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2m-Weak amenability of group algebras

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ABSTRACT

A common fixed point property for semigroups is applied to show that the group algebra $L^1(G)$ of a locally compact group G is 2m-weakly amenable for each integer $m \geq 1$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A linear mapping $D: \mathcal{A} \rightarrow X$ is called a *derivation* if it satisfies $D(ab) = aD(b) + D(a)b$ $(a, b \in \mathcal{A})$. Given any $x \in X$, the mapping $\mathrm{ad}_x: a \mapsto ax - \alpha(a \in \mathcal{A})$ is a continuous derivation, called an *inner derivation*.

If X is a Banach $\mathcal{A}$-bimodule, then the dual space X^* of X is naturally a Banach $\mathcal{A}$-bimodule with the $\mathcal{A}$-module actions defined by

$$\langle x, af \rangle = \langle xa, f \rangle \qquad \langle x, fa \rangle = \langle ax, f \rangle \quad (a \in \mathcal{A}, f \in X^*, x \in X).$$

Note that the Banach algebra $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule with the product giving the module actions. So $\mathcal{A}^{(n)}$, the n-th dual space of $\mathcal{A}$, is naturally a Banach $\mathcal{A}$-bimodule in the above sense for each $n \in \mathbb{N}$. The Banach algebra $\mathcal{A}$ is called *n-weakly amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is inner. If $\mathcal{A}$ is n-weakly amenable for each $n \in \mathbb{N}$ then it is called *permanently weakly amenable*.

Let G be a locally compact group. The integral of a function f on a measurable subset K of G against a fixed left Haar measure is denoted by $\int_K f dx$. Two functions on G are regarded identical if they are equal to each other almost everywhere with respect to the left Haar measure. The group algebra $L^1(G)$ is the Banach algebra consisting of all absolutely integrable functions on G (with respect to the left Haar measure), equipped with the convolution product and the usual L^1 norm

$$\|f\|_1 := \int_G |f(t)| dt.$$

When G is discrete, $L^1(G)$ is $\ell^1(G)$ consisting of all absolutely summable functions on G.

Johnson showed in [1] that $L^1(G)$ is always 1-weakly amenable for any locally compact group G. It was shown further in [2] that $L^1(G)$ is in fact n-weakly amenable for all odd numbers n. Whether this is still true for even numbers n was left

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open in [2]. For a free group G, Johnson proved later in [3] that $\ell^1(G)$ is indeed $2m$-weakly amenable for any $m \in \mathbb{N}$. The problem has been resolved affirmatively for general locally compact group G in [4] and in [5] independently, using a theory established in [6].

In this note we present a short proof to the n-weak amenability of $L^1(G)$ for even numbers n. Our proof is based on a common fixed point property for semigroups. In Section 2 we study this fixed point property. For the general theory concerning amenability and fixed point properties of locally compact groups we refer the reader to [7,8]. The proof to the main result will be given in Section 3.

2. Common fixed points for semigroups

Let S be a (discrete) semigroup. The space of all bounded complex valued functions on S is denoted by $\ell^\infty(S)$. It is a Banach space with the uniform supremum norm. In fact $\ell^\infty(S) = (\ell^1(S))^*$, the dual space of $\ell^1(S)$. For each $s \in S$ and each $f \in \ell^\infty(S)$ let $\ell_s f$ be the left translate of f by s, that is $\ell_s f(t) = f(st)$ $(t \in S)$ (the right translate $r_s f$ is defined similarly). A function $f \in \ell^\infty(S)$ is called *weakly almost periodic* if its left orbit $\mathcal{LO}(f) = \{\ell_s f : s \in S\}$ is relatively compact in the weak topology of $\ell^\infty(S)$. The space of all weakly almost periodic functions on S is denoted by $WAP(S)$, which is a closed subspace of $\ell^\infty(S)$ containing the constant function and invariant under the left and right translations. A linear functional $m \in WAP(S)^*$ is a *mean* on $WAP(S)$ if $\|m\| = m(1) = 1$. A mean m on $WAP(S)$ is a *left invariant mean* (abbreviated as LIM) if $m(\ell_s f) = m(f)$ for all $s \in S$ and all $f \in WAP(S)$. If S is a group, it is well known that $WAP(S)$ always has a LIM [7].

Let X be a Banach space and C a nonempty subset of X. A mapping $T: C \to C$ is called *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. When X is a separable locally convex topological space whose topology is determined by a family Q of seminorms on X, we will denote it by (X, Q) to highlight the topology Q.

