Matrix semigroup homomorphisms into higher dimensions

Damjana Kokol Bukovšek *

Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

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Abstract

We study non-degenerate irreducible homomorphisms from the multiplicative semigroup of all \( n \)-by-\( n \) matrices over an algebraically closed field of characteristic zero to the semigroup of \( m \)-by-\( m \) matrices over the same field. We prove that every non-degenerate homomorphism from the multiplicative semigroup of all \( n \)-by-\( n \) matrices to the semigroup of \((n + 1)\)-by-\((n + 1)\) matrices when \( n \geq 3 \) is reducible and that every non-degenerate homomorphism from the multiplicative semigroup of all \( 3 \)-by-\( 3 \) matrices to the semigroup of \( 5 \)-by-\( 5 \) matrices is reducible.

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1. Introduction

Let \( \mathbb{F} \) be an algebraically closed field of characteristic zero and let \( \mathcal{M}_n(\mathbb{F}) \) denote all \( n \)-by-\( n \) matrices with entries in \( \mathbb{F} \). In this paper we study non-degenerate matrix semigroup homomorphisms \( \varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F}) \), i.e., multiplicative maps, where \( n \geq 3 \) and \( m > n \). One way to obtain a semigroup homomorphism \( \varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F}) \) is to take a group homomorphism \( \varphi' : GL_n(\mathbb{F}) \to GL_m(\mathbb{F}) \) and trivially extend it to all matrices taking \( \varphi(A) = 0 \) for every \( A \) with \( \det A = 0 \). These trivial extensions are called degenerate and are known (see for example [13, pp. 115–136] or [1, p. 231]). The problem of homomorphisms \( \varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F}) \) is solved

* Fax: +386 1 25 17 281.
E-mail address: damjana.kokol@fmf.uni-lj.si

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for \( m \leq n \) in [5,2]. The case \( n = 1 \) for the field \( \mathbb{C} \) of complex numbers is studied in [7]. The case \( n = 2, m = 3 \) is explored in [3], and the case \( n = 2, m = 4 \) in [4]. Semigroup homomorphisms \( \varphi : \Delta_n(\mathbb{F}) \rightarrow \mathcal{M}_n(\mathbb{F}) \), where \( \Delta_n(\mathbb{F}) \) is a certain subsemigroup of \( \mathcal{M}_n(\mathbb{F}) \), are characterized in [14]. Semigroup homomorphisms \( \varphi : \Delta_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{K}) \), where \( \mathbb{K} \) is another field, are explored in [10–12,15] for the case \( m \leq n \). In [9] isomorphisms of subsemigroups of \( \mathcal{M}_n(\mathbb{F}) \) that contain all rank one matrices are studied. The problem of homomorphisms \( \varphi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F}) \) is connected to the problem of congruences on the semigroup \( \mathcal{M}_n(\mathbb{F}) \), which are characterized in [6].

A semigroup homomorphism \( \varphi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F}) \) is irreducible if the image of \( \varphi \) is an irreducible semigroup, i.e., it has no proper non-trivial invariant subspace of \( \mathbb{F}^m \) when it is viewed as a set of matrices acting on vector space \( \mathbb{F}^m \). We prove that every non-degenerate homomorphism \( \varphi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_{n+1}(\mathbb{F}) \) is reducible and that every non-degenerate homomorphism \( \varphi : \mathcal{M}_3(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F}) \), where \( m \) is 4 or 5, is reducible.

2. Singular matrices

We first look where an non-degenerate irreducible homomorphism sends singular matrices.

**Proposition 1.** Let \( \varphi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F}) \) a semigroup homomorphism, which sends 0 to 0 and identity to identity. Let

\[
\begin{align*}
  k &= \min\{\text{rank } A; \varphi(A) \neq 0\}. \\
  \binom{n}{k} &\leq m. 
\end{align*}
\]

If \( \text{rank } A = \text{rank } B \) then \( \varphi(A) = \varphi(B) \).

**Proof.** A semigroup homomorphism which sends \( I \) to \( I \), maps invertible matrices to invertible matrices. If \( \text{rank } A = \text{rank } B \), then there exist such invertible matrices \( P, Q \) that \( A = PBQ \). So \( \varphi(A) = \varphi(P)\varphi(B)\varphi(Q) \) and \( \text{rank } \varphi(A) = \text{rank } \varphi(B) \).

Let \( E_1, E_2, \ldots, E_t \) be \( \binom{n}{k} \) distinct diagonal idempotents of rank \( k \). Then \( \text{rank } \varphi(E_1) = \text{rank } \varphi(E_2) = \cdots = \text{rank } \varphi(E_t) \geq 1 \). Since \( E_iE_j \) for \( i \neq j \) has rank less than \( k \), we have \( \varphi(E_i)\varphi(E_j) = 0 \), and \( \varphi(E_1), \varphi(E_2), \ldots, \varphi(E_t) \) are orthogonal idempotents. We conclude that \( t(\text{rank } \varphi(E_1)) \leq m \), implying \( \binom{n}{k} \leq m. \)

**Proposition 2.** Assume that \( n \geq 3 \) and \( m < 2n \). Let \( \varphi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F}) \) be a semigroup homomorphism, which is non-degenerate and sends 0 to 0 and identity to identity. Suppose that \( \text{rank } A = 1 \) implies \( \text{rank } \varphi(A) = 1 \). Then \( \text{rank } A = 2 \) implies \( \text{rank } \varphi(A) = 2 \).

