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Sufficient Optimality Criteria in Continuous Time Programming

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Sufficient optimality conditions are obtained in the case of continuous time programming problems under the assumptions that (i) particular linear combinations of the components of the constraint function are quasiconvex and objective functional is pseudoconcave "almost everywhere," (ii) a particular linear combination of constraint function and objective functional is pseudoconcave "almost everywhere."

1. INTRODUCTION

The continuous time programming problem originated from Bellman's bottleneck problem [1]. Tyndall [2] and Levinson [3] studied duality for the linear continuous time programming problem and established well-known duality theorems. Hanson and Mond [4] generalized these duality theorems to the case where the objective functional is concave and derived the complementary slackness principle and Kuhn–Tucker necessary and sufficient conditions. Farr and Hanson [5] further generalized the continuous time programming problem by introducing nonlinear differentiable constraints and establishing the complementary slackness principle and Kuhn–Tucker theorem in their setup. Recently Singh and Farr [6] considered the continuous time programming problem

$$\text{Maximize } \; l(z) = \int_0^T h(z(t))\, dt \tag{MP}$$

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$$\text{Subject to } f(z(t)) \leqslant c(t) + \int_0^t g(s, t, z(s))\, ds, \qquad 0 \leqslant t \leqslant T,$$

$$z(t) \geqslant 0, \;\; 0 \leqslant t \leqslant T,$$

where $z(\cdot)$ is an $N \times 1$ vector-valued function defined on $[0, T]$, $f(\cdot)$ is an $M \times 1$ vector-valued function defined on the space $L_\infty^N[0, T]$ of all $N \times 1$ vector-valued, bounded, and measurable functions defined on $[0, T]$, $g(\cdot, \cdot, \cdot)$ is an $M \times 1$ vector-valued function defined on $[0, t] \times [0, T] \times L_\infty^N[0, T]$ for each $t \in [0, T]$, $c(\cdot)$ is an $M \times 1$ vector-valued function defined on $[0, T]$, $h(\cdot)$ is a real-valued function defined on $L_\infty^N[0, T]$, and all the integrals are in the Lebesgue sense. They established the optimality criteria of Kuhn–Tucker and Fritz–John type for this problem without assuming the differentiability of the functions involved. However, all the authors have taken the functions to be convex or concave in the nonlinear case. In [7], Singh has weakened the convexity (concavity) restrictions by considering quasiconvex/quasiconcave and pseudoconcave functions and has established a sufficient optimality criterion for (MP). But there is an obscurity in the proof of his main Theorem 2. This theorem is stated in somewhat straightforward manner and the sufficient optimality conditions are obtained under the weaker assumptions that (i) particular linear combinations of the components of the constraint function are quasiconvex and objective functional is pseudoconcave "almost everywhere," (ii) a particular linear combination of constraint function and objective functional is pseudoconcave "almost everywhere."

2. Preliminaries

Let $L_\infty^{+N}[0, T]$ be the collection of all $N \times 1$ vector-valued nonnegative, bounded, and measurable functions defined on $[0, T]$. Let

$$H(z(t)) = f(z(t)) - c(t) - \int_0^t g(s, t, z(s))\, ds.$$

Then $H(\cdot)$ is the constraint function defined on $L_\infty^N[0, T]$ with values in $L_\infty^M[0, T]$. Let

$$S = \{z(t) \in L_\infty^{+N}[0, T] / H(z(t)) \leqslant 0 \text{ for all } t \in [0, T]\},$$

i.e., S is the set of all feasible solutions of (MP). If there exists $\bar{z}(t) \in S$ such that $l(\bar{z}) = \text{Max}_{z(t) \in S}\, l(z)$ we say $\bar{z}(t)$ is an optimal solution of (MP).

