General solution for Eshelby's problem of 2D arbitrarily shaped piezoelectric inclusions

W.-N. Zou a,*, Q.-C. He a,b, Q.-S. Zheng a,c

a Institute for Advanced Study, Nanchang University, Nanchang 330031, China
b Université Paris-Est, Laboratoire de Modélisation et Simulation Multi Echelle, UMR 8208 CNRS, 5 Bd Descartes, 77454 Marne-la-Vallée, France
c Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China

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A B S T R A C T

Eshelby's problem of piezoelectric inclusions arises sometimes in exploiting the electromechanical coupling effect in piezoelectric media. For example, it intervenes in the nanostructure design of strained semiconductor devices involving strain-induced quantum dot (QD) and quantum wire (QWR) growth. Using the extended Stroh formalism, the present work gives a general analytical solution for Eshelby's problem of two-dimensional arbitrarily shaped piezoelectric inclusions. The key step toward obtaining this general solution is the derivation of a simple and compact boundary integral expression for the eigenfunctions in the extended Stroh formalism applied to Eshelby's problem. The simplicity and compactness of the boundary integral expression derived make it much less difficult to analytically tackle Eshelby's piezoelectric problem for a large variety of non-elliptical inclusions. In the present work, explicit analytical solutions are obtained and detailed for all polygonal inclusions and for the inclusions characterized by Jordan's curves and Laurent's polynomials. By considering the piezoelectric material GaAs (110), the analytical solutions provided are illustrated numerically to verify the coincidence between different expressions, and to clarify the jump across the boundary of the inclusion and the singularity around the corner of the inclusion.

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1. Introduction

Many semiconductor materials are piezoelectric. The coupling effect between mechanical and electric fields has an important contribution to the electronic and optical properties of semiconductor materials (Pan, 2002a,b). Piezoelectric materials have been widely used as sensors and actuators in intelligent advanced structures. For example, a crucial factor in the study of strained semiconductor quantum devices is the strain-induced quantum dot (QD) and quantum wire (QWR) growth (see, e.g., O'Reilly and Adams, 1994; Nishi et al., 1994; Gosling and Willis, 1995; Park and Chuang, 1998; Davies, 1998; Andreev et al., 1999; Faux and Pearson, 2000; Freund, 2000; Pearson and Faux, 2000; Pan and Yang, 2003; Pan et al., 2005). Along with considerable attention attracted by piezoelectric materials, suitable mathematical modeling becomes important to studying electromechanical behaviors. In particular, Green's function technique has been developed both for the three-dimensional (3D) case (Wang, 1992; Dunn and Taya, 1993; Dunn and Wienecke, 1997; Huang and Kuo, 1997; Kuvshinov, 2008) and for the two-dimensional (2D) case (Ting, 1996; Lu and Williams, 1998; Pan, 2002c). For 2D piezoelectric materials, another remarkable technique is the extended Stroh formalism. Because of its preservation of most essential features of the Stroh formalism, the extended Stroh formalism acts as a very powerful tool for the study of piezoelectricity (Ting, 1996; Yin, 2005; Hwu, 2008). Note that the classical Stroh formalism has also extended to solve some three-dimensional anisotropic problems (Wu, 1998; Barber and Ting, 2007).

Eshelby's piezoelectric inclusion problem includes the well-known Eshelby's elastic inclusion problem as a particular one (Eshelby, 1957), corresponding to an infinite homogeneous piezoelectric medium containing a subdomain \( \omega \), called an electroelastic inclusion, over which a uniform eigenstrain and/or eigenelectric field is prescribed (see, e.g., Wang, 1992; Ru, 2000; Pan, 2004). It is known that Eshelby's elastic inclusion problem is of prominent importance to a large variety of mechanical and physical phenomena and plays an important role in particular in micromechanics (see, e.g., Willis, 1981; Mura, 1982; Nemat-Nasser and Hori, 1993). So does Eshelby's piezoelectric inclusion problem for piezoelectric materials. Recently, we have obtained explicit analytical solutions to Eshelby's isotropic elastic and anisotropic thermal inclusion problems for a wide variety of non-elliptical inclusions (Zou et al., 2010a,b). For Eshelby's piezoelectric inclusion problem, most of the existing analytical studies concern elliptical/ellipsoidal shapes (Wang, 1992; Liang et al., 1995; Chung and Ting, 1996;
Dunn and Wienecke, 1997; Li and Dunn, 1998; Zeng and Rajapakse, 2003) and only a few ones are dedicated to non-elliptical inclusions (Ru, 2000, 2003; Wang and Shen, 2003; Pan, 2004).

In the work of Ru (2000), use was made of the conformal mapping which maps the exterior of a unit circle to the exterior of an inclusion. In the one of Pan (2004), Green’s method was adopted. The objective of the present work is to go further in analytically solving Eshelby’s problem of 2D arbitrarily shaped piezoelectric inclusions by applying the extended Stroh formalism which have been proven to be powerful in treating 2D anisotropic problems. The key step is the presentation of a boundary integral expression for the eigenfunctions. The simplicity and compactness of the boundary integral formula derived make it much less difficult to analytically tackle Eshelby’s piezoelectric problem for a large variety of non-elliptical inclusions. In the present work, explicit analytical solutions are obtained and detailed for all polygonal inclusions and the inclusions characterized by Laurent polynomials and Jordan curves. These results include those reported by Ru (2000) and Pan (2004) as the particular ones.

The paper is structured as follows. In Section 2, we apply the extended Stroh formalism to Eshelby’s piezoelectric inclusion problem and derive a simple and compact boundary integral expression for the relevant eigenfunctions. The general expressions for Eshelby’s tensor and its average over the inclusion are then provided. Section 3 is dedicated to obtaining explicit analytical solutions for all polygonal inclusions. In Section 4, analytical solutions for the inclusions with smooth boundaries, say characterized by Laurent polynomials and Jordan curves, are presented. By numerically analyzing the solutions of inclusions of different shapes embedded in the piezoelectric material GaAs (110), in Section 5, we test the validity of expansion solution for inclusions characterized by Laurent’s polynomials, and certify and compare our solutions in different expressions by describing a square with different curves, and in Section 6, the fields of regular cracked inclusions are shown, and singularities around vertices and jumps across boundaries are illustrated and discussed theoretically. A few concluding remarks are given in Section 7. The elements of the extended Stroh formalism for piezoelectricity are presented in Appendix A; Appendix B gives a proof of equivalency between our solutions and that from Green’s function method.

2. General solution to Eshelby’s piezoelectric inclusion problem

2.1. General integral expressions for the eigenfunctions

Basing on the extended Stroh formalism for piezoelectricity (see Appendix A), we further use the following matrix notations

\[ \sigma_p = [\sigma_{1p}, \sigma_{2p}, \sigma_{3p}, D_p]^T, \quad \epsilon_p = [\epsilon_{1p}, \epsilon_{2p}, \epsilon_{3p}, -0.5E_p]^T, \quad p = 1, 2 \] (1)

and

\[ f' = [f'_2(z_2), f'_3(z_2), f'_1(z_2)]^T, \] (2)

where \( f'_l(z_2), l = 1, 2, 3, 4 \) are the derivatives of eigenfunctions \( f_l(z_2) \) with respect to \( z_2 \) and the diagonal matrix composed of four elements, say \( \{p_1, p_2, p_3, p_4\} \), are denoted by \( p \).

Let \( \Omega \) be the \( x_1 - x_2 \) plane made of a homogeneous piezoelectric medium and containing a subdomain, say \( \omega \), which undergoes a uniform eigenstrain and a uniform eigenfield. Let \( \partial \Omega \) denote the supplement of \( \omega \) to the \( x_1 - x_2 \) plane, \( \Gamma = \partial \omega \) the curve separating \( \omega \) and \( \beta \) (Fig. 1), with \( \omega \) and \( \beta \) being defined as open sets. Throughout this paper, we indicate the quantities in \( \omega \) and \( \beta \) with the subscripts \( * \) and \( \_ \) respectively. By \( \mathbf{u} * \), we symbolize the additional displacement and electric fields in \( \omega \) induced by the eigenstrain \( \epsilon^* \) and eigenfield \( \mathbf{E} * \), namely

\[ \mathbf{u}^* = \begin{pmatrix} \epsilon_{11}^* x_1 + \epsilon_{12}^* x_2 \\ \epsilon_{22}^* x_2 + \epsilon_{12}^* x_1 \\ 2\epsilon_{13}^* x_1 + 2\epsilon_{23}^* x_2 \\ - (E_1^* x_1 + E_2^* x_2) \end{pmatrix}, \] (3)

where \( (\epsilon_{11}^*, \epsilon_{12}^*, \epsilon_{22}^*) \) are the in-plane eigenstrains, \( (\epsilon_{13}^*, \epsilon_{23}^*) \) the anti-plane eigenstrains, and \( (E_1^*, E_2^*) \) the eigenfield. It is convenient to introduce a diagonal matrix

\[ L = (1, 1, 2, 2), \] (4)

and notation

\[ \mathbf{e}_p = L \mathbf{e}_p, \quad p = 1, 2. \] (5)

Let \( (u_i, \phi) \) be the elastic displacement and electrical potential fields caused by the eigenstrains and eigenfield, \( \mathbf{n} \) the unit normal on the boundary \( \Gamma \) toward from \( \omega \) to \( \beta \). The continuity conditions for the displacement and traction vectors across the boundary are

\[ u_i^* = u_i^* + u_i, \quad n_i \sigma_i^* = n_i \sigma_i. \] (6)