**Proof.** Denote by \( E_{ij} \) the matrix which has 1 in the \( i \)th row and the \( j \)th column, and 0 elsewhere. Matrices \( \varphi(E_{11}), \varphi(E_{22}), \ldots, \varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F}) \) are orthogonal idempotents of rank 1. Let

\[
\begin{align*}
P_2 &= E_{11} + E_{22}, & P_3 &= E_{11} + E_{33}, & \ldots, & P_n &= E_{11} + E_{nn}.
\end{align*}
\]

\( \text{Rank } \varphi(P_2) \) cannot be 1, since \( \varphi(E_{11}) \) and \( \varphi(E_{22}) \) are orthogonal. Suppose \( \text{rank } \varphi(P_2) \geq 3 \). Then \( \varphi(P_2), \varphi(P_3), \ldots, \varphi(P_n) \) are commuting idempotents of equal rank by Proposition 1. Their products are \( \varphi(P_i)\varphi(P_j) = \varphi(E_{11}) \). So

\[
\text{rank}(\varphi(P_2) + \varphi(P_3) + \cdots + \varphi(P_n)) \geq 2(n - 1) + 1 \geq m.
\]
Now $\varphi(E_{22} + E_{33})$ has products of rank 1 with $\varphi(P_2)$ and $\varphi(P_3)$, and it is orthogonal to $\varphi(P_4), \ldots, \varphi(P_n)$, so
\[
\text{rank} (\varphi(P_2) + \varphi(P_3) + \cdots + \varphi(P_n) + \varphi(E_{22} + E_{33})) \\
\geq \text{rank} (\varphi(P_2) + \varphi(P_3) + \cdots + \varphi(P_n)) + 1,
\]
which is a contradiction. So rank $\varphi(P_2) = 2$ and, finally, rank $A = 2$ implies rank $\varphi(A) = 2$. □

Some parts of this proof could be done using congruences on matrices [6].

The next proposition is trivially true for $n = 3$ and $m < 6$. We prove it also for larger $n$.

**Proposition 3.** Assume that $n > 4$ and $m < 2n$ or that $n = 4$ and $m \leq 5$. Let $\varphi : \mathcal{M}_n(F) \rightarrow \mathcal{M}_m(F)$ be a semigroup homomorphism, which is non-degenerate and sends 0 to 0 and identity to identity. Then we have two possibilities:

(a) if rank $A = 1$ then rank $\varphi(A) = 1$, and if rank $A = 2$ then rank $\varphi(A) = 2$, or

(b) if rank $A < n - 1$ then $\varphi(A) = 0$, and if rank $A = n - 1$ then rank $\varphi(A) = 1$.

**Proof.** Let
\[
k = \min \{\text{rank} A; \varphi(A) \neq 0\}.
\]
Since $\varphi$ is non-degenerate, $1 \leq k \leq n - 1$. If $n > 4$, then $m < 2n \leq \binom{n}{2}$. If $n = 4$, then $m \leq 5 < \binom{4}{2}$. So by Proposition 1, $k = 1$ or $k = n - 1$.

Case (a): $k = 1$. The matrices $E_{11}, E_{22}, \ldots, E_{nn} \in \mathcal{M}_n(F)$ are idempotents of rank 1, so $\varphi(E_{11}), \varphi(E_{22}), \ldots, \varphi(E_{nn}) \in \mathcal{M}_m(F)$ are orthogonal idempotents of the same rank, say $l$. Since they are orthogonal, $nl \leq m$, so $l = 1$. Thus rank $A = 1$ implies rank $\varphi(A) = 1$. Proposition 2 now gives us the asserted result.

Case (b): $k = n - 1$. We have that rank $A < n - 1$ implies $\varphi(A) = 0$. Let $P_1, P_2, \ldots, P_n \in \mathcal{M}_n(F)$ be distinct diagonal idempotents of rank $n - 1$. Then $\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n) \in \mathcal{M}_m(F)$ are orthogonal idempotents with the same rank, say $l$. Since they are orthogonal, $nl \leq m$, so $l = 1$. Thus rank $A = n - 1$ implies rank $\varphi(A) = 1$. □

3. Two possibilities

We will now explore the two possibilities which appear in Proposition 3. The first one is that only 0 maps to 0.

**Proposition 4.** Assume that $n \geq 2$ and $m \geq n$. Let $\varphi : \mathcal{M}_n(F) \rightarrow \mathcal{M}_m(F)$ be a semigroup homomorphism, which is non-degenerate and sends 0 to 0 and identity to identity. Suppose that rank $A = 1$ implies rank $\varphi(A) = 1$ and that rank $A = 2$ implies rank $\varphi(A) = 2$. Then
\[
\varphi(A) = S \begin{bmatrix}
\hat{f}(A) & * \\
* & *
\end{bmatrix} S^{-1},
\]
where $f : F \rightarrow F$ is a field homomorphism and $S \in \mathcal{M}_m(F)$ is an invertible matrix.
This proposition generalizes Theorem 1 in [2] and Corollary 8 in [14] to the case \( m > n \). In the proof we follow the main ideas in [2]. The assumption that \( \varphi \) maps rank 2 matrices to rank 2 matrices is necessary, otherwise function \( f \) may not be additive, as the following example shows.

**Example.** A non-degenerate semigroup homomorphism \( \varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F}) \) defined by
\[
\varphi(A) = \text{Sym}^2 A
\]
maps rank 1 matrices to rank 1 matrices, rank 2 matrices to rank 3 matrices and is of the form
\[
\varphi(A) = \begin{bmatrix}
\hat{f}(A) & * \\
* & *
\end{bmatrix},
\]
where \( f(x) = x^2 \).

**Proof (of Proposition 4).** Denote by \( E_{ij} \) the matrix which has 1 in the \( i \)th row and the \( j \)th column, and 0 elsewhere.

