}_i, \phi) \overset{\text{def}}{=} \mathcal{B}(V^\varepsilon, \phi, \mathbf{u}_i) + \mathcal{B}(\mathbf{u}_i, \phi, V^\varepsilon), \qquad V^\varepsilon \overset{\text{def}}{=} \tilde{\mathbf{v}}_0^\varepsilon + \mathbf{v}^\varepsilon, \tag{48}$$

and

$$\langle \mathcal{F}_i, \phi\rangle \stackrel{\text{def}}{=} (\mathbf{f}_i, \phi) - \mu\big(\big(\widetilde{\mathbf{v}}_0^\varepsilon, \phi\big)\big) + \mathcal{B}\big(\widetilde{\mathbf{v}}_0^\varepsilon, \phi\big) + \mathcal{B}\big(\widetilde{\mathbf{v}}_0^\varepsilon, \phi, V^\varepsilon\big) + \mathcal{B}\big(\mathbf{v}^\varepsilon, \phi, \widetilde{\mathbf{v}}_0^\varepsilon\big) \tag{49}$$

for $i = 1, 2$ and $\phi \in H_0^1$. We recall that $\mathbf{v}^\varepsilon$ is the very weak solution of (25) with $\mathbf{v}^\varepsilon|_\Gamma = \mathbf{v}_0 - \mathbf{v}_0^\varepsilon$, where $\mathbf{v}_0^\varepsilon$ is a smooth approximation of $\mathbf{v}_0$ such that $\operatorname{div}\mathbf{v}_0^\varepsilon = 0$, $|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon|_3$ is very small with respect to μ (cf. (38)), and $\widetilde{\mathbf{v}}_0^\varepsilon$ is a Leray–Hopf extension of $\mathbf{v}_0^\varepsilon$ satisfying (35). From (38) we have

$$\mathcal{L}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leqslant \frac{\mu}{4}\|\mathbf{u}_1 - \mathbf{u}_2\|^2. \tag{50}$$

Then, writing (47) for $i = 1, 2$, taking the difference and setting $\phi = \mathbf{u}_1 - \mathbf{u}_2$, we obtain

$$\begin{aligned}
&\frac{3}{4}\mu\|\mathbf{u}_1 - \mathbf{u}_2\|^2 \\
&\quad\leqslant \mathcal{B}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2) + a\big(\mathbf{w}_1 - \mathbf{w}_2, \operatorname{rot}(\mathbf{u}_1 - \mathbf{u}_2)\big) \\
&\qquad + (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{u}_1 - \mathbf{u}_2) \\
&\quad\leqslant c\|\mathbf{u}_2\|\|\mathbf{u}_1 - \mathbf{u}_2\|^2 + a|\mathbf{w}_1 - \mathbf{w}_2|\|\mathbf{u}_1 - \mathbf{u}_2\| + c|\mathbf{f}_1 - \mathbf{f}_2|\|\mathbf{u}_1 - \mathbf{u}_2\|,
\end{aligned}$$

whence

$$\frac{3}{4}\mu\|\mathbf{u}_1 - \mathbf{u}_2\| \leqslant c\|\mathbf{u}_2\|\|\mathbf{u}_1 - \mathbf{u}_2\| + a|\mathbf{w}_1 - \mathbf{w}_2| + c|\mathbf{f}_1 - \mathbf{f}_2|.$$

From Lemma 5.1 we have that $\|\mathbf{u}_2\| \leqslant M$, where M is a constant that does not increase with μ; thus for μ large enough such that $c\|\mathbf{u}_2\| \leqslant \mu/4$, we obtain

$$\frac{\mu}{2}\|\mathbf{u}_1 - \mathbf{u}_2\| \leqslant a|\mathbf{w}_1 - \mathbf{w}_2| + c|\mathbf{f}_1 - \mathbf{f}_2|. \tag{51}$$