Let C be a subset of a locally convex topological vector space (X, Q). We say that $\mathfrak{S} = \{T_s : s \in S\}$ is a *representation* of S on C if for each $s \in S$, T_s is a mapping from C into C and $T_{st}(x) = T_s(T_t x)$ $(s, t \in S, x \in C)$. The representation is called continuous if each T_s $(s \in S)$ is Q–Q continuous; It is called equicontinuous if for each neighborhood $\mathcal{N}$ of 0 there is a neighborhood $\mathcal{O}$ of 0 such that $T_s(x) - T_s(y) \in \mathcal{N}$ whenever $x, y \in C$, $x - y \in \mathcal{O}$ and $s \in S$. The representation is called *affine* if C is convex and each T_s $(s \in S)$ is an affine mapping, that is $T_s(ax + by) = aT_s(x) + bT_s(y)$ for all constants $a, b \geq 0$ with $a + b = 1$, $s \in S$ and $x, y \in C$. We say that $x \in C$ is a *common fixed point* for (the representation of) S if $T_s(x) = x$ for all $s \in S$.

The following fixed point theorem was proved in [9].

Theorem 1. *Let S be a discrete semigroup and $\mathfrak{S}$ an equicontinuous affine representation of S on a weakly compact convex subset C of a separated locally convex space X. If $WAP(S)$ has a left invariant mean then C contains a common fixed point for S.*

Let B be a nonempty bounded subset of a Banach space X. By definition the *Chebyshev radius* of B in X is

$$r_B = \inf\left\{r \geq 0 : \exists x \in X \ \sup_{b \in B} \|x - b\| \leq r\right\}.$$

Clearly we have $0 \leq r_B < \infty$ and

$$\sup_{b \in B} \|x - b\| \geq r_B \quad \text{for each } x \in X. \tag{2.1}$$

The *Chebyshev center* of B in X is defined to be

$$C_B = \left\{x \in X : \sup_{b \in B} \|x - b\| \leq r_B\right\}.$$

Chebyshev center has been extensively used in the field of fixed point theory (see [10,11]). Some asymptotic version of it has been employed to study fixed point properties of semigroups [12–14].

We now recall that a Banach space X is *L-embedded* if the image of X under the canonical embedding into its bidual X^{**}, still denoted by X, is an ℓ_1 summand in X^{**}, that is if there is a subspace X_s of X^{**} such that $X^{**} = X \oplus_1 X_s$, where $\oplus_1$ denotes the ℓ_1 direct sum. The class of L-embedded Banach spaces includes all $L^1(\Sigma, \mu)$ (the space of all absolutely integrable functions on a measure space (Σ, μ)), preduals of von Neumann algebras, dual spaces of M-embedded Banach spaces and the Hardy space H_1. In particular, given a locally compact group G, the space $L^1(G)$ is L-embedded. So are its even duals $L^1(G)^{(2m)}$ $(m \in \mathbb{N})$. We refer to [15] for more details of the theory concerning this type of Banach spaces. We also refer to [16–18] for the study of fixed points of various mappings in an L-embedded Banach space. In [16], as an application of a fixed point theorem, a surprising short solution to the well-known derivation problem was given. The problem was first settled by Losert in [6].

We now give a common fixed point theorem for semigroups, which will provide the major machinery for our proof to the main result.

Theorem 2. *Let S be a discrete semigroup and $\mathfrak{S}$ a representation of S on an L-embedded Banach space X as nonexpansive affine mappings. Suppose that $WAP(S)$ has a LIM and suppose that there is a nonempty bounded set $B \subset X$ such that $B \subseteq \overline{T_s(B)}$ for all $s \in S$, then X contains a common fixed point for S.*

Proof. We use the idea of [16] to show that there is a nonempty weakly compact convex set in X that is S-invariant. We first regard B as a subset of X^{**}. Let r_B be the Chebyshev radius and C the Chebyshev center of B in X^{**}. Then C is nonempty, weak* compact and convex. In fact, for each $r > r_B$,

$$C_r := \left\{x \in X^{**} : \sup_{b\in B} \|x - b\| \le r\right\}$$

is nonempty by the definition of r_B. Note that $C_r = \cap_{b\in B} B[b, r]$, where $B[b, r]$ denotes the closed ball in X^{**} centered at b with radius r. The set C_r is convex and weak* compact since each $B[b, r]$ is. The collection $\{C_r : r > r_B\}$ is decreasing as r decreases. Thus $C = \cap_{r>r_B} C_r$ is nonempty and is still weak* compact and convex. By the L-embeddedness of X there is a subspace X_s of X^{**} such that $X^{**} = X \oplus_1 X_s$. Let $x \in C$. then there are $c \in X$ and $\xi \in X_s$ such that $x = c + \xi$. For each $b \in B$, $\|x - b\| = \|c - b\| + \|\xi\|$. So

$$r_B \ge \sup_{b\in B} \|x - b\| = \sup_{b\in B} \|c - b\| + \|\xi\| \ge r_B + \|\xi\|.$$