Matrices \( E_{11}, E_{22}, \ldots, E_{nn} \in \mathcal{M}_m(\mathbb{F}) \) are orthogonal idempotents of rank 1, so \( \varphi(E_{11}), \varphi(E_{22}), \ldots, \varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F}) \) are orthogonal idempotents of rank 1. It follows that they are simultaneously similar to
\[
E_{11}, E_{22}, \ldots, E_{nn} \in \mathcal{M}_m(\mathbb{F}).
\]
Thus we may assume without loss of generality that
\[
\varphi(E_{ii}) = E_{ii}.
\]
Let \( \delta_{ij} \) be the Kronecker symbol, \( \delta_{ij} = 1 \) if \( i = j \), and \( \delta_{ij} = 0 \) otherwise. We have
\[
\delta_{ki}\delta_{jl}\varphi(E_{ij}) = \varphi(\delta_{ki}\delta_{jl}E_{ij}) = \varphi(E_{kk}E_{ij}E_{ll}) = E_{kk}\varphi(E_{ij})E_{ll},
\]
so
\[
\varphi(E_{ij}) = \begin{bmatrix}
t_{ij}E_{ij} & 0 \\
0 & *
\end{bmatrix}.
\]
Since \( E_{ij}E_{ji} = E_{ij} \), we obtain \( t_{ij} \neq 0 \), and since \( \varphi(E_{ij}) \) has rank 1, we have \( * = 0 \). Thus
\[
\varphi(E_{ij}) = t_{ij}E_{ij}.
\]
We may now apply a simultaneous similarity with a diagonal matrix
\[
\text{diag}(1, t_{12}, \ldots, t_{1n}, 1, \ldots, 1)
\]
to obtain \( \varphi(E_{1j}) = E_{1j} \). Now
\[
E_{1j} = \varphi(E_{1j}) = \varphi(E_{11}E_{ij}) = E_{11}t_{ij}E_{ij} = t_{ij}E_{1j},
\]
so \( t_{ij} \) equals 1 for all \( i, j \) and therefore
\[
\varphi(E_{ij}) = E_{ij}.
\]
Let \( a \) be an element in \( \mathbb{F} \).
\[
\varphi(aE_{11}) = \varphi(E_{11}aE_{11}E_{11}) = E_{11}\varphi(aE_{11})E_{11},
\]
so the only non-zero entry of \( \varphi(aE_{11}) \) is at the \((1, 1)\) position. So there exists such mapping \( f : \mathbb{F} \to \mathbb{F} \) that
\[
\varphi(aE_{11}) = f(a)E_{11}.
\]
Mapping $f$ is obviously multiplicative. Furthermore

$$
\varphi(aE_{ij}) = \varphi(aE_{i1}E_{1j}) = E_{i1}\varphi(aE_{11})E_{1j} = E_{i1}f(a)E_{11}E_{1j} = f(a)E_{ij}.
$$

Now let $A = [a_{ij}]_{i,j=1}^n$ be a matrix in $\mathcal{M}_n(\mathbb{F})$. We have

$$
E_{ii}\varphi(A)E_{jj} = \varphi(E_{ii}AE_{jj}) = \varphi(a_{ij}E_{ij}) = f(a_{ij})E_{ij},
$$

so the $ij$th entry of $\varphi(A)$ is $f(a_{ij})$ and

$$
\varphi(A) = \begin{bmatrix}
\hat{f}(A) & * \\
* & *
\end{bmatrix}.
$$

Further, the matrix

$$
\varphi(E_{11} + E_{22}) = \begin{bmatrix}
E_{11} + E_{22} & * \\
* & *
\end{bmatrix}
$$

has rank 2; thus we may assume

$$
\varphi(E_{11} + E_{22}) = \begin{bmatrix}
E_{11} + E_{22} & * \\
0 & 0
\end{bmatrix}.
$$

Let $A = [a_{ij}]_{i,j=1}^n$ be a matrix in $\mathcal{M}_n(\mathbb{F})$, such that $a_{ij} = 0$ if $i \geq 3$. Then

$$
\varphi(A) = \varphi((E_{11} + E_{22})A) = \begin{bmatrix}
E_{11} + E_{22} & * \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\hat{f}(A) & * \\
* & *
\end{bmatrix} = \begin{bmatrix}
\hat{f}(A) & * \\
0 & 0
\end{bmatrix}.
$$

Let us now prove that $f$ is additive. For $a, b \in \mathbb{F}$ we have

$$
f(a + b)E_{11} = \varphi((a + b)E_{11}) = \varphi((aE_{11} + bE_{12})(E_{11} + E_{21}))
$$

$$
= \begin{bmatrix}
f(a)E_{11} + f(b)E_{12} & * \\
0 & 0
\end{bmatrix} \begin{bmatrix}
E_{11} + E_{21} & * \\
0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(f(a) + f(b))E_{11} & * \\
0 & 0
\end{bmatrix},
$$

so $f(a + b) = f(a) + f(b)$, and thus $\hat{f}$ is multiplicative. □

The second possibility is that only almost full rank matrices map to non-zero matrices. For a matrix $A \in \mathcal{M}_n(\mathbb{F})$ we denote by Cof$(A)$ the so called cofactor matrix of all $(n-1)$-by-$(n-1)$ minors of the matrix $A$.

**Proposition 5.** Assume that $n \geq 3$ and $m \geq n$. Let $\varphi : \mathcal{M}_n(\mathbb{F}) \rightarrow \mathcal{M}_m(\mathbb{F})$ be a semigroup homomorphism, which is non-degenerate and sends 0 to 0 and identity to identity. Suppose that rank $A < n - 1$ implies $\varphi(A) = 0$ and that rank $A = n - 1$ implies rank $\varphi(A) = 1$. Then

$$
\varphi(A) = S \begin{bmatrix}
\hat{f}(\text{Cof}(A)) & * \\
* & *
\end{bmatrix} S^{-1},
$$

where $f : \mathbb{F} \rightarrow \mathbb{F}$ is a homomorphism of the multiplicative semigroup $(\mathbb{F}, \cdot)$ and $S \in \mathcal{M}_m(\mathbb{F})$ is an invertible matrix.