We assume that $H(z(t))$ is Lebesgue measurable and the space $L_\infty^N[0, T]$ is suitably normed. Throughout this paper, F^c denotes the complement of F

with respect to $[0, T]$ for any subset F of $[0, T]$. The definitions of quasiconvexity, quasiconcavity, and pseudoconcavity "almost everywhere" (a.e.) of functions used in this paper are given in Singh's paper [7]. It is also proved in [7] that if $H(\cdot)$ is Fréchet differentiable and quasiconvex at $z_1(t) \in L_\infty^N[0, T]$, then for $z_2(t) \in L_\infty^N[0, T]$

$$H(z_2(t)) \leqslant H(z_1(t)) \Rightarrow dH(z_1(t); z_2(t) - z_1(t)) \leqslant 0 \qquad \text{for all } t \in [0, T], \tag{2.1}$$

where $dH(z_1(t); z_2(t) - z_1(t))$ is the Fréchet differential of $H(\cdot)$ at $z_1(t)$ with increment $z_2(t) - z_1(t)$.

3. Sufficient Conditions for Optimality

In Ref. [7], Theorem 2 gives sufficient conditions of optimality for the maximization problem (MP). But there is an obscurity in the proof of this theorem. It is claimed that quasiconvexity of $f(\cdot)$ on $L_\infty^N[0, T]$ and quasiconcavity of $g(\cdot, \cdot, \cdot)$ with respect to the third component implies that $H(\cdot)$ is quasiconvex on $L_\infty^N[0, T]$. This is not always correct because of a well-known result that a linear combination of quasiconvex functions is not quasiconvex [9]. However, if we assume $H(z(t))$ to be quasiconvex on $L_\infty^N[0, T]$, the same proof of Theorem 2 in Ref. [7] applies (with a few changes in Case 2 in view of the above remarks) and we obtain sufficient optimality conditions for (MP) in the form of the following theorem.

THEOREM 1. *Let $H(z(t))$ be Fréchet differentiable and quasiconvex on $L_\infty^N[0, T]$ and $h(z(t))$ be pseudoconcave on $L_\infty^N[0, T]$ a.e. on $[0, T]$. Then $z^0(t) \in L_\infty^{+N}[0, T]$ is an optimal solution of* (MP) *if there exists $u^0(t) \in L_\infty^M[0, T]$ satisfying the conditions*

$$-dh(z^0(t); z(t) - z^0(t)) + u^{0\prime}(t)\, dH(z^0(t); z(t) - z^0(t)) \geqslant 0, \tag{3.1}$$

$$H(0) = 0 \qquad \textit{and} \qquad H(z^0(t)) \leqslant 0 \qquad \textit{for all} \quad t \in [0, T], \tag{3.2}$$

$$u^0(t) \geqslant 0 \qquad \textit{for all} \quad t \in [0, T], \tag{3.3}$$

$$\int_0^T u^{0\prime}(t)\, H(z^0(t))\, dt = 0. \tag{3.4}$$

The following theorem generalizes the above theorem in the sense that the requirement of quasiconvexity on all the components of $H(z(t))$ is somewhat weakened.

For this we define the following sets.

For each $t \in [0, T]$, let

$$I_t = \{i / H_i(z^0(t)) = 0\},$$
$$J_t = \{i / H_i(z^0(t)) < 0\}.$$

Then $I_t \cup J_t = \{1, 2, ..., M\}$ for each $t \in [0, T]$. Let $I = \bigcup_{t \in [0,T]} I_t$, $J = \bigcap_{t \in [0,T]} J_t$, $I_1 = \bigcap_{t \in [0,t]} I_t$, and $J_1 = \bigcup_{t \in [0,T]} J_t$. Then $I \cap J = \emptyset$, $I \cup J = \{1, 2, ..., M\}$, $I_1 \cap J_1 = \emptyset$, $I_1 \cup J_1 = \{1, 2, ..., M\}$, and $J \subseteq J_1$, $I_1 \subseteq I$.

THEOREM 2. *Let $H(z(t))$ be Fréchet differentiable and $H_I(z(t))$ be quasiconvex on $L_\infty^N[0, T]$ and $h(z(t))$ be pseudoconcave on $L_\infty^N[0, T]$ a.e. on $[0, T]$. Then $z^0(t) \in L_\infty^{+N}[0, T]$ is an optimal solution of* (MP) *if there exists $u^0(t) \in L_\infty^M[0, T]$ satisfying conditions* (3.1)–(3.4) *of Theorem* 1.