The last inequality is due to (2.1). Therefore, we must have $\xi = 0$. This shows that $C \subset X$. The weak* compactness of C (in X^{**}) is the same as the weak compactness of it (in X). So C is a nonempty, weakly compact and convex subset of X.

Now for $s \in S$, $b \in B$ and $x \in C$ we have

$$\|T_s(x) - T_s(b)\| \le \|x - b\| \le r_B$$

since T_s is nonexpansive. This implies that $\|T_s(x) - a\| \le r_B$ for $a \in \overline{T_s(B)}$ $(s \in S, x \in C)$. In particular, this holds for all $a \in B$ since $B \subseteq \overline{T_s(B)}$. Thus $T_s(x) \in C$ whenever $x \in C$ and $s \in S$, showing that C is S-invariant. Note that a nonexpansive representation of S is indeed equicontinuous. By Theorem 1, there is a common fixed point for S in C. The proof is complete. □

Theorem 1 has been extended to the general semitopological semigroup setting in [19]. A more general version of Theorem 2 and some discussion on when there is a set B such that $T_s(B) = B$ for all $s \in S$ can also be found there.

3. 2m-weak amenability of $L^1(G)$

Let X be a Banach space. Denote the space of all bounded linear operators on X by $B(X)$. The space $B(X)$ is a Banach algebra with the operator norm topology and the composition product. So is $B(X) \times B(X)^{op}$ with the product topology and coordinatewise operations, where $B(X)^{op}$ is the algebra formed by reversing the order of the product in $B(X)$. The *strong operator topology* (or briefly *so*-topology) on $B(X) \times B(X)^{op}$ is the topology induced by the family of seminorms $\{p_x : x \in X\}$, where

$$p_x(S, T) = \max\{\|S(x)\|, \|T(x)\|\} \quad (S, T \in B(X))$$

(see [20, p. 327]).

Given a locally compact group G, let $M(G)$ be the space of all bounded complex valued regular Borel measures on G. With the convolution product of measures and with the norm induced by the total variation, $M(G)$ is a Banach algebra containing $L^1(G)$ as a closed ideal. In fact, $M(G)$ is the multiplier algebra of $L^1(G)$, and as the multiplier algebra of $L^1(G)$, $M(G)$ is a subalgebra of $B(L^1(G)) \times B(L^1(G))^{op}$ with each $\mu \in M(G)$ being identified with (the double multiplier) $(\ell_\mu, r_\mu) \in B(L^1(G)) \times B(L^1(G))^{op}$, where ℓ_μ and r_μ denote, respectively, the left multiplier operator and the right multiplier operator on $L^1(G)$ implemented by μ. We refer to [20] for the standard theory about multipliers and multiplier algebras.

It is well-known that $lin\{\delta_t : t \in G\}$, the linear space generated by the point measures δ_t $(t \in G)$, is dense in $M(G)$ in the *so*-topology [20, Proposition 3.3.41(i)]. In particular, for each $h \in L^1(G)$ there is a net $(u_\alpha) \subset lin\{\delta_t : t \in G\}$ such that $\|(u_\alpha - h) * a\|_1 \to 0$ and $\|a * (u_\alpha - h)\|_1 \to 0$ for all $a \in L^1(G)$.

Recall that if $\mathcal{A}$ is a Banach algebra, then its bidual $\mathcal{A}^{**}$ is a Banach algebra equipped with the Arens product $\Box$ defined

$$\langle f, u\Box v\rangle = \langle v \cdot f, u\rangle, \quad v \cdot f \in \mathcal{A}^* : \qquad \langle a, v \cdot f\rangle = \langle fa, v\rangle$$

for $u, v \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. If X is a Banach $\mathcal{A}$-bimodule, then its bidual X^{**} is naturally a Banach $\mathcal{A}^{**}$-bimodule with the module actions given by

$$\langle F, u \cdot M\rangle = \langle M \cdot F, u\rangle, \quad M \cdot F \in A^* : \qquad \langle a, M \cdot F\rangle = \langle F \cdot a, M\rangle$$