The continuity ones for the tangential electric field and normal electric displacement read

\[ \mathbf{n} \times \mathbf{E}^* = \mathbf{n} \times (\mathbf{E}^* + \mathbf{E}'), \quad \mathbf{n} \cdot \mathbf{D}^* = \mathbf{n} \cdot \mathbf{D}', \] (7)

As illustrated in Fig. 1, the increasing direction of \( dy \) is to keep \( \omega \) on the left-hand side as the Cartesian coordinate system is counterclockwise oriented. This implies

\[ n_1 ds = dx_2, \quad n_2 ds = -dx_1. \] (8)

where \( ds \) is an infinitesimal arc length element at the boundary point \( (x_1, x_2) \). Substituting (123), (8) and \( E_1 = -\psi, E_2 = -\psi \) into (6) and (7) deliver

\[ \frac{d}{ds}(\psi_i^* - \psi_i) = 0 \quad \text{with} \quad i = 1, 2, 3, 4, \quad \frac{d}{ds}(\psi - \phi^* - \phi^+) = 0. \] (9)

According to the continuity of the relevant qualities, we must have

\[ \psi_i = \psi_i^* \quad \text{with} \quad i = 1, 2, 3, 4, \quad \phi^* = \phi^* + \phi^+. \] (10)

Combining (10) and (6), gives the equivalent conditions of the generalized displacement and stress function across the interface:

\[ \mathbf{u}_*(y) = \mathbf{u}_+(y) + \mathbf{u}^*(y), \quad \psi^+_y (y) = \psi^*_y (y), \] (11)

where \( y = x_1 + ix_2 \in \Gamma \).

Accounting for the general solution (120) of extended Stroh formalism, the continuity condition (11) can be expressed by

\[ \mathbf{A}_{\mathbf{f}_*}(y) + \mathbf{A}_{\mathbf{f}_+}(y) = \mathbf{A}_{\mathbf{f}_*}(y) + \mathbf{A}_{\mathbf{f}_+}(y) + \mathbf{u}^*, \] (12)

\[ \mathbf{B}_{\mathbf{f}_*}(y) + \mathbf{B}_{\mathbf{f}_+}(y) = \mathbf{B}_{\mathbf{f}_*}(y) + \mathbf{B}_{\mathbf{f}_+}(y). \]
where \( y \in \Gamma \) and the overline denotes complex conjugation. Multiplying the two conditions of (12) by \( B^\top \) and \( A^\top \), respectively, and adding the resulting equations, we obtain
\[
f_\Gamma(y) = f_\Gamma(y) + B^\top u(y), \quad y \in \Gamma,
\]
where use is made of the orthogonality relation (see Chung and Ting, 1996)
\[
\begin{bmatrix} B^\top & A^\top \\ B^\top & A^\top \end{bmatrix} \begin{bmatrix} A \hline \bar{A} \end{bmatrix} = \begin{bmatrix} A \hline \bar{A} \end{bmatrix} \begin{bmatrix} B^\top \hline B^\top \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
with \( I \) being the \( 4 \times 4 \) identity matrix.

Since \( f_j(z)(j = 1, 2, 3, 4) \) are four functions which are sectionally analytic with respect to \( z \) in the entire complex plane except for \( \Gamma \), it is helpful to write \( B^\top u(y) \) as functions of \( y \) to coordinate with the functions \( f_j(z) \). Using
\[
x_1 = \frac{p_1 y_1 - p_1 y_1}{p_1 - p_1}, \quad x_2 = \frac{y_1 - y_1}{p_1 - p_1},
\]
we have expression from (3) and (5)
\[
B^\top u(y) = B^\top \tilde{e}_1 x_1 + B^\top \tilde{e}_2 x_2 = - \left( \begin{array}{c} c_1 y_1 + d_1 y_1 \\ c_2 y_1 + d_2 y_1 \\ c_3 y_1 + d_3 y_1 \\ c_4 y_1 + d_4 y_1 \end{array} \right), \quad y \in \Gamma.
\]
and so obtain a decoupled form of the conditions (13):
\[
f^\top = f^\top + \begin{bmatrix} c_1 y_1 + d_1 y_1 \\ c_2 y_1 + d_2 y_1 \\ c_3 y_1 + d_3 y_1 \\ c_4 y_1 + d_4 y_1 \end{bmatrix}, \quad y \in \Gamma.
\]
Utilizing the following Lemma (Henrici, 1986; Ablowitz and Fokas, 2003): Let \( f^\top \) be a simple, closed, regular, positively oriented curve enclosing the origin, and let \( b(t) \) (\( t \in \Gamma \)) be a Hölder continuous function {namely for \( t, \zeta \in \Gamma \) satisfying \( |b(t) - b(\zeta)| \leq C|t - \zeta|^{\alpha} \), \( C > 0 \), \( \alpha \in (0,1) \)} on \( \Gamma \), the degenerated Privalov (or Riemann–Hilbert) problem \( f^\top(t) = f^\top(t) + b(t) \) has the general solution
\[
f(z) = \frac{1}{2 \pi i} \int_\Gamma \frac{b(t)}{t - z} \, dt,
\]
the jumping relations (17) over the boundary
\[
\Gamma_I = \left\{ y_i = x_1 + p_i x_2 | y_i = x_1 + i x_2 \in \Gamma \right\}, \quad I = 1, 2, 3, 4,
\]
directly yield
\[
f_j(z_i) = \frac{1}{2 \pi i} \int_{\Gamma I} \frac{c_i y_i + d_i y_i}{y_i - z_i} \, dy_i
\]
\[
= c_i z_i \chi^\alpha + \frac{d_i}{2 \pi i} \int_{\Gamma I} \frac{\nabla y_i}{y_i - z_i} \, dy_i, \quad I = 1, 2, 3, 4,
\]
where \( \chi^\alpha \) is the characteristic function of \( \alpha \) that equals to 1 or 0 according as \( z \) is an inner or outer point of \( \alpha \). Note that Ru (2000) obtained the decoupled relation (17) but he did not provide the compact integral expression (20). The connection between (20) and the solution based Green’s function method is given in Appendix B.

Then the generalized strain and stress components are then given by
\[
u_1 = 2 \text{Re}[A_f^\top], \quad \nu_2 = 2 \text{Re}[A_f^\top f],
\]
\[
sigma_2 = 2 \text{Re}[B_f^\top], \quad \sigma_1 = -2 \text{Re}[B_f^\top f],
\]
where
\[
f_j(z_i) = c_i z_i \chi^\alpha + \frac{d_i}{2 \pi i} \int_{\Gamma I} \frac{\nabla y_i}{y_i - z_i} \, dy_i, \quad I = 1, 2, 3, 4
\]
with
\[
g(p_i; z_i) = \frac{1}{2 \pi i} \int_{\Gamma I} \frac{dy_i}{y_i - z_i}.
\]
Sometimes, it is convenient to write \( g(p_i; z_i) \) as \( g(p_i; z, \bar{z}) \) which takes the form
\[
g(p_i; z, \bar{z}) = \frac{1}{2 \pi i} \int_{\Gamma I} \frac{(1 + i p_i \bar{z}) dy_i + (1 - i p_i \bar{z}) dy_i}{(1 - i p_i)(y - z) + (1 + i p_i)(y - \bar{z})}.
\]

2.2. General expressions of Eshelby’s tensor in piezoelectric inclusion problem

In this subsection, summation convention for repeated indices does not apply. From (16) and (23), we know that
\[
f_j(z_i) = - \sum_{\kappa, \gamma} B_{\kappa \gamma} \tilde{e}_\kappa \tilde{F}_\gamma (\eta)
\]
in which
\[
\tilde{F}_\gamma (\eta) = \frac{1}{2 \pi i} \int_{\Gamma I} \frac{\nabla y_i}{y_i - \gamma} \, dy_i
\]
Resulting in (26) and (5) into (132) results in
\[
\epsilon_i = - \sum_{L, M \in K \delta} \text{Re} \left\{ A_{LM} B_{LM} K_{LM} + K_{LM} A_{LM} B_{LM} \right\} \epsilon_{\eta, \eta},
\]
from which we deduce the Eshelby’s tensor \( \Sigma_{\eta, \eta} \) defined by \( \epsilon_i = \Sigma_{\eta, \eta} \epsilon_{\eta, \eta} \) as
\[
\Sigma_{\eta, \eta} = - \sum_{L, M \in K \delta} \text{Re} \left\{ A_{LM} B_{LM} L_{LM} M_{LM} F_{LM}(z_M) \right\} \epsilon_{\eta, \eta},
\]
Similarly, Substituting (26) and (5) into (129) gives
\[
\sigma_i = -2 \sum_{M \in K \delta} \text{Re} \left\{ B_{LM} L_{LM} M_{LM} F_{LM}(z_M) B_{LM} \right\} \epsilon_{\eta, \eta},
\]
which delivers the eigenstiffness tensor \( \Omega_{\eta, \eta} = \Omega_{\eta, \eta} \epsilon_{\eta, \eta} \) as
\[
\Omega_{\eta, \eta} = -2 \sum_{M \in K \delta} \text{Re} \left\{ B_{LM} L_{LM} M_{LM} F_{LM}(z_M) B_{LM} \right\}.
\]
Recall that the eigenstiffness tensor rather than Eshelby’s tensor is directly involved in various micromechanics schemes for composites of inclusion–matrix types (cf. Zheng and Du, 2001; Zheng et al., 2006). Here, we can see that the formula for the eigenstiffness tensor \( \Omega_{\eta, \eta} \) is more compact than that for Eshelby’s tensor \( \Sigma_{\eta, \eta} \).