Proof. According to the hypothesis, $H(z^0(t)) \leqslant 0$ for all $t \in [0, T]$. For each $i = 1, 2, ..., M$ define the sets

$$A_i = \{t \in [0, T] / H_i(z^0(t)) = 0\},$$
$$B_i = \{t \in [0, T] / H_i(z^0(t)) < 0\}.$$

Let $A = \bigcap_{i=1}^M A_i$, $B = \bigcup_{i=1}^M B_i$.

It can be shown [7] that $A \cup B = [0, T]$ and $A \cap B = \emptyset$. The definitions of the sets I_t and J_t show that if $t \in A$, then $I_t = \{1, 2, ..., M\}$ and $J_t = \emptyset$.

Using (3.4) we have

$$\begin{aligned}
0 &= \int_0^T u^{0\prime}(t)\, H(z^0(t))\, dt \\
&= \int_0^T \sum_{i=1}^M u_i^0(t)\, H_i(z^0(t))\, dt \\
&= \sum_{i=1}^M \int_0^T u_i^0(t)\, H_i(z^0(t))\, dt \\
&= \sum_{i=1}^M \left[\int_A u_i^0(t)\, H_i(z^0(t))\, dt + \int_B u_i^0(t)\, H_i(z^0(t))\, dt \right] \\
&= \sum_{i=1}^M \int_B u_i^0(t)\, H_i(z^0(t))\, dt \qquad (\text{since } H_i(z^0(t)) = 0 \text{ for all } t \in A) \\
&= \int_B [u_{I_1}^{0\prime}(t)\, H_{I_1}(z^0(t)) + u_{J_1}^{0\prime}(t)\, H_{J_1}(z^0(t))]\, dt \\
&= \int_B u_{J_1}^{0\prime}(t)\, H_{J_1}(z^0(t))\, dt \qquad (\text{since } H_{I_1}(z^0(t)) = 0 \text{ for all } t \in [0, T]).
\end{aligned}$$

This implies that $\int_B u_i^0(t) H_i(z^0(t))\,dt = 0$ for each $i \in J_1$ because of nonnegativity of $u_{J_1}^0(t)$ and $H_{J_1}(z^0(t)) \leqslant 0$ for all $t \in [0, T]$.

Hence either $\mu(B) = 0$ or $u_i^0(t) H_i(z^0(t)) = 0$ a.e. on B for all $i \in J_1$ where μ is the Lebesgue measure restricted to the σ-field of Lebesgue measurable subsets of $[0, T]$.

Case 1. Suppose $\mu(B) = 0$. Let

$$\begin{aligned} z^*(t) &= z^0(t) \quad &\text{if} \quad t \in A \\ &= 0 \quad &\text{if} \quad t \in B. \end{aligned}$$

Then

$$z^*(t) = z^0(t) \qquad \text{a.e. on} \quad [0, T]. \tag{3.5}$$

If $t \in A$, then by construction of A

$$H(z^*(t)) = H(z^0(t)) = 0$$

and if $t \in B$, then

$$H(z^*(t)) = H(0) = 0.$$

Hence $H(z^*(t)) = 0$ for all $t \in [0, T]$. Therefore, for any feasible solution $z(t)$,

$$H(z(t)) \leqslant 0 = H(z^*(t)) \qquad \text{for all} \quad t \in [0, T].$$

In particular, the inequality

$$H_I(z(t)) \leqslant 0 = H_I(z^*(t)) \tag{3.6}$$

holds for all $t \in [0, T]$. Since $H_I(z(t))$ is quasiconvex on $L_\infty^N[0, T]$, therefore it follows from (2.1) that

$$dH_I(z^*(t); z(t) - z^*(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T]. \tag{3.7}$$

Also Fréchet differentiability of $H_J(z(t))$ gives that

$$H_J(z(t)) = H_J(0) + dH_J(0; z(t)) + \|z(t)\| \in (0; z(t)) \qquad \text{for all} \quad t \in [0, T],$$

where $\varepsilon(0; z(t)) \to 0$ as $z(t) \to 0$.