This proposition generalizes Theorem 2 in [2] to the case $m > n$ and the methods are similar. Here the obtained homomorphism $f$ is not necessarily additive, as the following example shows.
Example. A non-degenerate semigroup homomorphism \( \varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F}) \) defined by \( \varphi(A) = \text{Sym}^2(\text{Cof}(A)) \) maps rank 1 matrices to 0, rank 2 matrices to rank 1 matrices and is of the form 
\[
\varphi(A) = \begin{bmatrix}
\hat{f}(\text{Cof}(A)) & * \\
* & *
\end{bmatrix},
\]
where \( f(x) = x^2 \).

Proof (of Proposition 5). Denote by \( E_{ij} \) the matrix which has 1 in the \( i \)th row and the \( j \)th column, and 0 elsewhere. Introduce \( P_{ii} = I - E_{ii} \in \mathcal{M}_n(\mathbb{F}) \), and let \( I_i \) be the identity matrix in \( \mathcal{M}_i(\mathbb{F}) \). Further, let \( N_i \) be the matrix in \( \mathcal{M}_i(\mathbb{F}) \), defined by \( N_i = E_{12} + \cdots + E_{i-1,i} \). Denote \( P_{ij} = I_{i-1} \oplus N^{T}_{i-j+1} \oplus I_{n-j} \) if \( i < j \), and \( P_{ij} = I_{j-1} \oplus N_{i-j+1} \oplus I_{n-i} \) if \( i > j \).

The matrices \( P_{11}, P_{22}, \ldots, P_{nn} \in \mathcal{M}_n(\mathbb{F}) \) are orthogonal idempotents of rank \( n - 1 \), so \( \varphi(E_{11}), \varphi(E_{22}), \ldots, \varphi(E_{nn}) \in \mathcal{M}_m(\mathbb{F}) \) are orthogonal idempotents of rank 1. So they are simultaneously similar to \( E_{11}, E_{22}, \ldots, E_{nn} \in \mathcal{M}_m(\mathbb{F}) \). Without loss of generality we may thus assume that 
\[
\varphi(P_{ii}) = E_{ii}.
\]

Observe that \( P_{ij} = P_{ik}P_{kj} \) and \( P_{ik}P_{lj} \) has rank less than \( n - 1 \) if \( k \neq l \). We now have
\[
\delta_{ki}\delta_{jl}\varphi(P_{ij}) = \varphi(\delta_{ki}\delta_{jl}P_{ij}) = \varphi(P_{kk}P_{ij}P_{ll}) = E_{kk}\varphi(P_{ij})E_{ll},
\]
so
\[
\varphi(P_{ij}) = \begin{bmatrix}
t_{ij}E_{ij} & 0 \\
0 & *
\end{bmatrix}.
\]
The matrix \( \varphi(P_{ij}) \) has rank 1, so \( t_{ij} \neq 0 \) and \( * = 0 \). This implies
\[
\varphi(P_{ij}) = t_{ij}E_{ij}.
\]
We may now apply a simultaneous similarity with a diagonal matrix
\[
\text{diag}(1, t_{12}, \ldots, t_{1n}, 1, \ldots, 1)
\]
to obtain \( \varphi(P_{1j}) = E_{1j} \). Now
\[
E_{1j} = \varphi(P_{1j}) = \varphi(P_{1i}P_{ij}) = E_{1i}t_{ij}E_{ij} = t_{ij}E_{1j},
\]
so \( t_{ij} = 1 \) for all \( i, j \) and
\[
\varphi(P_{ij}) = E_{ij}.
\]
For a matrix \( A \in \mathcal{M}_n(\mathbb{F}) \) we denote by \( A_{ij} \in \mathcal{M}_{n-1}(\mathbb{F}) \) the matrix \( A \) with \( i \)th row and \( j \)th column deleted. Let \( A \in \mathcal{M}_{n-1}(\mathbb{F}) \) be arbitrary matrix and \( A' = 0_1 \oplus A \in \mathcal{M}_n(\mathbb{F}) \). Then
\[
\varphi(A') = \varphi(P_{11}A'P_{11}) = E_{11}\varphi(A')E_{11},
\]
so the only non-zero entry of \( \varphi(A') \) is at the \( (1, 1) \) position. Thus we have a multiplicative mapping \( \varphi' : \mathcal{M}_{n-1}(\mathbb{F}) \to \mathbb{F} \). So there exists a multiplicative mapping \( f : \mathbb{F} \to \mathbb{F} \) such that
\[
\varphi'(A) = f(\det A)
\]
and
\[
\varphi(A') = f(\det A)E_{11} = f(\det A'_{11})E_{11}.
\]
Now let \( B \in \mathcal{M}_n(\mathbb{F}) \). We have
\[
E_{ii}\varphi(B)E_{jj} = \varphi(P_{ii}BP_{jj}) = \varphi(P_{ii}P_{1i}BP_{j1}P_{j1}) = E_{ii}\varphi(P_{ii}BP_{j1})E_{jj}.
\]
The matrix \( P_{ii}BP_{j1} \) has the form of \( A' \), so \( \varphi(P_{ii}BP_{j1}) = \hat{f}(\det B_{ij})E_{11} \) and
\[
E_{ii}\varphi(B)E_{jj} = E_{ii}\hat{f}(\det B_{ij})E_{11}E_{1j} = E_{ii}\hat{f}(\det B_{ij})E_{jj}.
\]
Thus the \( ij \)th entry of \( \varphi(A) \) is \( \hat{f}(\det A_{ij}) \) and
\[
\varphi(A) = \begin{bmatrix}
\hat{f}(\text{Cof}(A)) & * \\
* & *
\end{bmatrix}.
\]
This ends the proof. \( \square \)

4. Case \( m = n + 1 \)

We will now prove our main theorem. We will assume that \( m = n + 1 \) and show that in this case either of the two possibilities of the previous section gives us reducibility.