2.3. Average of the eigenfunctions

Utilizing the formula (cf. Laurentiﬀ and Shabat, 2002)
\[
\frac{1}{2 \pi i} \int_{\Gamma I} \omega(t, \bar{t}) dt \tau = \int_{\omega} \frac{\partial \omega(x, \bar{x})}{\partial \bar{x}} \, dx
\]
we can calculate the average of \( \omega \) over the inclusion \( \omega \) by
\[
\tilde{g}(p_i) = \langle g(p_i; z_i) \rangle = \frac{1}{2 \pi i} \int_{\Gamma I} \frac{\nabla y_i - \gamma}{y_i - z_i} \, dy_i.
\]
or equivalently
\[
g(p_i) = (g(p_i; z, \bar{z})) = \frac{1 + ip_i}{4\pi i} \oint_{C_i} \left( 1 + \frac{i}{y - z} \right) (dy + \tau_i dy) dz,
\]
where $|\omega|$ stands for the area of the inclusion and
\[
\epsilon_i = \frac{1 + ip_i}{1 - ip_i}.
\]
So, the average of $f_i(z_i)$ is determined by
\[
(f_i(z_i)) = c_i + d_i g(p_i), \quad i = 1, 2, 3, 4.
\]
A similar derivation can be done for the average Eshelby’s tensor $\Sigma^u$ and the average eigenstiffness tensor $\Omega^u$.

3. Analytical solutions for polygonal inclusions

In the following of this paper, we ascribe the solution of Eshelby’s problem to the undetermined function $g(p; \omega)$ and its average $\bar{g}(p)$. As the formulae (24), (25), (33) and (34) show, the solution must be form invariant with respect to a given $p_i$. Therefore, we will use an universal parameter $p$ instead of $p_i$, and all variables dependent on $p$ are indicated implicitly, say $y, z, s, \ldots$, which can be used for any $p_i$ by replacing $p$ by $p_i$ in a group.

### 3.1. General solution

Let $\omega$ be an arbitrary polygonal inclusion with the boundary consisting of $N$ rectilinear sides $\partial\omega_k$ with $k = 1, 2, \ldots, N$. As illustrated in Fig. 2, denoting by $y_k$ and $y_{k+1}$ the two end points of the $k$th side $\partial\omega_k$, we can parameterize all points of this side in the following form
\[
y = y_k + [y_{k+1} - y_k] t, \quad (0 \leq t \leq 1).
\]
Then, it follows
\[
\int_{\partial\omega_k} \frac{dy}{y - z} = \int_0^1 \frac{\bar{S}_k}{w_k + s_k} dt = \frac{\bar{S}_k}{s_k} \ln \frac{w_{k+1}}{w_k},
\]
where $w_k = y_k - z, \bar{S}_k = y_{k+1} - y_k - w_k$ and $\ln z = \ln |z| + \arg(z)$ with $-\pi < \arg(z) \leq \pi$. We sum the integrals of all sides to obtain the explicit expression of the general solution (24) as follows:
\[
g(z) = \frac{1}{2\pi i} \sum_{k=1}^N \frac{\bar{S}_k}{s_k} \ln \frac{w_{k+1}}{w_k}.
\]

Care must be taken in using these solutions in which the logarithmic terms cannot be in general combined freely. To be able to operate the terms freely, the arguments $\theta_k$ of $w_k$ need to be prescribed as follows. Referring to Fig. 2, we first assign the range of $\theta_k$ to be $(-\pi, \pi]$. If the direction of $w_{k+1}$ is counter-clockwise/clockwise rotated from the direction of $w_k$ through an angle less than $\pi$, then we assign $\theta_k$ to be larger/smaller than $\theta_{k+1}$. Analogously, we assign $\theta_{k+1}$ to be larger/smaller than $\theta_k$ when the counter-clockwise/clockwise rotation from the direction of $w_k$ to that of $w_{k+1}$ is an angle less than $\pi$. For a simply connected polygonal inclusion with $N$ sides, the complex point $w_{N+1}$ can be superposed with $w_1$ but should possess argument $2\pi + \arg(w_{N+1})$ if $z$ is an interior point. The ranges of $\phi_k$ are definite in the same way, which will be crucial in calculating the average Eshelby tensor. By virtue of these prescriptions and the foregoing discussion, the general solutions can be written as
\[
g(z) = \frac{1}{2\pi i} \sum_{k=1}^N e^{-i2\pi k} \left\{ \ln \frac{R_{k+1}}{R_k} + i(\theta_{k+1} - \theta_k) \right\},
\]
where $R_k, L_k$ and $\theta_k, \phi_k$ are the norms and arguments of $w_k$ and $s_k$ specified through
\[
w_k = R_k e^{i\phi_k}, \quad s_k = L_k e^{i\phi_k}.
\]
Compared with the expressions of solutions established by Pan (2004) for polygonal inclusions, the formulas (39) or (40) together with (21) and (22) are much simpler and much more compact.

### 3.2. Solutions for special polygonal inclusions

#### 3.2.1. Rectangle

Consider a rectangular inclusion of size $2a \times 2b$ with $a \geq b$. Since Eshelby’s tensor is size-independent, without loss of generality we pose $a = 1$ and set the associated side to be parallel to the basis vector $i_1$. As illustrated in Fig. 3, we can specify the four vertices of $\Gamma$, by
\[
y_{21} = 1 + p \tan \varphi, \quad y_{22} = 1 + p \tan \varphi, \quad y_{31} = -1 + p \tan \varphi, \quad y_{32} = -1 - p \tan \varphi,
\]
with $\varphi = \arctan(b)$, and introduce the local geometric parameters
\[
x_1 = -\arg(w_{11}), \quad x_2 = \arg(w_{21}), \quad x_3 = \pi - \arg(w_{31}), \quad x_4 = \pi + \arg(w_{41}),
\]
so that angles $x_i$ belong to $[0, \pi]$ for all interior points. Substituting (42) into (40) gives
\[
s_1 = 2p \tan \varphi, \quad s_2 = 2, \quad s_3 = -2p \tan \varphi, \quad s_4 = 1.
\]
Correspondingly,
\[
g(z) = \frac{1}{2\pi i} \frac{p}{p - 1} \left[ (x_1 + x_2 + x_3 + x_4 - 2\pi) + i \ln \frac{R_{11} R_{33}}{R_{22} R_{44}} \right],
\]
where $\chi(z)$ is the characteristic function of $\omega$. The solutions (44) are applicable for both interior and exterior points. Besides, substituting (42) into (39), the solution for the function can be expressed in an alternative way:
\[
g(z) = \chi^u + \frac{1}{2\pi i} \frac{p}{p - 1} \left[ \ln \frac{(1 + p \tan \varphi)^2 - z^2}{(1 - p \tan \varphi)^2 - z^2} + 2\pi i \text{Id} \chi^u \right],
\]
where the indicator function is defined by

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![Fig. 2. Prescriptions of arguments.](image)

![Fig. 3. Parameters of rectangle.](image)
\[ I_d = \begin{cases} 1, & \text{if } \text{Im} \left( \frac{1 + \tan \alpha + \frac{a}{b}}{1 - \tan \alpha - \frac{a}{b}} \right) > \pi, \\ 0, & \text{else}. \end{cases} \] (46)

Of particular interest is a square-shaped inclusion. By posing \( \varphi = \pi/4 \) in (45), it follows that
\[ g(z) = \chi^\alpha + \frac{1}{2\pi i} \left( \frac{p}{\bar{p}} - 1 \right) \ln \left( \frac{1 + p}{1 - p} \right)^2 - \frac{2\pi i}{\alpha} \chi^\alpha \left( \frac{p}{\bar{p}} - 1 \right). \] (47)

At the center of the inclusion, \( z = 0 \), the value \( f(0) \) is evaluated by
\[ g(0) = 1 + \frac{1}{2\pi i} \left( \frac{p}{\bar{p}} - 1 \right) \ln \left( \frac{1 + p}{1 - p} \right). \] (48)

3.2.2. Regular polygonal inclusions

Consider an \( N \)-fold (\( N \geq 3 \)) regular polygonal inclusion inscribed into a unit circle with the following vertices
\[ y_k = \cos \left( k - \frac{1}{2} \right) \theta + p \sin \left( k - \frac{1}{2} \right) \theta, \quad k = 1, 2, \ldots, N, \] (49)

where \( \varphi = \frac{2\pi}{N} \). Substituting (49) and \( s_k = \cos (k + \frac{1}{2}) \theta - \cos (k - \frac{1}{2}) \theta + pi \sin (k + \frac{1}{2}) \theta - \sin (k - \frac{1}{2}) \theta \) into (39), we obtain
\[ g(z) = \frac{1}{2\pi i} \sum_{k=1}^{N} \ln \frac{\kappa - \bar{p}}{\kappa - p} \cos \left( k + \frac{1}{2} \right) \theta + p \sin \left( k + \frac{1}{2} \right) \theta - \frac{\pi}{\alpha} \cdot \frac{\sin \kappa}{\kappa} \left( \frac{p}{\bar{p}} - 1 \right). \] (50)

For a square-shaped inclusion, \( k = 4 \) and \( \theta = \pi/4 \), it follows the same result as (48).