Therefore, feasibility of $z(t)$ and $H(0) = 0$ imply that

$$dH_J(z^*(t); z(t) - z^*(t)) = dH_J(0; z(t)) \leqslant 0 \qquad \text{for all} \quad t \in B.$$

Also $t \in A$ implies that the sets J_t and J are empty, therefore

$$dH_J(z^*(t); z(t) - z^*(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T]. \tag{3.8}$$

Thus, from (3.7) and (3.8), we have

$$dH(z^*(t); z(t) - z^*(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T],$$

which shows that

$$dH(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on} \quad [0, T].$$

It now follows from the last inequality, (3.1) and (3.3) that

$$dh(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on} \quad [0, T]. \tag{3.9}$$

The function $h(z(t))$ is pseudoconcave at $z^0(t)$ a.e. on $[0, T]$; therefore (3.9) yields $h(z(t)) \leqslant h(z^0(t))$ a.e. on $[0, T]$. Thus $l(z) = \int_0^T h(z(t))\,dt \leqslant \int_0^T h(z^0(t))\,dt = l(z^0)$, i.e., $z^0(t)$ is an optimal solution of (MP).

Case 2. Suppose $u_i^0(t)\,H_i(z^0(t)) = 0$ a.e. on B for each $i \in J_1$. Since $H_i(z^0(t)) = 0$ for each $i = 1, 2,..., M$ and all $t \in A$, therefore

$$u_i^0(t)\,H_i(z^0(t)) = 0 \qquad \text{a.e. on } [0, T] \text{ for each } i \in J_1. \tag{3.10}$$

In particular, $u_i^0(t)\,H_i(z^0(t)) = 0$ a.e. on $[0, T]$ for each $i \in J \subseteq J_1$. The definition of the set J shows that

$$u_i^0(t) = 0 \qquad \text{a.e. on } [0, T] \text{ for each } i \in J,$$

i.e.,

$$u_J^0(t) = 0 \qquad \text{a.e. on } [0, T]. \tag{3.11}$$

Since $H_{I_1}(z^0(t)) = 0$ for all $t \in [0, T]$ and $I \sim I_1 \subseteq J_1$, where $I \sim I_1 = \{i/i \in I$ and $i \notin I_1\}$, therefore it follows from (3.10) that $u_i^0(t)\,H_i(z^0(t)) = 0$ a.e. on $[0, T]$ for each $i \in I$.

Thus $u_i^0(t)\,H_i(z(t)) \leqslant 0 = u_i^0(t)\,H_i(z^0(t))$ a.e. on $[0, T]$ for each $i \in I$ and for any feasible solution $z(t)$. For an arbitrary but fixed $i \in I$, let

$$E_i = \{t \in [0, T]/u_i^0(t)\,H_i(z(t)) > u_i^0(t)\,H_i(z^0(t))\}.$$

Then $\mu(E_i) = 0$ and $u_i^0(t)\,H_i(z(t)) \leqslant u_i^0(t)\,H_i(z^0(t))$ for all $t \in E_i^c$.

Now define

$$\begin{aligned} u_i^*(t) &= u_i^0(t) \qquad \text{if} \quad t \in E_i^c \\ &= 0 \qquad\quad \text{if} \quad t \in E_i \end{aligned}$$

so that $u_i^*(t) = u_i^0(t)$ a.e. on $[0, T]$. It follows that

$$u_i^*(t)\, H_i(z(t)) \leqslant u_i^*(t)\, H_i(z^0(t)) \qquad \text{for all} \quad t \in [0, T]. \tag{3.12}$$

Let $M_i = \{t \in [0, T]/u_i^*(t) = 0\}$. Then $u_i^*(t) > 0$ for $t \in M_i^c$.