Now, we use Eq. (17). Assume at first that $\mathbf{w}_{01} = \mathbf{w}_{02}$. Then from (17) written for $\mathbf{w} = \mathbf{w}_i$, $i = 1, 2$, we have

$$\begin{aligned}
(\mathbf{w}_1, h_i) = a(\mathbf{v}_1, \operatorname{rot}\psi_i) + (\mathbf{g}_1, \psi_i) - \alpha \int_\Gamma \mathbf{w}_0 \frac{\partial \psi_i}{\partial \mathbf{n}}\, ds \\
- \beta \int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \operatorname{div}\psi_i\, ds,
\end{aligned} \tag{52}$$

where $h_i = L_{\mathbf{v}_i}(\psi_i)$ for $L_\mathbf{v} \stackrel{\text{def}}{=} -\alpha\Delta\psi - (\mathbf{v}\cdot\nabla)\psi - \beta\nabla\operatorname{div}\psi + \gamma\psi$. Making the difference in (52) for $i = 1, 2$, we obtain

$$\begin{aligned}
|\mathbf{w}_1 - \mathbf{w}_2|^2 = a(\mathbf{v}_1 - \mathbf{v}_2, \operatorname{rot}\psi_2) + a\big(\mathbf{v}_1, \operatorname{rot}(\psi_1 - \psi_2)\big) + (\mathbf{g}_1 - \mathbf{g}_2, \psi_2) \\
+ (\mathbf{g}_1, \psi_1 - \psi_2) + \alpha \int_\Gamma \mathbf{w}_0 \frac{\partial}{\partial \mathbf{n}}(\psi_1 - \psi_2)\, ds \\
+ \beta \int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \operatorname{div}(\psi_1 - \psi_2)\, ds.
\end{aligned} \tag{53}$$

Now, we estimate the terms on the right-hand side of (53). The first term is easily estimated:

$$a(\mathbf{v}_1 - \mathbf{v}_2, \operatorname{rot}\psi_2) = a(\mathbf{u}_1 - \mathbf{u}_2, \operatorname{rot}\psi_2) = a\big(\operatorname{rot}(\mathbf{u}_1 - \mathbf{u}_2), \psi_2\big) \leqslant a\|\mathbf{u}_1 - \mathbf{u}_2\||\psi_2| \leqslant \frac{a}{\gamma}\|\mathbf{u}_1 - \mathbf{u}_2\||\mathbf{w}_1 - \mathbf{w}_2|, \tag{54}$$

where we used (11). The second term can be estimated as follows:

$$a\big(\mathbf{v}_1, \operatorname{rot}(\psi_1 - \psi_2)\big) \leqslant a|\mathbf{v}_1|\|\psi_1 - \psi_2\|. \tag{55}$$

We have $-\alpha\Delta\psi_i + (\mathbf{v}_1 \cdot \nabla)\psi_i - \beta\nabla \operatorname{div}\psi_i + \gamma\psi_i = \mathbf{w}_1 - \mathbf{w}_2$, $i = 1, 2$; then making the difference for $i = 1, 2$, we get

$$-\alpha\Delta(\psi_1 - \psi_2) + (\mathbf{v}_1 \cdot \nabla)(\psi_1 - \psi_2) - \beta\nabla \operatorname{div}(\psi_1 - \psi_2) + \gamma(\psi_1 - \psi_2) = -\big((\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla\big)\psi_2 = -\big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla\big)\psi_2. \tag{56}$$

Multiplying by $\psi_1 - \psi_2$ and integrating in Ω we obtain, in particular, $\|\psi_1 - \psi_2\| \leqslant c\|\mathbf{u}_1 - \mathbf{u}_2\|\|\psi_2\|$, so, using again (11), it follows that

$$\|\psi_1 - \psi_2\| \leqslant c\|\mathbf{u}_1 - \mathbf{u}_2\||\mathbf{w}_1 - \mathbf{w}_2|.$$

Using this estimate in (55) we obtain

$$\big|a\big(\mathbf{v}_1, \operatorname{rot}(\psi_1 - \psi_2)\big)\big| \leqslant c|\mathbf{v}_1|\|\mathbf{u}_1 - \mathbf{u}_2\||\mathbf{w}_1 - \mathbf{w}_2|. \tag{57}$$

Next, we have

$$(\mathbf{g}_1 - \mathbf{g}_2, \psi_2) \leqslant |\mathbf{g}_1 - \mathbf{g}_2|\gamma^{-1}|\mathbf{w}_1 - \mathbf{w}_2| \tag{58}$$

and

$$(\mathbf{g}_1, \psi_1 - \psi_2) \leqslant c|\mathbf{g}_1|\|\psi_1 - \psi_2\| \leqslant c|\mathbf{g}_1|\|\mathbf{u}_1 - \mathbf{u}_2\||\mathbf{w}_1 - \mathbf{w}_2|. \tag{59}$$