3.3. Average over the inclusion

For an arbitrary polygonal inclusion, a point \( z \) on the \( j \)th side and a point \( y \) on the \( k \)th side can be parametrized by
\[ z = y_j + s_j \tau, \quad y = y_k + s_k \tau \] with \( \tau \in [0, 1] \),
so that
\[ w = y - z = s_j \tau + s_k \tau - s_j \tau. \] (51)

Above, use is made of the notation
\[ s_j = y_j - y_j, \quad s_j = s_{j+1}. \] (53)

Then, starting from (33) and after making some calculations, we can obtain the following formula:
\[ g(p) = -\frac{1}{2\pi i(p - p) \alpha} \left\{ \sum_{j=1}^{\infty} s_j \ln s_j + \sum_{j=1}^{\infty} \frac{e^{-\alpha s_j} - e^{-\alpha s_j}}{2} + C_{sj} s_j \right\}, \] (54)

where the expression of \( C_{sj} \) is symmetric in \( j \) and \( k \), and takes on the following forms:
\[ C_{sj} = \frac{e^{-\alpha s_j} - e^{-\alpha s_j}}{2} \left( s_j \ln s_j + \frac{s_j}{s_j} \ln s_j \right). \] (55)

if \( k > j + 1 \), and
\[ C_{sj} = \frac{e^{-\alpha s_j} - e^{-\alpha s_j}}{2} s_j \left( \frac{1}{s_j} \ln s_j + \frac{1}{s_j} \ln s_j \right) + \frac{1}{s_j} \ln s_j \] (56)

if \( k > j + 1 \). Remark that, in the foregoing formulae, when \( j \) (or \( k \)) is equal to \( N \), we have to set \( j + 1 = 1 \) (or \( k + 1 = 1 \)).

For a rectangular inclusion with \( tanp > 0 \) and \( |\alpha| = 4\tan\varphi \), we have
\[ g(p) = 1 - \frac{p - \bar{p}}{\alpha} \left[ \ln \left( 1 + \frac{p}{\alpha} \tan \varphi \right) + \frac{1}{2\alpha \tan \varphi} \ln \left( 1 - p^2 \tan^2 \varphi \right) \right] \]
\[ + \frac{p \tan \varphi}{2} \ln \left( 1 - p^2 \tan^2 \varphi \right). \] (57)

4. Analytical solutions of smooth inclusions

4.1. Inclusions described by the Laurent polynomials

4.1.1. General solution

By the Riemann mapping theorem (Henrici, 1986), the shape of any given inclusion \( \omega \) can be approached by the Laurent polynomial of \( w \):
\[ y(w) = f_0 + R \left( w + \sum_{k=1}^{N} b_k w^{-k} \right), \quad |w| = 1, \] (58)

where \( R > 0 \) and \( f_0 \) is a unique inner point of the domain \( \omega \) bounded by \( \Gamma \). The parameters \( R \) and \( f_0 \) characterize the “size” and “center” of \( \omega \). For Esthely’s inclusion problems, without loss of generality we can shift and zoom \( \omega \) and set the parameters \( R \) and \( f_0 \) to be \( f_0 = 0 \) and \( R = 1 \). Some useful information on the shape expression (58) can be found in Zou et al. (2010a).

It is easy to verify that \( |\epsilon| = \frac{|1 - p|}{\alpha} \leq 1 \) when \( \text{Im} \) \( p \) is positive. Then, starting from (25) and using \( |\epsilon| = \frac{|1 - p|}{\alpha} \leq 1 \), we can derive
\[ g(p, z, w) = \frac{1 - ip}{1 - ip} \left[ 1 + \frac{\alpha |1 - p|}{\alpha} \sum_{k=1}^{N} e^{-\alpha k} J_k \right] \] (59)

with
\[ J_k = \frac{1}{2\pi i} \oint_{\Gamma} \frac{1 - \zeta - \bar{\zeta}}{1 - \zeta - \bar{\zeta}} \frac{\beta}{\zeta - \bar{\zeta}} \] (60)

If \( z \) is an inner point of the inclusion, the preconditions of (58) as a meromorphic and one-to-one mapping from the exterior of a unit disk to the exterior of a simple-connected domain guarantees that \( y^{-1}(y(w) - z) \) is non-zero outside the unit disk (or the origin after mapping must belong to the inside of the inclusion) (Henrici, 1986). In other words, \( (1 - zw^{-1} + \sum_{k=1}^{N} b_k w^{-k}) \) can be expanded with powers of \( w^{-1} \).

For an ellipse with \( y(w) = w + b_1 w^{-1} \) with \( |b_1| \leq 1 \), using
\[ s_0 = \frac{y - z}{y - z} = \frac{w^{-1} + b_1 w - z}{w + b_1 w^{-1} - z}, \quad s_k = \frac{w^{-1} + b_1 w - z}{w + b_1 w^{-1} - z} \]
\[ = \frac{1}{w} + \sum_{k=1}^{\infty} \frac{(zw^{-1} - b_1 w^{-2})^k}{w} \] (61)

direct calculations yield
\[ J_k = b_1^k, \quad k = 1, 2, \ldots. \] (62)

Substituting (62) into (59) brings forth the result
\[ g(p, z, w) = \frac{1 - ip}{1 - ip} \left( 1 + \frac{1 + ip}{b_1} b_1^k \right) \]
\[ = \frac{1 - ip}{1 - ip} \left( 1 + \frac{1 + ip}{b_1} \right) \] (63)

which coincides with that of Ru (2000).

Unfortunately, the above simplification process cannot be extended to shapes other than ellipse. For example, for the inclusion described by \( y(w) = w + b_1 w^{-1} + b_2 w^{-2} \) with \( |w| = 1 \), we have
This is much more complicated compared with (62), and there is no compact form like (63). For the inclusion given by \( y(w) = w + b_1 w^{-1} + b_2 w^{-n} \) with \(|w| = 1\), the integral \( I_k \) can be calculated by

\[
I_k = \frac{1}{k(n-1)!} J_k(w) \quad \text{for} \quad w = 1,
\]

(65)

where

\[
P_k(w) = (b_1 - b_2 w^{n-1} - b_2 w^{n+1})^k \frac{1}{1 - \sum_{m=1}^{k-1} (b_1 w^{-1} - b_2 w^{n+1})^m}.
\]

The explicit solutions for more complex inclusions can be also obtained.

### 4.1.2. Average over the inclusion

From (34), and the property \(|e| < 1\), we can obtain

\[
g(p) = \frac{1 + ip}{1 - ip} \left[ \frac{1}{(2k+1)^{1/2}} e^{k-1 \xi_k} \right],
\]

(67)

where \( \xi_k \) is defined by

\[
\xi_k = \frac{1}{4 \pi (1 - ip)} \int f_i f_j \frac{y - z}{(y - z)^k} dy dz
\]

(68)

and \( f_i f_j \frac{y - z}{(y - z)^k} dy dz = 4 \pi (1 - ip) f_i f_j \frac{y - z}{(y - z)^k} dy dz \) are used. The basic technique used to work out \( \xi_k \) was presented in Appendix A of Zou et al. (2010a). For example, if the shape is defined by \( y(w) = w + b_1 w^{-1} + b_2 w^{-n} \) with \(|w| = 1\), the integral \( I_k \) can be calculated by

\[
I_k = \frac{\pi}{k(n-1)!} \frac{d^{2k(n-1)+n+1} P_k(u, v)}{d[u^{n+1} v^{n+1}]} \quad \text{for} \quad u = 0, v = 0,
\]

(69)

where the area \(|| \) is equal to \( \pi (1 - b_1 - b_2) \) and the polynomial \( P_k(u, v) \) is defined by

\[
P_k(u, v) = \left[ u^{n-1} - b_1 u^{n-1} - b_2 \right] \left( 1 - b_1 v^2 - b_2 v^{n+1} \right) \left[ 1 + \sum_{m=1}^{k-1} (b_1 u v - b_2 u^{m-1} v^{n+1})^m \right]^k.
\]

(70)

The solutions similar to (69) and (70) can be obtained for inclusions of more general shapes.

### 4.2. Inclusions whose boundaries can be described as Jordan’s curves

Let \( \Gamma \) be a simple closed curve, called a Jordan curve, composed of straight line segments and circular arcs which are one by one smoothly connected, say, \( y_1, y_2, y_3, \ldots, y_{2M} \), where \( y_{2M} = y_1 \), and the arc length coordinate of point \( y \) between two ends \( y_j, y_{j+1} \) is calculated by

\[
y_j(y) = \frac{L_j y_j}{2 \pi} + \frac{L_j}{2 \pi} \int \frac{1}{y - y_j} \left[ 1 - \frac{1}{y_j} \right] dy - \frac{L_j}{2 \pi} \int \frac{1}{y - y_j} \left[ 1 - \frac{1}{y_j} \right] dy.
\]

(71)

The centers of circular arcs are

\[
Q_j = (2k+1) - r_j' \left( \frac{1}{y_j} \right),
\]

(72)

where \( r_j' = \frac{L_j}{2 \pi} \frac{1}{y_j} \) are the signed arc radii, namely

\[
r_j > 0 \quad \text{if} \quad \phi(y_j) > \phi(y_j),
\]

(73)

\[
r_j < 0 \quad \text{if} \quad \phi(y_j) < \phi(y_j).
\]

A Jordan curve with \( 2M \) segments can be constructed by smoothing a \( M \)-sided polygon. Suppose that the vertices of the polygon are

\[
V_1, V_2, V_3, \ldots, V_{2M},
\]

(74)

where \( V_{2M} = V_1 \). Letting the vertices \( V_1 \) and the arc radii \( r_1 \) be given, then \( y_1, y_2, y_3, \ldots, y_{2M} \) can be calculated from

\[
y_1 = V_{2M} - r_1 \left( \frac{1}{y_1} \right),
\]

(75)

\[
y_2 = V_{2M} - r_1 \left( \frac{1}{y_2} \right) + \frac{1}{2} \frac{y_2}{y_2 - y_1}.
\]

The arc length coordinate \( y_j(y) \) must be satisfied.