From (3.12),. we obtain the inequality

$$H_i(z(t)) \leqslant H_i(z^0(t)) \qquad \text{for all} \quad t \in M_i^c.$$

Let

$$\begin{aligned} \hat{z}(t) &= z^0(t) \qquad && \text{if} \quad t \in M_i^c \\ &= 0 && \text{if} \quad t \in M_i. \end{aligned}$$

Therefore

$$H_i(z(t)) \leqslant H_i(\hat{z}(t)) \qquad \text{for all} \quad t \in [0, T]. \tag{3.13}$$

The relation in (3.13) further yields

$$dH_i(\hat{z}(t); z(t) - \hat{z}(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T]$$

in view of the fact that $H_I(z(t))$ is quasiconvex on $L_\infty^N[0, T]$. This shows that $u_i^*(t)\, dH_i(\hat{z}(t); z(t) - \hat{z}(t)) \leqslant 0$ for all $t \in [0, T]$ i.e., $u_i^*(t)\, dH_i(z^0(t); z(t) - z^0(t)) \leqslant 0$ for all $t \in [0, T]$, because $u_i^*(t) = 0$ for all $t \in M_i$ and $\hat{z}(t) = z^0(t)$ for all $t \in M_i^c$. The last inequality gives that $u_i^0(t)\, dH_i(z^0(t); z(t) - z^0(t)) \leqslant 0$ a.e. on $[0, T]$.

Arguing similarly for each $i \in I$, we observe that

$$u_I^{0\prime}(t)\, dH_I(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T]. \tag{3.14}$$

Also (3.11) gives

$$u_J^{0\prime}(t)\, dH_J(z^0(t); z(t) - z^0(t)) = 0 \qquad \text{a.e. on } [0, T]. \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$u^{0\prime}(t)\, dH(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T]. \tag{3.16}$$

Hence from (3.1), we obtain

$$dh(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T],$$

which is inequality (3.9) and we can proceed as in Case 1 above to prove that $z^0(t)$ is an optimal solution of (MP).

Remark 1. We note that in Theorem 2, there is a quasiconvexity assumption on the constraint function $H_I(z(t))$, that is, on each component of

$H_I(z(t))$. The following theorem replaces this requirement of a quasiconvexity assumption on each component of $H_I(z(t))$ by quasiconvexity assumption on particular linear combinations of the components of $H_I(z(t))$.

THEOREM 3. *Let $h(\cdot)$ be pseudoconcave on $L_\infty^N[0, T]$ a.e. on $[0, T]$ and $H(z(t))$ be Fréchet differentiable on $L_\infty^N[0, T]$. Then $z^0(t) \in L_\infty^{+N}[0, T]$ is an optimal solution of* (MP) *if there exists $u^0(t) \in L_\infty^M[0, T]$ satisfying conditions* (3.1)–(3.4) *of Theorem* 1 *and $\hat{u}_I'(t) H_I(z(t))$ is quasiconvex on $L_\infty^N[0, T]$, where $\hat{u}_I(t) = u_I^0(t)$ a.e. on $[0, T]$.*

Proof. The proof is the same as that of Theorem 2 above except that in Case 1 the arguments to get inequality (3.9) are as follows: From (3.3) and (3.6) we have

$$u_I^{0\prime}(t)\, H_I(z(t)) \leqslant 0 = u_I^{0\prime}(t)\, H_I(z^*(t)) \qquad \text{for all} \quad t \in [0, T].$$

The quasiconvexity of $u_I^{0\prime}(t)\, H_I(z(t))$ on $L_\infty^N[0, T]$ and (2.1) give that

$$u_I^{0\prime}(t)\, dH_I(z^*(t); z(t) - z^*(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T]. \tag{3.17}$$

Also from (3.3) and (3.8) we obtain

$$u_J^{0\prime}(t)\, dH_J(z^*(t); z(t) - z^*(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T]. \tag{3.18}$$

Combining (3.17) and (3.18), we get that

$$u^{0\prime}(t)\, dH(z^*(t); z(t) - z^*(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T].$$

This gives that

$$u^{0\prime}(t)\, dH(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T]$$

because of (3.5). Then inequality (3.9) results from (3.1) and we can proceed as in Case 1 of Theorem 2 to show that $z^0(t)$ is an optimal solution of (MP).