and

$$\langle F, M \cdot u\rangle = \langle u \cdot F, M\rangle, \quad u \cdot F \in X^* : \qquad \langle x, u \cdot F\rangle = \langle F \cdot x, u\rangle,$$
$$F \cdot x \in A^* : \quad \langle a, F \cdot x\rangle = \langle x \cdot a, F\rangle$$

for $u \in \mathcal{A}^{**}$, $M \in X^{**}$, $F \in X^*$, $x \in X$ and $a \in \mathcal{A}$. In particular, for any integer $m \in \mathbb{N}$, $\mathcal{A}^{(2m)}$ is a Banach $\mathcal{A}^{**}$-bimodule.

A Banach $\mathcal{A}$-bimodule X is called *neo-unital* if $X = \mathcal{A}X\mathcal{A}$, that is every element $x \in X$ may be written in the form $x = ayb$ for some $a, b \in \mathcal{A}$ and $y \in X$. If $\mathcal{A}$ has a bounded approximate identity (e_α) and X is a neo-unital Banach $\mathcal{A}$-bimodule, then we may extend the $\mathcal{A}$ bimodule actions on X to $M(\mathcal{A})$, the multiplier algebra of $\mathcal{A}$. The extension is defined as follows.

$$\mu x = \lim_\alpha (\mu e_\alpha)x = (\mu a)yb, \qquad x\mu = \lim_\alpha x(e_\alpha \mu) = ay(b\mu)$$

for $\mu \in M(\mathcal{A})$ and $x = ayb \in X$. Here we note that $\mu a, b\mu \in \mathcal{A}$ since $\mathcal{A}$ is (always) an ideal of $M(\mathcal{A})$. These operations make X a unital Banach $M(\mathcal{A})$-bimodule. In this case a continuous derivation D: $\mathcal{A} \to X^*$ may be extended to a continuous derivation from $M(\mathcal{A})$ to X^* by defining

$$D(\mu) = \text{wk}^*\text{-}\lim_\alpha D(\mu e_\alpha) \quad (\mu \in M(\mathcal{A})).$$

Moreover this extended D is *so*-weak* continuous. In fact, if $\mu_\alpha \to \mu$ in $M(\mathcal{A})$ in the *so*-topology and $x = ayb \in X$ for $a, b \in \mathcal{A}$ and $y \in X$, then

$$\begin{aligned}\lim_\alpha \langle x, D(\mu_\alpha)\rangle &= \lim_\alpha \langle ay, D(b\mu_\alpha)\rangle - \lim_\alpha \langle \mu_\alpha ay, D(b)\rangle \\ &= \langle ay, D(b\mu)\rangle - \langle \mu ay, D(b)\rangle = \langle x, D(\mu)\rangle.\end{aligned}$$

We refer to the seminar paper [21] and the monograph [20] for more details of the above extensions.

We now can prove the main result of the paper.

Theorem 3. *Let G be a locally compact group. Then the group algebra $L^1(G)$ is $2m$-weakly amenable for each $m \in \mathbb{N}$.*

Proof. Denote $\mathcal{A} = L^1(G), X = \mathcal{A}^{(2m)}$ and $Y = \mathcal{A}^{(2m-1)}$. Then, as we have indicated, X is a Banach $\mathcal{A}^{**}$-bimodule. Let (e_α) be a bounded approximate identity of $\mathcal{A}$ and let E be a weak* cluster point of (e_α) in $\mathcal{A}^{**}$. Then $Ea = aE = a$ for all $a \in \mathcal{A}$. We have the $\mathcal{A}$-bimodule decomposition $X = X_1 \oplus X_2 \oplus X_3$, where

$$X_1 = \ell_E \circ r_E(X), \qquad X_2 = (I - r_E)(X), \qquad X_3 = (I - \ell_E) \circ r_E(X).$$

Here I denotes the identity operator, ℓ_E is the left multiplication by E and r_E the right multiplication by E. It is readily seen that

$$X_2 = (\mathcal{A}Y)^\perp \cong (Y/\mathcal{A}Y)^*, \qquad X_1 \oplus X_3 = r_E(X) \cong (\mathcal{A}Y)^*$$

as Banach $\mathcal{A}$-bimodules. Similarly, in $(\mathcal{A}Y)^*$

$$(I - \ell_E)\left((\mathcal{A}Y)^*\right) = (\mathcal{A}Y\mathcal{A})^\perp \cong (\mathcal{A}Y/\mathcal{A}Y\mathcal{A})^*$$

and

$$\ell_E\left((\mathcal{A}Y)^*\right) \cong (\mathcal{A}Y\mathcal{A})^*$$

as Banach $\mathcal{A}$-bimodules. We have

$$X_3 \cong (\mathcal{A}Y/\mathcal{A}Y\mathcal{A})^* \quad \text{and} \quad X_1 \cong (\mathcal{A}Y\mathcal{A})^*.$$