It is convenient to use the arc length coordinate to label the points on \( \Gamma \). Letting the arc length coordinate of end \( y_j \) be \( l_j \), the arc length coordinate of point \( y \) between two ends \( y_j, y_{j+1} \) is calculated by

\[
l = \begin{cases} 
  l_j + r_1 \left( \arg(y - c_j) - \phi_j \right), & \text{if } J = 2K, \\
  l_j + \frac{y_j - y_{j+1}}{y_j - y_j}, & \text{if } J = 2K - 1.
\end{cases}
\]

(78)

Inversely, a point \( y \) with arc length coordinate \( s \) has the following Cartesian coordinate:

\[
y = c_j + r_1 \left( \frac{e^{i (s - s_j)}}{2 \pi} \right),
\]

(79)

where \( y_j = \frac{L_j}{2 \pi} (l_j - l_j) \), if \( J = 2K - 1 \).

After the foregoing preparations, we can now calculate the integral (24) of arc \( \partial \Gamma_{2M} \) through

\[
I_{2M} \quad \text{for} \quad \int_{\partial \Gamma_{2M}} \frac{dy}{w},
\]

(80)

where

\[
\int_{\partial \Gamma_{2M}} \frac{dy}{w} = \int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy + \frac{1}{y - z} dy.
\]

(81)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(82)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(83)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(84)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(85)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(86)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(87)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(88)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(89)

\[
\int_{\partial \Gamma_{2M}} \frac{1}{y - z} dy = \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy + \frac{1}{y - z} \int_{\partial \Gamma_{2M}} \left( 1 - \frac{1}{y} \right) dy.
\]

(90)
where $\zeta_k = e^{i(\phi_{K} - \frac{\pi}{2})}$, $\zeta_{k+1} = e^{i(\phi_{K} - \frac{\pi}{2})}$, 
and the undetermined integral defined by 
\[
H = \int_{\zeta_k}^{\zeta_{k+1}} \frac{dz}{4\zeta} \left( \frac{1}{\zeta + \alpha + \zeta} \right)^{\frac{1}{2}} 
\]
has branches 
\[
H = \frac{\sqrt{\frac{1}{4}\lambda^2 - \epsilon + \frac{1}{2}}}{2\epsilon} \ln \frac{\sqrt{\frac{1}{4}\lambda^2 - \epsilon} + \frac{1}{2}}{\sqrt{\frac{1}{4}\lambda^2 - \epsilon} - \frac{1}{2}} \theta \left( \frac{1}{2} \right), 
\]
if $\epsilon \neq 0$, and 
\[
H = \frac{1}{2\epsilon} \ln \left( \frac{1}{\sqrt{\frac{1}{4}\lambda^2 - \epsilon}} \right), 
\]
if $\epsilon = 0$. Combining (80)-(83) with the known integral of a straight segment 
\[
\delta_{2K-1}(p;z) = \frac{1}{2\pi i} \sum_{k=1}^{N} \frac{S_{k+1} - S_{k}}{S_{k} - S_{k+1}} \ln \frac{W_{2K}(z)}{W_{2K-1}(z)}, 
\]
we can rearrange the eigenfunction solution as 
\[
g(z) = \chi^0 + \frac{1}{2\pi i} \sum_{k=1}^{N} \left( \ln \frac{W_{2K}(z)}{W_{2K-1}(z)} + \frac{1}{2\pi i} \chi_k \right), 
\]
where the property 
\[
1 = \frac{1}{2\pi i} \sum_{k=1}^{N} \left( \ln \frac{W_{2K}(z)}{W_{2K-1}(z)} + \frac{1}{2\pi i} \chi_k \right) = \chi^n 
\]
is used and the integral $I_k$ herein is specified in the following. 
Introducing the notation 
\[
h_{k}(z) = \frac{c_{k}(z) - z}{c_{k}(z) - a(z)}, 
\]
\[
q_{k}(z) = \frac{1}{\sqrt{c_{k}(z) - a(z)}}, 
\]
the integral $I_k$ can be expressed as follows: (i) if $\epsilon = 0$, then 
\[
I_k = \left\{ \begin{array}{ll}
\frac{\sqrt{r_{k}^2}}{\epsilon (\epsilon - 2)} & \ln \frac{\gamma_{2K-1}}{\gamma_{2K}} - i \left( \phi_{k+1} - \phi_{k} - 2\pi \frac{r_{k}}{r_{k+1}} \chi_{k} \right) \\
+ \frac{\sqrt{r_{k}^2}}{\epsilon (\epsilon - 2)} (e^{-i\phi_{k+1}} - e^{-i\phi_{k}}), & \text{if } \lambda \neq 0. \\
- \frac{1}{2} (e^{-i\phi_{k+1}} - e^{-i\phi_{k}}), & \text{if } \lambda = 0.
\end{array} \right. 
\]
(ii) if $\epsilon \neq 0$, then 
\[
I_k = \frac{p - p}{i(1 + p^2)} \left( 1 + q_{k} \right) \left[ \ln \frac{h_{k}(z) - i(1 - ip)r_{k}e^{i\phi_{k+1}}}{h_{k}(z) - i(1 - ip)r_{k}e^{i\phi_{k}}} - i(\phi_{k+1} - \phi_{k} - 2\pi \frac{r_{k}}{r_{k+1}} \chi_{k} \right] \\
+ \frac{p - p}{i(1 + p^2)} (1 - q_{k}) \\
\left[ \ln \frac{h_{k}(z) - i(1 - ip)r_{k}e^{i\phi_{k+1}}}{h_{k}(z) - i(1 - ip)r_{k}e^{i\phi_{k}}} - i(\phi_{k+1} - \phi_{k} - 2\pi \chi_{k} \right] \\
+ \frac{2\pi i(p - p)}{1 + p^2} \left( 1 - \frac{r_{k}}{r_{k+1}} \right) \chi_{k} 
\]
or equivalently 
\[
I_k = \frac{p - p}{i(1 + p^2)} \left[ \ln \frac{\gamma_{2K+1}}{\gamma_{2K}} - i \left( \phi_{K+1} - \phi_{K} - 2\pi \frac{r_{K}}{r_{K+1}} \chi_{k} \right) \\
+ \frac{2\pi i(q_{k} - \chi_{k})}{r_{k}} \right] \\
+ \frac{p - p}{i(1 + p^2)} q_{k} \left[ \ln \frac{h_{k}(z) - i(1 - ip)r_{k}e^{i\phi_{k+1}}}{h_{k}(z) - i(1 - ip)r_{k}e^{i\phi_{k}}} - i(\phi_{k+1} - \phi_{k} - 2\pi \chi_{k} \right] \\
+ \frac{2\pi i(p - p)}{1 + p^2} \left( 1 - \frac{r_{k}}{r_{k+1}} \right) \chi_{k} 
\]
where the indicator functions $\chi_{k}^{1}, \chi_{k}^{2}, \chi_{k}^{3}$ and $\chi_{k}^{4}$ are defined by 
\[
\chi_{k}^{1} = \begin{cases} 
1, & \text{if } \arg \left( \frac{\gamma_{2K+1}}{\gamma_{2K}} \right) < \frac{\pi - \phi_{k+1} - 2\pi \chi_{k} \chi_{k}^{1}}{2}, & \text{with } r_{k} > 0, \\
0, & \text{else},
\end{cases} 
\]
\[
\chi_{k}^{2} = \begin{cases} 
1, & \text{if } \arg \left( \frac{\gamma_{2K+1}}{\gamma_{2K}} \right) < \frac{\phi_{k+1} - 2\pi \chi_{k} \chi_{k}^{2}}{2}, & \text{with } r_{k} < 0, \\
0, & \text{else},
\end{cases} 
\]
\[
\chi_{k}^{3} = \begin{cases} 
1, & \text{if } \arg \left( \frac{\gamma_{2K+1}}{\gamma_{2K}} \right) < \frac{\phi_{k+1} - 2\pi \chi_{k} \chi_{k}^{3}}{2}, & \text{with } r_{k} < 0, \\
0, & \text{else},
\end{cases} 
\]
\[
\chi_{k}^{4} = \begin{cases} 
1, & \text{if } \arg \left( \frac{\gamma_{2K+1}}{\gamma_{2K}} \right) < \frac{\phi_{k+1} - 2\pi \chi_{k} \chi_{k}^{4}}{2}, & \text{with } r_{k} < 0, \\
0, & \text{else},
\end{cases} 
\]

The singularity analysis of the function $g(z)$ of a Jordan curve around the end point $y_{j}$ is very complicated, but direct numerical calculations show that, other than the logarithmic singularity of polygon around its vertices, there is no longer singularity at the boundary of an inclusion whose boundary corresponds to a Jordan curve.