In Case 2, the argument to get inequality (3.16) runs as follows. Let $E = \{t \in [0, T] / u_I^{0\prime}(t)\, H_I(z(t)) > u_I^{0\prime}(t)\, H_I(z^0(t))\}$. Then we claim that $E \subseteq \bigcup_{i \in I} E_i$. Let $t \in E$ and assume $t \notin \bigcup_{i \in I} E_i$. Then $t \in E_i^c$ for each $i \in I$, which implies that

$$u_i^0(t)\, H_i(z(t)) \leqslant u_i^0(t)\, H_i(z^0(t)) \qquad \text{for each } i \in I.$$

i.e.,

$$u_I^{0\prime}(t)\, H_I(z(t)) \leqslant u_I^{0\prime}(t)\, H_I(z^0(t)),$$

which contradicts our assumption that $t \in E$. Since each set E_i is of measure zero, therefore $\mu(E)=0$ and $u_I^{0\prime}(t)\, H_I(z(t)) \leqslant u_I^{0\prime}(t)\, H_I(z^0(t))$ for all $t \in E^c$. Now define

$$\begin{aligned} \hat{u}_I(t) &= u_I^0(t) \qquad \text{if} \quad t \in E^c \\ &= 0 \qquad\quad\ \ \text{if} \quad t \in E. \end{aligned}$$

Clearly $\hat{u}_I(t) = u_I^0(t)$ a.e. on $[0, T]$ and $\hat{u}_I'(t)\, H_I(z(t)) \leqslant \hat{u}_I'(t)\, H_I(z^0(t))$ for all $t \in [0, T]$.

Therefore quasiconvexity of $\hat{u}_I'(t)\, H_I(z(t))$ and (2.1) yield

$$\hat{u}_I'(t)\, dH_I(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{for all} \quad t \in [0, T],$$

i.e.,

$$u_I^{0\prime}(t)\, dH_I(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on} \quad [0, T],$$

which is inequality (3.14) and we can proceed as in Case 2 of Theorem 2 to prove that $z^0(t)$ is an optimal solution of (MP).

Remark 2. We note that in Theorem 3 there is still a pseudoconcavity assumption on objective functional $h(\cdot)$ and a quasiconvexity assumption on $\hat{u}_I'(t)\, H_I(z(t))$, separately. The following theorem replaces this by pseudoconcavity assumption on a particular linear combination of the objective functional $h(\cdot)$ and the constraint function $H(\cdot)$.

THEOREM 4. *$z^0(t) \in L_\infty^{+N}[0, T]$ is an optimal solution of* (MP) *if there exists $u^0(t) \in L_\infty^M[0, T]$ satisfying conditions* (3.1)–(3.4) *of Theorem* 1 *and $(h - u_I^{0\prime}(t)\, H_I)(z(t))$ is pseudoconcave on $L_\infty^N[0, T]$ a.e. on $[0, T]$.*

Proof. The proof of this theorem is also the same as that of Theorem 2 except that arguments in Cases 1 and 2 are as follows.

Case 1. From (3.1) and (3.8), we obtain

$$\begin{aligned} dh(z^0(t); z(t) - z^0(t)) &- u^{0\prime}(t)\, dH(z^0(t); z(t) - z^0(t)) \\ &\leqslant 0 \qquad \text{for all} \quad t \in [0, T] \end{aligned} \tag{3.19}$$

and $u_J^{0\prime}(t)\, dH_J(z^*(t); z(t) - z^*(t)) \leqslant 0$ for all $t \in [0, T]$. Therefore

$$u_J^{0\prime}(t)\, dH_J(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T]. \tag{3.20}$$

From (3.19) and (3.20), we have

$$dh(z^0(t); z(t) - z^0(t)) - u_I^{0\prime}(t)\, dH_I(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T],$$

i.e.,

$$d(h - u_I^{0\prime}(t)\, H_I)(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T].$$