**Theorem 6.** Assume that \( n \geq 3 \). Every non-degenerate semigroup homomorphism \( \varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{n+1}(\mathbb{F}) \) is reducible.

**Proof.** Suppose \( \varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{n+1}(\mathbb{F}) \) is an irreducible non-degenerate semigroup homomorphism. An irreducible semigroup homomorphism maps 0 to 0, \( I \) to \( I \) and invertible matrices to invertible matrices. By Proposition 3 we have two possibilities:

(a) rank \( A = 1 \) implies rank \( \varphi(A) = 1 \) and rank \( A = 2 \) implies rank \( \varphi(A) = 2 \) or
(b) rank \( A < n - 1 \) implies \( \varphi(A) = 0 \) and rank \( A = n - 1 \) implies rank \( \varphi(A) = 1 \).

In case (a)
\[
\varphi(A) = S \begin{bmatrix}
\hat{f}(A) & * \\
* & *
\end{bmatrix} S^{-1},
\]
where \( f : \mathbb{F} \to \mathbb{F} \) is a field homomorphism and \( S \in \mathcal{M}_{n+1}(\mathbb{F}) \) is an invertible matrix. So for arbitrary \( A \in \mathcal{M}_n(\mathbb{F}) \) we now have
\[
\varphi(A) = \begin{bmatrix}
\hat{f}(A) & \varphi_{12}(A) \\
\varphi_{21}(A) & \varphi_{22}(A)
\end{bmatrix}.
\]
If also \( B \in \mathcal{M}_n(\mathbb{F}) \), then
\[
\varphi(AB) = \begin{bmatrix}
\hat{f}(AB) & \varphi_{12}(AB) \\
\varphi_{21}(AB) & \varphi_{22}(AB)
\end{bmatrix} = \begin{bmatrix}
\hat{f}(A) & \varphi_{12}(A) \\
\varphi_{21}(A) & \varphi_{22}(A)
\end{bmatrix} \begin{bmatrix}
\hat{f}(B) & \varphi_{12}(B) \\
\varphi_{21}(B) & \varphi_{22}(B)
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\hat{f}(A)\hat{f}(B) + \varphi_{12}(A)\varphi_{21}(B) & * \\
* & *
\end{bmatrix}.
\]
So \( \varphi_{12}(A)\varphi_{21}(B) = 0 \) for all \( A, B \in \mathcal{M}_n(\mathbb{F}) \). If \( \varphi_{12}(A) \neq 0 \) for some \( A \in \mathcal{M}_n(\mathbb{F}) \), we have a non-zero linear functional, which is zero on the image of \( \varphi \). So \( \varphi \) is reducible (see [8, p. 27]) If \( \varphi_{12}(A) = 0 \) for every \( A \in \mathcal{M}_n(\mathbb{F}) \), \( \varphi \) is reducible by the same argument.
In case (b)

\[ \varphi(A) = S \begin{bmatrix} \hat{f}(\text{Cof}(A)) & * \\ * & * \end{bmatrix} S^{-1}, \]

where \( f : \mathbb{F} \to \mathbb{F} \) is a semigroup homomorphism and \( S \in \mathcal{M}_{n+1}(\mathbb{F}) \) is an invertible matrix.

We consider the images under \( \varphi \) of the permutation matrices. Denote by \( R_i \) the transposition matrix \( I_i - 1 \oplus (E_{12} + E_{21}) \oplus I_{n-i-1} \) for \( i = 1, 2, \ldots, n-1 \). If \( j < i \) or \( j > i + 1 \), we have \( P_{jj}R_i = P_{jj}R_iP_{jj} \), so \( E_{jj}\varphi(R_i) = E_{jj}\varphi(R_i)E_{jj} \), thus the only non-zero element in the \( j \)th row of \( \varphi(R_i) \) is in the \( j \)th position. The same holds for the \( j \)th column. On the other hand, \( P_{ii}R_i = P_{ii(i+1)} \), so \( E_{ii}\varphi(R_i) = E_{ii(i+1)} \), thus the only non-zero element in the \( i \)th row of \( \varphi(R_i) \) is in the \((i+1)\)st position and vice versa. The same holds for the \( i \)th and the \((i+1)\)st column.

We have thus seen that

\[ \varphi(R_i) = S \begin{bmatrix} \hat{f}(\text{Cof}(R_i)) & 0 \\ 0 & * \end{bmatrix} S^{-1}. \]

The entry in the last row and column must be \( \pm 1 \), since \( R_i \) is an involution. Since the matrices \( R_i \) generate the whole group of permutation matrices, we have for every permutation matrix \( P \)

\[ \varphi(P) = S \begin{bmatrix} \hat{f}(\text{Cof}(P)) & 0 \\ 0 & \pm 1 \end{bmatrix} S^{-1}. \]

Now let \( A = A' \oplus I_{n-2} \), where

\[ A' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_2(\mathbb{F}). \]

So

\[ \varphi(A) = S \begin{bmatrix} \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) & f(ad - bc)I_{n-2} & * \\ * & * \end{bmatrix} S^{-1}. \]

Multiplying \( A \) by \( P_{33}, \ldots, P_{nn} \) on the left or on the right side we obtain

\[ \varphi(A)_{n+1,i} = 0 \quad \text{and} \quad \varphi(A)_{i,n+1} = 0 \]

for \( i = 3, \ldots, n \). Thus

\[ \varphi(A) = S \begin{bmatrix} \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) & 0 & * \\ 0 & f(ad - bc)I_{n-2} & 0 \end{bmatrix} S^{-1}. \]