5. Numerical examples: certification and comparison

In two-dimensional anisotropic piezoelectric Eshelby’s problems, the choice of reference plane differs subject of $(p)$, and for a given $p$ from $(p)_{1}$, the effect of the shape of the inclusion is determined only by the function $g_{1}(p_{1};z)$. In this section, we choose piezoelectric GaAs $(110)$, which is the same as in Pan (2004), to numerically illustrate our analytical results for inclusions of different shapes. First, by approaching an elliptical inclusion by a $N$-sided equilateral polygonal inclusion, we verify the solution calculated by (39) converging to that of (63), and using the same configuration considering the connection proved in Appendix B, we have found that the disturbances of dimensionless stresses and electric displacements calculated from (39) coincide with those listed in Table 1 and 2 of Pan (2004).

5.1. Parameters in the extended Stroh formalism

With the properties of piezoelectric GaAs $(110)$ (cf. Pan, 2002a,b,c, 2004; Pan et al., 2005), we calculate the eigenvalues $(p)$ from (124) and corresponding parameters $(e)$ as: 
\[
(p) = \{-0.3812 + 1.5815i, 0.3104 + 1.0598i, -0.0196 + 0.9961i, 0.08173 + 0.5851i\}, 
\]
\[
(e) = \{-0.2418 - 0.1120i, -0.05059 + 0.1431i, 0.001924 - 0.005501i, 0.2584 + 0.06489i\}. 
\]
The biggest module of $(e)$ is 0.26645. Note that $(e)$ is completely dependent on the material properties on the reference plane. In general, the more isotropic the material properties on the reference plane are, the smaller the biggest module of $(e)$ is.
The corresponding normalized matrices $A$ and $B$ in the extended Stroh formalism, which appear for instance in expressions (120) and (129), (132), can be calculated simultaneously from (124).

5.2. Validity of the solution for inclusions characterized by the Laurent polynomial

In order to verify the validity of the expansion formulae (59) and (67) for inclusions of shapes characterized by Laurent polynomials, we consider an inclusion whose shape is described by

$$y(w) = w - \frac{1}{6}w^3 + \frac{1}{55}w^7, \quad |w| = 1,$$

(98)

which can be used to model a “square” shape (Kachanov et al., 1994) centered at the origin. As shown in Fig. 4(a), the area of it is $|\omega| = \pi(1 - 3b_1b_2 - 7b_1b_2) = \frac{1229}{1344}\pi$.

(99)

which results in an equivalent side length

$$a = \sqrt{|\omega|} \approx 1.695.$$

(100)

We calculate the integrals $J_k(z)$ in (65) at five interior points along the diagonal of the square, namely $z = (0,0), (0.2,0.2), (0.4,0.4), (0.6,0.6), (0.8,0.8)$ and $T_k$ in (69). Then utilizing eigenvalue $p$ with the biggest $\epsilon$ to test the convergence of the expansions (59) of $g(p;z,z)$ and (67) of $g(p)$ due to different truncations, the results listed in Table 1. They show a very quick convergence with $k$.

These calculations tell us that the convergence of expansions (59) and (67) is mainly controlled by the small parameters $\epsilon$. The biggest value of $\epsilon$ we have studied is 0.354 for a left-hand quartz material (Pan, 2002c). For present example, six order truncation of $g(p;z,z)$ and $g(p)$ are exactly enough. For other cases, a truncation of order less than ten would be good enough, if the corresponding maximal small parameter is less than 0.5.

5.3. Solutions of a square-shaped inclusion described in different ways

Now we test and compare the results for a square-shaped inclusion described by a Laurent polynomial, a polygon, and a Jordan curve, as shown in Fig. 4. The arc radius of the Jordan curve (Fig. 4(c)) is choosed to be the same, namely $r = \frac{a}{2\pi}$. Taking the eigenvalue to be

$$p = -0.3812 + 1.5815i,$$

we calculate the values of $g(p;z,z)$ along the diagonal and $g(p)$ by the general formulae (39) and (54) for the polygonal inclusion, and the formulae (59), (67) for the Laurent polynomial. The field values $g(p;z,z)$ of the Jordan curve are calculated from (86). The results of $g(p;z,z)$ at five points and $g(p)$ are listed in Table 2, more results of $g(p;z,z)$ are shown in Fig. 5. From these results, we see that: (i) the averages are very close to each other; (ii) the fields of the polygon and the Jordan curve are different obviously only around the corners; (iii) the field of the Laurent polynomial presents a notable departure from other two fields and has a gently variance around the corner. So it could be pointed out that the more smooth the shape of the inclusion is, the more gently the field inside the inclusion is, but the average of the field is almost invariant.

It should be noticed that the solution for the Laurent polynomial has to be confined to the interior of the inclusion and the diagonal line of the smoothing Jordan curve is shorter than that of a real square. Besides, Fig. 5 also shows the interface jump for the Jordan curve and the vertex singularity for the square, which will be discussed in the next section.

6. Numerical examples: fields, singularities and jumps

It is well-known that the solutions for Eshelby’s elastic polygonal inclusion problems exhibit logarithmical singularity around the vertices. This is of course also true for Eshelby’s piezoelectric ones. The logarithmical singularity of a special field, say, the strain field $\varepsilon$, inside a polygonal inclusion depends on many factors, such as the material properties relative to the reference plane, the inclusion orientation, and the eigenstrains under consideration. But the key to work out the stress and strain fields is the complex eigenfunctions or equivalently the functions $g(p;z)$ which depends almost only on the shape of the inclusion, while other factors just redistribute them in an interleaved way. In this section, as an example, we consider the field $g(p;z)$ for a regular crux inclusion and discuss its singularities and jumps.

6.1. Fields and singularities of regular crux inclusion

A regular crux inclusion defined by parameters $a$ and $t$ is shown in Fig. 6. Here we take the half-width to be $a = 1$ and the half-thickness $t = 0.3$.

Using the formula (39), we calculate the functions $g(p;z)$ with four different eigenvalues listed in (97) through (39). Shown in Figs. 7, 8 are the contours of the real and image parts of $g(p;z)$.

These figures show: (1) the jumps of $\text{Re}[g(p;z)]$ across the whole boundary and the jumps of $\text{Im}[g(p;z)]$ across the vertical parts; (2) the singularities of $\text{Im}[g(p;z)]$ around all vertices; (3) the different $p$ results in different distribution orientations. Further, it is seen that that the span of $\text{Im}[g(p;z)]$ is larger than that of $\text{Re}[g(p;z)]$. This means that the maxima of $\text{Im}[g(p;z)]$ are more local around the vertices and consolidate the singularities of $\text{Im}[g(p;z)]$.

Next, it seems that the fields $\text{Re}[g(p;z)]$ have no singularity. Since the contours are illustrated from the values at discrete points not on the boundary, the singularity approaching vertices must be weakened. Using (41) and (39), the singularity around vertex $k$ is determined by

$$g(p;z) \sim \frac{1}{2\pi i} \left( S_{k-1} \frac{S_{k}}{S_{k-1}} \right) \ln R_k,$$

(101)

In the case of a crux inclusion, let the side $\partial\Omega_{k-1}$ be horizontal and $\partial\Omega_k$ vertical, the formula (101) becomes

$$g(p;z) \sim \frac{1}{2\pi i} \left( 1 - \frac{p}{\bar{p}} \right) \ln R_k \sim \frac{1}{2\pi i} \ln R_k,$$

(102)

where $\alpha$ corresponding to for different $p$ takes value from

$$\alpha \in \{-1.890 \pm 0.4556, -1.842 \pm 0.5395, -2.000 \pm 0.0220, -1.9621 \pm 0.2740\}.$$

(103)

It is observed that the singularity coefficients of the image parts are larger than those of the real parts. For example, Fig. 9 illustrated the distribution of $g(p;z)$ along the line $(\lambda z); x_1 = x_2$ with $p = -1.9621 + 0.2740$, where both the real and image parts have

![Fig. 4. Square shapes characterized by (a) Laurent polynomial, (b) polygon, (c) Jordan curve smoothed by four arcs.](image-url)
the singularity when approaching the vertex but the former is easier to be concealed.

6.2. Jumps across the interface

For Eshelby’s inclusion problem in isotropic elasticity, the interface jump of Eshelby’s tensor was theoretically clarified (from (2.11), (2.12) of Eshelby (1957); see also Mura, 1982). For some specially cases of polygonal inclusions, Rodin (1996) checked it again and Nozaki et al. (2001) confirmed it through investigating the interface jump of stress between the matrix and inclusion through numerical calculation. In comparison, the interface jump of Eshelby’s problem in anisotropic elasticity has been seldom studied (Kuvshinov, 2008).