By the pseudoconcavity of $(h - u_I^{0\prime}(t)\, H_I)$, we have

$$(h - u_I^{0\prime}(t)\, H_I)(z(t)) \leqslant (h - u_I^{0\prime}(t)\, H_I)(z^0(t)) \qquad \text{a.e. on } [0, T]. \quad (3.21)$$

We know that $H_{I_1}(z^0(t)) = 0$, $H_I(z(t)) \leqslant 0$ and $u_I^0(t) \geqslant 0$ for all $t \in [0, T]$; therefore (3.21) reduces to

$$h(z(t)) \leqslant h(z^0(t)) - u_{I_2}^{0\prime}(t)\, H_{I_2}(z^0(t)) \qquad \text{a.e. on } [0, T],$$

where $I_2 = I \sim I_1$. Let

$$K = \{t \in [0, T]/h(z(t)) > h(z^0(t)) - u_{I_2}^{0\prime}(t)\, H_{I_2}(z^0(t))\}.$$

Then $\mu(K) = 0$ and

$$h(z(t)) \leqslant h(z^0(t)) - u_{I_2}^{0\prime}(t)\, H_{I_2}(z^0(t)) \qquad \text{for all} \quad t \in K^c. \qquad (3.22)$$

Let $K_1 = \{t \in [0, T]/h(z(t)) > h(z^0(t))\}$. We assert that $K_1 \subseteq B \cup K$. This can be shown as follows.

Let $t \in K_1$ and $t \notin B \cup K$. Then $t \notin B$ and $t \notin K$, which imply that $t \in A$ and $t \in K^c$. Hence from (3.22), it follows that $h(z(t)) \leqslant h(z^0(t))$, since $H_{I_2}(z^0(t)) = 0$ for $t \in A$. This contradicts our assumption that $t \in K_1$. Both sets B and K are of measure zero and as a consequence $\mu(K_1) = 0$. Thus $h(z(t)) \leqslant h(z^0(t))$ a.e. on $[0, T]$, i.e., $l(z) \leqslant l(z^0)$. Hence $z^0(t)$ is an optimal solution of (MP).

Case 2. From (3.11) of Theorem 2 (Case 2) it follows that $u_J^0(t) = 0$ a.e. on $[0, T]$. Inequality (3.19) now yields the inequality

$$dh(z^0(t); z(t) - z^0(t)) - u_I^{0\prime}(t)\, dH_I(z^0(t); z(t) - z^0(t)) \leqslant 0 \qquad \text{a.e. on } [0, T].$$

Arguing as in Case 1 above and noting the pseudoconcavity of $(h - u_I^{0\prime}(t)\, H_I)$, we obtain inequality (3.22).

From (3.10) and the fact that $I_2 \subseteq J_1$, we have

$$u_{I_2}^{0\prime}(t)\, H_{I_2}(z^0(t)) = 0 \qquad \text{a.e. on } [0, T].$$

Let $L = \{t \in [0, T]/u_{I_2}^{0\prime}(t)\, H_{I_2}(z^0(t)) < 0\}$ so that $\mu(L) = 0$ and

$$u_{I_2}^{0\prime}(t)\, H_{I_2}(z^0(t)) = 0 \qquad \text{for all} \quad t \in L^c. \qquad (3.23)$$

We now claim that $K_1 \subseteq K \cup L$. To show this let $t \in K_1$ and $t \notin K \cup L$. Then $t \notin K$ and $t \notin L$, which imply that $t \in K^c$ and $t \in L^c$. From (3.22) and (3.23), we now obtain $h(z(t)) \leqslant h(z^0(t))$ showing that $t \notin K_1$, a contradiction of our assumption. Thus $\mu(K_1) = 0$ as the sets K and L are both of measure zero. Therefore $h(z(t)) \leqslant h(z^0(t))$ a.e. on $[0, T]$, i.e., $l(z) \leqslant l(z^0)$, showing that $z^0(t)$ is an optimal solution of (MP).

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