Let \( C_{1,2,(n+1)} \) be a compression to the first, second and last rows and columns of a matrix. Define

\[ \psi(A') = C_{1,2,(n+1)}(S^{-1}\varphi(A' \oplus I_{n-2})S). \]

It is obvious that \( \psi \) is multiplicative and we have just seen that

\[ \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}. \]

By Theorem 1 in [3] we have two possibilities:

(i)

\[ \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}. \]
and $f$ is additive. In this case we have

$$
\varphi(A) = S \left[ \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ f(ad - bc)I_{n-2} & 0 \end{bmatrix} \right] S^{-1} = S \left[ \hat{f}(\text{Cof}(A)) \begin{bmatrix} 0 & 0 \\ * & \end{bmatrix} \right] S^{-1}
$$

for $A = A' \oplus I_{n-2}$. The same holds for permutation matrices. Since matrices of the form $A = A' \oplus I_{n-2}$ and permutation matrices generate the complete $\mathcal{M}_n(F)$, we obtain

$$
\varphi(A) = S \left[ \hat{f}(\text{Cof}(A)) \begin{bmatrix} 0 & 0 \\ * & \end{bmatrix} \right] S^{-1}
$$

for all $A \in \mathcal{M}_n(F)$, and consequently $\varphi$ is reducible.

(ii) $\psi\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \hat{g} \left( \begin{bmatrix} d^2 & c^2 & dc \\ b^2 & a^2 & ba \\ 2db & 2ca & da + cb \end{bmatrix} \right)$, where $f(x) = g(x^2)$ and $g$ is additive. In this case we have

$$
\varphi(A) = S \hat{g} \left[ \begin{bmatrix} d^2 & c^2 & 0 & dc \\ b^2 & a^2 & 0 & ba \\ 0 & 0 & (ad - bc)^2I_{n-2} & 0 \\ 2db & 2ca & 0 & da + cb \end{bmatrix} \right] S^{-1}
$$

for $A = A' \oplus I_{n-2}$. Now let

$$
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3}
$$

and $B = R_2AR_2$, so

$$
B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3}.
$$

We have

$$
AB = BA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3},
$$

but on the other hand

$$
\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1},
$$

$$
\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \end{bmatrix} \pm 1.
$$
\[ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix} S^{-1} = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \pm 1 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ \pm 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}, \]

so

\[
\varphi(A)\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \pm 1 \\ 0 & 0 & 0 & I_{n-3} & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}
\]

and

\[
\varphi(B)\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \pm 1 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}.
\]

This is a contradiction, so that the possibility (ii) cannot occur. \(\square\)

**Remark 1.** Case (a) in the proof is general: If \(n \geq 3, m > n\) and \(\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})\) is a non-degenerate semigroup homomorphism such that rank \(A = 1\) implies rank \(\varphi(A) = 1\) and rank \(A = 2\) implies rank \(\varphi(A) = 2\), then \(\varphi\) is reducible.

5. Case \(n = 3\) and \(m = 4, 5\)

We will now explore the case \(n = 3\) a little further.

**Theorem 7.** Assume that \(m = 4\) or \(m = 5\). Every non-degenerate semigroup homomorphism \(\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})\) is reducible.

**Proof.** If \(m = 4\), this is a special case of Theorem 6, so let \(m = 5\). Suppose \(\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_5(\mathbb{F})\) is an irreducible non-degenerate semigroup homomorphism.

Again we have two possibilities:

(a) rank \(A = 1\) implies rank \(\varphi(A) = 1\) and rank \(A = 2\) implies rank \(\varphi(A) = 2\) or

(b) rank \(A = 1\) implies \(\varphi(A) = 0\) and rank \(A = 2\) implies rank \(\varphi(A) = 1\).

In case (a) the same proof as in Theorem 6 works.

In case (b)

\[
\varphi(A) = S \begin{bmatrix} \hat{f}(\text{Cof}(A)) & * \\ * & * \end{bmatrix} S^{-1},
\]

where \(f : \mathbb{F} \to \mathbb{F}\) is a semigroup homomorphism and \(S \in \mathcal{M}_5(\mathbb{F})\) is an invertible matrix. Similarly as in Theorem 6 we prove, that if \(P\) is a permutation matrix, then...
\[ \varphi(P) = S \begin{bmatrix} \hat{f}(\text{Cof}(P)) & 0 \\ 0 & * \end{bmatrix} S^{-1}, \] (1)

and if
\[ A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

then
\[ \varphi(A) = S \begin{bmatrix} \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) & 0 & * \\ 0 & f(ad - bc) & 0 \\ * & 0 & * \end{bmatrix} S^{-1}. \]

Let \( C_{1,2,4,5} \) be a compression to the first, second and fourth and fifth rows and columns of a matrix. Define \( \psi : M_2(\mathbb{F}) \rightarrow M_4(\mathbb{F}) \)
\[ \psi(A') = C_{1,2,4,5}(S^{-1}\varphi(A' \oplus I_1)S). \]

It is obvious that \( \psi \) is multiplicative and we have just seen that
\[ \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) & * \\ * & * \end{bmatrix}. \]

The map \( \psi \) may be irreducible or reducible. If it is irreducible it has one of the forms (a) or (b) of Theorem 3 in [4]. If it is reducible, its image has an irreducible invariant subspace of dimension at least two, so this irreducible subspace is of dimension two or three. Thus we have four possibilities to explore:

(i)
\[ \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \hat{g} \left( \begin{bmatrix} d^3 & c^3 & c^2d & cd^2 \\ b^3 & a^3 & a^2b & ab^2 \\ 3b^2d & 3a^2c & a^2d + 2abc & 2abd + b^2c \\ 3bd^2 & 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd \end{bmatrix} \right), \]

where \( f(x) = g(x^2) \) and \( g \) is additive. In this case we have
\[ \varphi(A) = S\hat{g} \begin{bmatrix} d^3 & c^3 & c^2d & cd^2 \\ b^3 & a^3 & a^2b & ab^2 \\ 0 & 0 & (ad - bc)^3 & 0 \\ 3b^2d & 3a^2c & a^2d + 2abc & 2abd + b^2c \\ 3bd^2 & 3ac^2 & 2acd + bc^2 & ad^2 + 2bcd \end{bmatrix} S^{-1} \]

for \( A = A' \oplus I_1 \). Furthermore, we have
\[ \varphi(R_1) = S \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} S^{-1}, \]

and, using (1), it follows that
$\varphi(R_2) = S \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_1 & e_2 \\ 0 & 0 & 0 & e_3 & e_4 \end{bmatrix} S^{-1},$

where the lower-right corner $E = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$ is an involution similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the product $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} E$ is of order three or one. In particular, $e_1 + e_4 = 0$ and $e_2 + e_3 \neq 0$, so that $e_1 + e_2 + e_3 + e_4 \neq 0$. Now let

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and

$B = R_2 AR_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

The matrices $A$ and $B$ commute, but

$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1},$

$\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ -3 & 0 & 0 \end{bmatrix} E,$

so the upper-left 3-by-3 corner of $S^{-1}\varphi(A)\varphi(B)S$ is equal to

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ -3 & 0 & 0 \\ 1 & 2 \end{bmatrix} E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

and the upper-left 3-by-3 corner of $S^{-1}\varphi(B)\varphi(A)S$ is equal to

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 \end{bmatrix} E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 + 3 & 1 & 0 \end{bmatrix}$

This is a contradiction, possibility (i) cannot occur.

(ii)

$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} g(d)h(d) & g(c)h(c) & g(c)h(d) & g(d)h(c) \\ g(b)h(b) & g(a)h(a) & g(a)h(b) & g(b)h(a) \\ g(b)h(d) & g(a)h(c) & g(a)h(d) & g(b)h(c) \\ g(d)h(b) & g(c)h(a) & g(c)h(b) & g(d)h(a) \end{bmatrix} S^{-1},$
where \( f(x) = g(x)h(x) \) and \( g, h \) are additive. In this case we have

\[
\varphi(A) = S \begin{bmatrix}
g(d)h(d) & g(c)h(c) & 0 & g(c)h(d) & g(d)h(c) \\
g(b)h(b) & g(a)h(a) & 0 & g(a)h(b) & g(b)h(a) \\
0 & 0 & g(ad - bc)h(ad - bc) & 0 & 0 \\
g(b)h(d) & g(a)h(c) & 0 & g(a)h(d) & g(b)h(c) \\
g(d)h(b) & g(c)h(a) & 0 & g(c)h(b) & g(d)h(a)
\end{bmatrix} S^{-1}
\]

for \( A = A' \oplus I_1 \). Again

\[
\varphi(R_2) = S \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & e_1 & e_2 \\
0 & 0 & 0 & e_3 & e_4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} S^{-1},
\]

where lower-right corner \( E = \begin{bmatrix}
e_1 & e_2 \\
e_3 & e_4
\end{bmatrix} \) is involution similar to \( \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \), and \( e_1 + e_2 + e_3 + e_4 \neq 0 \). For

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and

\[
B = R_2 A R_2 = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

we have

\[
\varphi(A) = S \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix} S^{-1}
\]

and

\[
\varphi(B) = S \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
0 & 0 \\
1 & 1
\end{bmatrix} E
\]

so the upper-left 3-by-3 corner of \( S^{-1} \varphi(A) \varphi(B) S \) is equal to

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 + e_1 + e_2 + e_3 + e_4 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

and the upper-left 3-by-3 corner of \( S^{-1} \varphi(B) \varphi(A) S \) is equal to

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 + e_1 + e_2 + e_3 + e_4 & 0 & 1
\end{bmatrix}
\]
Since $A$ and $B$ commute, this is a contradiction, possibility (ii) cannot occur.

(iii) \[ \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \hat{f} \left( \begin{bmatrix} d & c \\ b & a \end{bmatrix} \right) \\ \ast \end{bmatrix} \]

and $f$ is additive. In this case we have

\[ \varphi(A) = S \begin{bmatrix} \hat{f}(\text{Cof}(A)) & \ast \\ 0 & \ast \end{bmatrix} S^{-1} \]

for $A = A' \oplus I_1$. The same holds for permutation matrices. Since matrices of the form $A = A' \oplus I_1$ and permutation matrices generate complete $\mathcal{M}_3(F)$, we obtain

\[ \varphi(A) = S \begin{bmatrix} \hat{f}(\text{Cof}(A)) & \ast \\ 0 & \ast \end{bmatrix} S^{-1}, \]

for all $A \in \mathcal{M}_3(F)$, and consequently $\varphi$ is reducible.