In our problem, the interface jump of \( g(p, z) \) can be derived from its integral formula (25). Rewrite (25) in the form

\[
g(p, z) = \frac{1 + ip}{1 + ip} X^0 + \frac{p - p}{\pi(1 + ip)} \int_r \frac{k(y; z)dy}{y - z}
\]

(104)
with

$$\kappa(y, z) = \frac{1}{1 - ip + (1 + ip) \frac{p}{z}}. \quad \text{(105)}$$

Assuming that the boundary line around a boundary point $b$ is continuously differentiable and has a local normal $n_b$, we can get the interface jump of $g(p; z)$ with the aid of the Cauchy principal-value integral (cf. Woods, 1976). When two points approach $b$ from two sides of the boundary, the second Plemelj formula gives the jump of $g(p; z)$ (outside values minus inside values) as follows:

$$\Delta g = \frac{1 + ip - 2i(p - \bar{p})}{1 + ip} \kappa(\gamma \rightarrow b, b) = \frac{(1 + ip)n_b - (1 - ip)n_b}{(1 - ip)n_b - (1 + ip)n_b}. \quad \text{(106)}$$

A remarkable property of the interface jump is its unit norm for an arbitrary boundary orientation, namely $|\Delta g| = 1$. Specially, for a vertical, a horizontal and a 45 degree oblique, we have

$$\Delta g = \begin{cases} -1, & n_b = \pm t, \\ -\frac{\bar{p}}{p}, & n_b = \pm 1, \\ -\frac{1 - ip - (1 - op)\bar{p}}{1 + ip - (1 + op)p}, & n_b = \pm \frac{1 + ip}{1 - ip}. \end{cases} \quad \text{(107)}$$

Taking $p = -0.3812 + 1.5815i$, we get $\Delta g = 0.1346 - 0.9909i$ for the 45 degree oblique, which has been exhibited for the solution of the Jordan curve in Fig. 5. For horizontal boundaries, there are only constant unit jumps of $\text{Re}[g]$; for vertical boundaries, the jumps of $g$ are

$$\Delta g_{\text{vertical}} = (0.8902 - 0.4556i, 0.8420 + 0.5395i, 0.9998 - 0.02200i, 0.9617 + 0.2740i) \quad \text{(108)}$$

for different eigenvalues $p$ in (97). These theoretical analyses are in agreement with our numerical results shown in Fig. 7 and 8. Since the jumps of $\text{Im}[g]$ across the vertical boundaries are very small, it is difficult to find them without the above theoretical analysis.

Further, denote the normal $n_b$ by $e^{i\theta}$, under the linear transformation

$$z \mapsto x_1 + px_2 = \frac{1 - ip}{2}z + \frac{1 + ip}{2}z, \quad \text{(109)}$$

induced by eigenvalue $p$, $n_b$ will deform to be $n_b'$ (Fig. 10)

$$n_b' = \frac{1 - ip}{2}e^{i\theta} + \frac{1 + ip}{2}e^{-i\theta}. \quad \text{(110)}$$

The interface jump (106) can be rewritten as

$$\Delta g = \frac{(1 + ip)e^{i\theta} - (1 - ip)e^{-i\theta}}{(1 - ip)e^{i\theta} - (1 + ip)e^{-i\theta}} = \frac{\overline{n_b'}}{n_b}. \quad \text{(111)}$$

which posts the geometrical meaning of (106), namely the jump $\Delta g$ is determined by the local deformed normal vector $n_b'$. It should be noticed that the local deformed normal vector $n_b'$ is in general not normal to the deformed boundary $\Gamma'$, since the linear transformation is not necessarily conformal.

**Fig. 6.** Parameters of regular crux.

**Fig. 7.** Real part of $g(p; z)$ for different $p$: (a) $(-0.3812, 1.5815)$, (b) $(0.3104, 1.0598)$, (c) $(-0.0196, 0.9961)$, (d) $(0.08173, 0.5851)$. 

7. Concluding remarks

Eshelby’s problem of arbitrarily shaped inclusions in a piezoelectric plane has been solved under the extended Stroh formalism. The presentation of a new compact boundary integral expression of eigenfunction make it much less difficult to work out explicit analytical solutions for various non-elliptical inclusions. The explicit analytical solutions obtained for non-elliptical inclusions in these paper unified all known results, and can be used in particular for checking the validity of Eshelby’s equivalent inclusion idea widely adopted in micromechanics when anisotropic materials are concerned. The results of inclusions of smooth shape can also serve as benchmarks of numerical study.

The methodology elaborated for solving Eshelby’s problem can be extended to more complex coupled phenomena, such as thermo-magnetoelectroelasticity. Besides, though the four eigenvalues in the extended Stroh formalism applied to piezoelectricity are assumed to be distinct in this paper, the cases were the repeated eigenvalues appear can be also treated by using slightly perturbed material coefficients, since the resulting errors are negligible (Pan, 2004). For a strict treatment of degenerate cases, one can refer to Guo and Zheng (2003).

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Appendix A. Elements of the extended Stroh formalism

A.1. Equations of piezoelectric media

The basic equations for a linear piezoelectric solid are given by

\[
\begin{align*}
    \sigma_{ij} &= C_{ijkl} u_{kl} + \epsilon_{ijkl} \phi_k, \\
    \sigma_{ij} &= 0, \\
    D_k &= e_{ijkl} u_{ij} - \epsilon_{ijkl} \phi_k.
\end{align*}
\]

(112)

where \( i, j, k = 1, 2, 3 \), repeated indices mean summation, a comma followed by \( i \) (\( i = 1,2,3 \)) stands for the partial derivative with respect

Fig. 8. Image part of \( g(p,z) \) for different \( p \): (a) \((-0.3812,1.5815)\), (b) \((0.3104,1.0598)\), (c) \((-0.01096,0.9961)\), (d) \((0.08173,0.5851)\).

Fig. 9. Singularity approaching the vertex.

Fig. 10. Deformation of boundary and its normal: (a) \( z = x_1 + i x_2 \), (b) \( z = x_1 + px_2 \).
to the ith spatial coordinate, \( u_i \) and \( \phi \) are the displacements and electrical potential, \( \sigma_{ij} \) and \( D_i \) are the stress and electrical displacements, and \( C_{ijkl}, e_{ij} \) and \( \varepsilon_{ij} \) are the elastic, piezoelectric and dielectric constants, respectively. Defining the extended displacement and stress components by

\[
\mathbf{u} = \begin{cases} u_i, & i = 1, 2, 3 \\ \phi, & i = 4 \end{cases}, \quad \mathbf{\sigma} = \begin{cases} \sigma_{ij}, & i = 1, 2, 3 \\ \varepsilon_{ij}, & i = 4 \end{cases}, \quad \mathbf{D} = \begin{cases} D_i, & i = 4 \end{cases},
\]

(113)

and adopting the notation

\[
C_{ijkl}, \quad I, K, k = 1, 2, 3, \\
\varepsilon_{ik}, \quad K = 4, I = 1, 2, 3, \\
\varepsilon_{ik}, \quad I = 4, K = 1, 2, 3, \\
-\varepsilon_{ij}, \quad I = 4.
\]

(114)

Eq. (112) for piezoelectricity can be recast into

\[
\sigma_{ij} = C_{ijkl}u_{kli}, \quad \sigma_{ij} = 0.
\]

(115)

which are reminiscent of the Hooke law and equilibrium equation of classical elasticity.

A.2. Extended Stroh formalism

The general solution of Eq. (115) governing a generalized two-dimensional problem where all physical quantities depend only on \( x_1 \) and \( x_2 \) can be obtained according to the extended Stroh formalism (Lothe and Barnett, 1976; Kuo and Barnett, 1991; Suo et al., 1992; Liang and Hwu, 1996; Ting, 1996; Tanuma, 2007). More precisely, we seek the solution of the form

\[
\mathbf{u} = (u_1, u_2, u_3, \phi)^T = \mathbf{a}(x_1 + px_2),
\]

(116)

where \( \mathbf{a} \) is a constant four-dimensional (4D) vector, \( p \) is a complex number, \( f(z) \) is an analytic function of the variable \( \zeta \) and the superscript \( T \) denotes the transpose of a matrix or vector. Substituting (116) into (115) will transform the problem to solve differential equations to an algebraic one to work out quadratic eigenvalue as follows, and the analytic functions will be determined according to the boundary conditions. Thus, all the Eq. (115) are satisfied for an arbitrary analytic function \( f \) if (see, e.g., Chung and Ting, 1996)

\[
\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T} \mathbf{a} = \mathbf{0},
\]

(117)

where the \( 4 \times 4 \) matrix \( \mathbf{R} \) and the \( 4 \times 4 \) symmetric matrices \( \mathbf{Q} \) and \( \mathbf{T} \) are defined by

\[
\mathbf{R}_{ik} = C_{ikx2}, \mathbf{Q}_{ik} = C_{ikx1}, \mathbf{T}_{ik} = C_{ikx2}.
\]

(118)

For the existence of a non-zero vector \( \mathbf{a} \), the characteristic equation of the eigenvalue problem (117), namely

\[
\det(\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}) = 0,
\]

(119)

must be verified. For a stable material with positive energy density, the roots of (119) form four conjugate pairs with non-zero imaginary parts (cf. Eshelby et al., 1953). Choosing four distinct roots \( p_i \) \((i = 1, 2, 3, 4)\) with positive imaginary parts and \( \mathbf{a}_i \) \((i = 1, 2, 3, 4)\) be the associated eigenvectors, then the general solution (the generalized displacement \( \mathbf{u} \) and the generalized stress function \( \psi \)) to (115) can be written as

\[
\mathbf{u} = (u_1, u_2, u_3, \phi)^T = 2\text{Re}(\mathbf{A} f(z)), \\
\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T = 2\text{Re}(\mathbf{B} f(z)),
\]