Let D: $\mathcal{A} \to X$ be a continuous derivation. Then $D = D_1 + D_2 + D_3$, where

$$\begin{aligned}&D_1 = \ell_E \circ r_E \circ D : \mathcal{A} \to X_1, \qquad D_2 = (I - r_E) \circ D : \mathcal{A} \to X_2, \\ &D_3 = (I - \ell_E) \circ r_E \circ D : \mathcal{A} \to X_3.\end{aligned}$$

Since ℓ_E and r_E are $\mathcal{A}$-bimodule morphisms, D_1, D_2 and D_3 are continuous derivations. Note that the left $\mathcal{A}$-module action on $Y/\mathcal{A}Y$ and the right $\mathcal{A}$-module action on $\mathcal{A}/\mathcal{A}Y\mathcal{A}$ are trivial. From [21, Proposition 1.5], D_2 and D_3 are inner. We now show that D_1 is also inner. Then D must be inner.

Since $\mathcal{A}Y\mathcal{A}$ is neo-unital, we may extend D_1 to a continuous derivation from $M(G)$, the multiplier of $\mathcal{A}$, to X_1. So we may consider Δ: $G \to X_1 \subset X$ defined by

$$\Delta(t) = D_1(\delta_t) \cdot \delta_{t^{-1}} \quad (t \in G).$$

It is readily seen that

$$\Delta(ts) = \delta_t \cdot \Delta(s) \cdot \delta_{t^{-1}} + \Delta(t) \quad (t, s \in G). \tag{3.1}$$

Let $B = \Delta(G)$. Then B is a nonempty bounded subset of X. For each $t \in G$, let T_t be the self mapping on X defined by

$$T_t(x) = \delta_t \cdot x \cdot \delta_{t^{-1}} + \Delta(t) \quad (x \in X).$$

Using (3.1) one may check that $\mathfrak{S} = \{T_t : t \in G\}$ defines a representation of G on X which is clearly nonexpansive and affine. Moreover, $T_t(\Delta(s)) = \Delta(ts)$ $(t, s \in G)$ and $T_e = I$. Since G is a group, the above implies $T_t(B) = B$ for each $t \in G$. Here G is regarded as a discrete group.

Since $WAP(G)$ has a LIM and X is L-embedded, by Theorem 2, there is $\xi \in X$ such that

$$\delta_t \cdot \xi \cdot \delta_{t^{-1}} + \Delta(t) = \xi \quad \text{for all } t \in G.$$

So $D_1(\delta_t) = \xi \cdot \delta_t - \delta_t \cdot \xi = \mathrm{ad}_{-\xi}(\delta_t)$ $(t \in G)$. Let $x = \ell_E \circ r_E(-\xi)$. Then $x \in X_1$. Also $D_1(\delta_t) \in X_1$. For any $ayb \in \mathcal{A}Y\mathcal{A}$ with $a, b \in \mathcal{A}$ and $y \in Y$, we have

$$\begin{aligned}\langle ayb, D_1(\delta_t)\rangle &= \langle ayb \cdot \delta_t - \delta_t \cdot ayb, -\xi\rangle = \langle E(ayb \cdot \delta_t - \delta_t \cdot ayb)E, -\xi\rangle \\ &= \langle ayb \cdot \delta_t - \delta_t \cdot ayb, x\rangle = \langle ayb, \mathrm{ad}_x(\delta_t)\rangle \quad t \in G.\end{aligned}$$

So it is true that $D_1(\delta_t) = \mathrm{ad}_x(\delta_t)$ for all $t \in G$. From what we have shown before stating the current theorem, both D_1 and ad_x, as continuous derivations from $M(G)$ into the dual of a neo-unital $\mathcal{A}$-bimodule, are *so*-weak* continuous. Since $lin(\delta_t : t \in G)$ is dense in $M(G)$ in the *so*-topology, we finally have

$$D_1(f) = \mathrm{ad}_x(f) \quad (f \in \mathcal{A} = L^1(G)),$$

therefore D_1 is inner. The proof is complete. □

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