(iv) \[ \psi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \hat{g} \begin{bmatrix} d^2 & c^2 & dc & \ast \\ b^2 & a^2 & ba & \ast \\ 2db & 2ca & da + cb & \ast \\ 0 & 0 & 0 & \ast \end{bmatrix}, \]

where $f(x) = g(x^2)$ and $g$ is additive. In this case we have

\[ \varphi(A) = S\hat{g} \begin{bmatrix} d^2 & c^2 & 0 & dc & \ast \\ b^2 & a^2 & 0 & ba & \ast \\ 0 & 0 & (ad - bc)^2 & 0 & 0 \\ 2db & 2ca & 0 & da + cb & \ast \\ 0 & 0 & 0 & 0 & \ast \end{bmatrix} S^{-1} \]

for $A = A' \oplus I_1$. Now

\[ \varphi(R_1) = S \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & \pm1 \end{bmatrix} S^{-1}. \]

If the last entry in the last row is equal to 1, then $a = 0$ and the lower-right 2-by-2 corner of every permutation matrix is equal to $I_2$, and consequently $\varphi$ is reducible. So the last entry in the last row is equal to $-1$. We may now apply a simultaneous similarity with a matrix of the form $I + \alpha E_{45}$ to obtain $a = 0$ and without disturbing the first four columns. Further,

\[ \varphi(R_2) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 & \ast \\ 0 & 0 & e_3 & e_4 & \ast \end{bmatrix} S^{-1}, \]

where the lower-right corner

\[ E = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \]
is an involution similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and the product $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E$ is of order three or one. So either $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or it has the form

$$E = \begin{bmatrix} -\frac{1}{2} & b \\ \frac{3}{4b} & \frac{1}{2} \end{bmatrix}$$

where $b \neq 0$. In the first case again $\varphi$ is reducible, in the second case we may apply a simultaneous similarity with a diagonal matrix of the form $I_4 \oplus [\beta]$ to obtain $b = \frac{1}{2}$.

So

$$\varphi(R_2) = S \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} \end{bmatrix} S^{-1}.$$

For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = R_2AR_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we now have

$$\varphi(A) = S \begin{bmatrix} 1 & 0 & 0 & x \\ 1 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} S^{-1}.$$

and

$$\varphi(B) = S \begin{bmatrix} 1 & 0 & 0 & \frac{3x}{2} & \frac{x}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -\frac{1}{2} + \frac{3y}{2} & \frac{1}{2} + \frac{y}{2} \\ -1 & 0 & 0 & 1 - \frac{3y}{4} & -\frac{z}{4} \\ 3 & 0 & 0 & \frac{9z}{4} & 1 + \frac{3y}{4} \end{bmatrix} S^{-1}.$$

Since $A$ and $B$ commute, $\varphi(A)$ and $\varphi(B)$ must also commute. Thus we obtain $x = 0$, $y = \frac{1}{2}$ and $z = 0$. Now let

$$C = R_1R_2R_1AR_1R_2R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$
The matrices $A$ and $C$ commute, but
\[
\varphi(A) = S \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} S^{-1}
\]
and
\[
\varphi(C) = S \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -\frac{1}{3} \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & -3 & 0 & 1
\end{bmatrix} S^{-1}
\]
do not commute. Again we get a contradiction and this ends the proof. □

6. Case $m = 6$

In this concluding section we will give five examples of irreducible non-degenerate homomorphisms, which go to the dimension 6. We have seen in previous section that every non-degenerate homomorphism from dimension 3 to dimension 5 is reducible. But there exist an irreducible non-degenerate homomorphism from dimension 4 to dimension 6, and two different irreducible non-degenerate homomorphisms from dimension 3 to dimension 6. We also give two non-degenerate homomorphisms from dimension 2 to dimension 6. We define an equivalence relation $R$ on the set of non-degenerate homomorphisms $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ as the transitive closure of the following relation $S$. Two non-degenerate homomorphisms $\varphi_1, \varphi_2 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_m(\mathbb{F})$ are $S$-related, if either

1. $\varphi_2(A) = \hat{f}(\varphi_1(A))$, where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism, or
2. $\varphi_2(A) = S\varphi_1(A)S^{-1}$, where $S$ is an invertible matrix, or
3. $\varphi_1(A) = \varphi_3(A) \otimes \varphi_4(A)$ and $\varphi_2(A) = \varphi_5(A) \otimes \varphi_4(A)$, where $\varphi_3, \varphi_5 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_k(\mathbb{F})$ are $S$-related as in (1) or (2), and $\varphi_4 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{m/k}(\mathbb{F})$.

**Example 1.** There exist two $R$-unrelated irreducible non-degenerate semigroup homomorphisms $\varphi : \mathcal{M}_2(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F})$:

(a) Symmetric power:
\[
\varphi(A) = \text{Sym}^5 A;
\]
(b) Tensor product:
\[
\varphi(A) = \hat{f}(A) \otimes (A \wedge A),
\]
where $f : \mathbb{F} \to \mathbb{F}$ is a field homomorphism and $f \neq id$.

**Example 2.** There exist two $R$-unrelated irreducible non-degenerate semigroup homomorphisms $\varphi : \mathcal{M}_3(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F})$:

(a) Symmetric square:
\[
\varphi(A) = \text{Sym}^2 A;
\]
(b) Symmetric square of exterior power:
\[ \varphi(A) = \text{Sym}^2(A \wedge A). \]

**Example 3.** There exists an irreducible non-degenerate semigroup homomorphism \( \varphi : \mathcal{M}_4(\mathbb{F}) \to \mathcal{M}_6(\mathbb{F}) \), the exterior power:
\[ \varphi(A) = A \wedge A. \]

To prove that these homomorphisms are irreducible let \( A = \text{diag}(1, 2) \) in Example 1(a), \( A = \text{diag}(1, 2, 3) \) in Example 2, \( A = \text{diag}(1, 2, 3, 4) \) in Example 3, and \( A = \text{diag}(1, a) \) in Example 1(b), where \( a \in \mathbb{F} \) is such that \( f(a) \neq a \), \( f(a) \neq 1/a \) and \( f(a) \neq a^2 \). Then \( \varphi(A) \) is a diagonal matrix with six different diagonal entries. If \( \varphi \) was reducible, then the common invariant subspace would be standard. That is obviously not the case.

**Acknowledgment**

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**References**

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