(120)

where “Re” stands for the real part and the constant matrices \( \mathbf{A} \) and \( \mathbf{B} \) are defined through \( \mathbf{a}_i \) as follows:

\[
\mathbf{b}_i = (\mathbf{R}^T + p_i \mathbf{T}) \mathbf{a}_i = -p_i^{-1}(\mathbf{Q} + p_i \mathbf{R}) \mathbf{a}_i, \quad i = 1, 2, 3, 4,
\]

\[
\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4), \quad \mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4).
\]

(121)

In (120), the 4D vector \( f(z) \) formed by the four analytic functions \( f_i(z) \) \((i = 1, 2, 3, 4)\), i.e.,

\[
\mathbf{f}(z) = [f_1(z_0), f_2(z_0), f_3(z_0), f_4(z_0)]^T,
\]

(122)

and the generalized stress function \( \psi \) is related to the extended stress components by

\[
\sigma_{11} = -\psi_{11}, \quad \sigma_{12} = \psi_{11}, \quad i = 1, 2, 3, 4.
\]

(123)

The eigenvalues \( p_i \) and eigenvectors \( (\mathbf{a}_i, \mathbf{b}_i) \) depend on the generalized material stiffness matrix \( C_{ijkl} \) and can be equivalently determined by the following eigenrelation (Chung and Ting, 1996):

\[
\mathbf{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \quad \zeta = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix},
\]

(125)

and

\[
\mathbf{N}_1 = -\mathbf{T}^{-1} \mathbf{R}, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R} \mathbf{T}^{-1} \mathbf{R} - \mathbf{Q}.
\]

(126)

with \( \mathbf{Q}, \mathbf{R}, \mathbf{T} \) being three \( 4 \times 4 \) real matrices defined by (118). Another approach to compute the eigenvalues \( p_i \) and eigenvectors \((\mathbf{a}_i, \mathbf{b}_i)\) is called the Lehnitzkii formalism (Ting, 1999, 2000a), where the eigenvectors can be given explicitly as soon as the eigenvalues \( p_i \) are worked out.

A.3. Symmetric expressions

It is often useful to write the stress and strain solutions explicitly in a symmetric form (Mantic and Paris, 1997; Ting, 1998, 2000b). From

\[
\sigma_{ij} = 2\text{Re} \left[ \sum_M B_{0M} f_M(z_M) \right], \quad \sigma_{ii} = -2\text{Re} \left[ \sum_M B_{0M} p_M f_M(z_M) \right], 
\]

(127)

and

\[
B_{1M} = -p_M B_{2M} \text{ (no summation)}
\]

(128)

we can write

\[
\sigma_{ij} = 2\text{Re} \left[ \sum_M B_{1M} f_M(z_M) B_{0M}^* \right].
\]

(129)

It should be noticed that (129) cannot be applied to \( \sigma_{33}, \sigma_{44}, \sigma_{34} \) and \( \sigma_{44} \) since this application is either useless or meaningless.

Similarly, from

\[
u_{i1} = 2\text{Re} \left[ \sum_M A_{0M} f_M(z_M) \right], \quad \nu_{i2} = 2\text{Re} \left[ \sum_M A_{0M} p_M f_M(z_M) \right], 
\]

(130)

using the notation

\[
\begin{pmatrix}
B_{21} & -B_{11} & 0 & 0 \\
B_{22} & -B_{12} & 0 & 0 \\
B_{23} & -B_{13} & 0 & 0 \\
B_{24} & -B_{14} & 0 & 0
\end{pmatrix}
= \mathbf{B}^* \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \mathbf{B}^* \mathbf{K}
\]

(131)

and the geometric relations \( \varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) \) with \( u_{0i} = 0 \), we can derive that

\[
\varepsilon_{ij} = \text{Re} \left[ \sum_M A_{0M} B_{2M}^* f_M(z_M) B_{0M} K_{0i} + A_{0M} B_{2M}^* f_M(z_M) B_{0M} K_{0j} \right].
\]

(132)
The formula (132) does not apply to $\varepsilon_{33}$, $\varepsilon_{44}$, $\varepsilon_{43}$ and $\varepsilon_{44}$ because it is useless or meaningless for these components.

**Appendix B. Connection with the solution of Pan (2004) derived via Green's function**

The Green's function method plays a key role in the study of various Eshelby's problems (Mura, 1982; Ting, 1996). This appendix aims to establish a connection between (20) and the Green's function solution of Pan (2004).

**B.1. Green's function solution of Pan (2004)**

We first recall the solution of Pan (2004) as follows. Let $u^f_i(x, X)$ be the $i$th Green's function of electric displacement/electric potential at $x$ due to a unit point-force/point-charge in the $j$th direction applied at $x$. Then, $u^f_i(x, X) = n_i(x)C_{ijkl}e^f_{lm}ds(x)$ must be the electric displacement/electric potential at $x$ due to a force-face/face-charge along $n$-direction at $x$ induced by the interface traction corresponding to the generalized eigenstrain $\varepsilon^f_{lm}$. That means that the generalized displacement $u^f_i(X)$ determined by

$$u^f_i(X) = \int_{\partial V} u^f_i(x, X)n_i(x)C_{ijkl}e^f_{lm}ds(x) = \int_{\partial V} u^f_i(x, X)C_{ijkl}e^f_{lm}dx$$

(133)

satisfies the field equation

$$C_{ijkl}u^f_{kl} = -f_j = C_{ijkl}e^f_{kl}, \quad l = 1, 2, J = 1, 2, 3, 4. \quad (134)$$

For a uniform generalized eigenstrain $\varepsilon^f_{lm}$ over the inclusion $\omega$, the equilibrium relation (134) indicates a point-force/point-charge $-C_{ijkl}e^f_{kl}$ along the interface $\Gamma$. The corresponding solutions for the elastic strain and electric fields are

$$\gamma_{ij}(X) = \frac{1}{2}C_{ijkl}e^f_{lm}\int_{\partial V} u^f_{ij}(x, X) + u^f_{ji}(x, X) n_i(x)ds(x), \quad (135)$$

$$E_p(X) = -C_{ijkl}e^f_{lm}\int_{\partial V} u^f_{ij}(x, X) n_i(x)ds(x). \quad (136)$$

Following Ting (1996), the Green's function $u^f_i(x, X)$ can be derived as (also see Pan, 2002c)

$$u^f_i(y, z) = \frac{1}{\pi} \text{Im}\{A_p \ln(y_1 - z_1)\}/A_x. \quad (137)$$

So the induced generalized displacement $u^f_i(X)$ can be calculated by

$$u^f_i(X) = C_{ijkl}e^f_{lm}\int_{\Gamma} \text{Im}\{A_p \ln(y_1 - z_1)\}/A_x n_i(y)ds(y), \quad (138)$$

which is integrable for polygonal inclusions (Pan, 2004).

**B.2. Connection between (20) and (138)**

From (16) and the definition $c_iy_i + d_iy_i = A_{ij}f_i(z_i)$, we have

$$A_{ij}f_i(z_i) = A_{ij} \int_{\Gamma} \frac{c_iy_i + d_iy_i}{y_1 - z_1}dy_1$$

$$= -A_{ij}B_1 \int_{\Gamma} \frac{y_1}{y_1 - z_1}y_m d\ln(y_1 - z_1)$$

$$= -A_{ij}B_1 \int_{\Gamma} \frac{y_1}{y_1 - z_1}y_m d\ln(y_1 - z_1)$$

$$= -A_{ij}B_1 \int_{\Gamma} \frac{y_1}{y_1 - z_1}y_m d\ln(y_1 - z_1)$$

$$-A_{ij}B_1 \int_{\Gamma} \frac{y_1}{y_1 - z_1}y_m d\ln(y_1 - z_1)$$

(139)

Using (118) and (121), we get

$$B = R^fA + TA(p) = -QA(p^{-1}) - RA. \quad (140)$$

It follows that

$$\left\{ \begin{array}{l} \varepsilon^f_{ij}B = \varepsilon^f_{ij}R^fA + TA(p) \\ -\varepsilon^f_{ij}B = \varepsilon^f_{ij}QA(p^{-1}) + RA \end{array} \right. \quad (141)$$

or

$$\left\{ \begin{array}{l} B_{ij}^{\varepsilon_{ij}} = (C_{ijkl}e^f_{ij})A_{kl} + [T_{ij}e_{ik} - \frac{1}{2}Q_{ijkl}e^f_{ij}p_A]A_{kl} \\ -B_{ij}^{\varepsilon_{ij}} = (C_{ijkl}e^f_{ij})A_{kl} - [T_{ij}e_{ik} - \frac{1}{2}Q_{ijkl}e^f_{ij}p_A]A_{kl} \end{array} \right. \quad (142)$$

Combining (8) and (142) yields

$$B_{ij}^{\varepsilon_{ij}}n_i = C_{ijkl}e^f_{ij}n_i ds + [T_{ij}e_{ik} - \frac{1}{2}Q_{ijkl}e^f_{ij}]A_{ij}n_i ds. \quad (143)$$

Thus, the generalized displacement field can be expressed by

$$u^f_i = 2\text{Re}\{A_p f_i(z_i)\} = 2\text{Re}\{A_p B_{ij}^{\varepsilon_{ij}}n_i ds - A_p B_{ij}^{\varepsilon_{ij}}n_i ds \}.$$
Pan, E., 2002a. Elastic and piezoelectric fields in substrates GaAs (001) and GaAs.