The boundary integrals give, by (56) and (12),

$$\begin{aligned}\alpha\int_\Gamma \mathbf{w}_0\frac{\partial}{\partial\mathbf{n}}(\psi_1 - \psi_2)\,ds &\leqslant \alpha|\mathbf{w}_0|_{L^2(\Gamma)}c\|\psi_1 - \psi_2\|_2\\ &\leqslant c|\mathbf{w}_0|_{L^2(\Gamma)}\big|\big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla\big)\psi_2\big|\\ &\leqslant c|\mathbf{w}_0|_{L^2(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\|\|\psi_2\|_2\\ &\leqslant c|\mathbf{w}_0|_{L^2(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\|\big(1 + \|\mathbf{u}_2\|^2\big)|\mathbf{w}_1 - \mathbf{w}_2|\\ &\leqslant c|\mathbf{w}_0|_{L^2(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\|(1 + M^2)|\mathbf{w}_1 - \mathbf{w}_2|\\ &= c|\mathbf{w}_0|_{L^2(\Gamma)}\|\mathbf{u}_1 - \mathbf{u}_2\||\mathbf{w}_1 - \mathbf{w}_2|\end{aligned} \tag{60}$$

and

$$\beta\int_\Gamma (\mathbf{w}_0 \cdot \mathbf{n}) \operatorname{div}(\psi_1 - \psi_2)\,ds \leqslant \beta|\mathbf{w}_0|_{L^2(\Gamma)}c\|\mathbf{u}_1 - \mathbf{u}_2\||\mathbf{w}_1 - \mathbf{w}_2|. \tag{61}$$

From (53)–(61) we obtain

$$|\mathbf{w}_1 - \mathbf{w}_2| \leqslant c\big(\|\mathbf{u}_1 - \mathbf{u}_2\| + |\mathbf{g}_1 - \mathbf{g}_2|\big). \tag{62}$$

Using this estimate in (51) we have $\|\mathbf{u}_1 - \mathbf{u}_2\| \leqslant c(|\mathbf{g}_1 - \mathbf{g}_2| + |\mathbf{f}_1 - \mathbf{f}_2|)$ for μ large enough. Then, from (62), it follows an estimate of the same type for $|\mathbf{w}_1 - \mathbf{w}_2|$. Therefore, we can write

$$|\mathbf{v}_1 - \mathbf{v}_2|_3 + |\mathbf{w}_1 - \mathbf{w}_2| \leqslant c\big(|\mathbf{f}_1 - \mathbf{f}_2| + |\mathbf{g}_1 - \mathbf{g}_2|\big), \tag{63}$$

as $|\mathbf{v}_1 - \mathbf{v}_2|_3 = |\mathbf{u}_1 - \mathbf{u}_2|_3 \leqslant c\|\mathbf{u}_1 - \mathbf{u}_2\|$. Estimate (63) gives the continuous dependence of solutions $(\mathbf{v}, \mathbf{w})$ on the data $\mathbf{f}$, $\mathbf{g}$, provided μ is large enough.

Now, to prove (6), we observe that if $\mathbf{w}_{01} \neq \mathbf{w}_{02}$, then in (52) we have $\mathbf{w}_{01}$ and $\mathbf{w}_{02}$ instead of $\mathbf{w}_0$, respectively, and subtracting these equations we obtain two new terms, namely

$$\alpha \int_\Gamma (\mathbf{w}_{01} - \mathbf{w}_{02}) \frac{\partial}{\partial \mathbf{n}} \psi_2 \, ds \quad \text{and} \quad \beta \int_\Gamma \big((\mathbf{w}_{01} - \mathbf{w}_{02}) \cdot \mathbf{n}\big) \operatorname{div} \psi_2 \, ds,$$

which can be estimate from above by $c|\mathbf{w}_{01} - \mathbf{w}_{02}||\mathbf{w}_1 - \mathbf{w}_2|$, whence we have (6) in view of the above considerations.

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