Note

A characterization of inverse Radon transform on the Laguerre hypergroup

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Abstract

Let $K = [0, \infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this note we give another characterization for a subspace of $\mathcal{S}(K)$ (Schwartz space) such that the Radon transform $R_\alpha$ on $K$ is a bijection. We show that this characterization is equivalent to that in [M.M. Nessibi, K. Trimèche, Inversion of the Radon transform on the Laguerre hypergroup by using generalized wavelets, J. Math. Anal. Appl. 208 (1997) 337–363]. In addition, we establish an inversion formula of the Radon transform $R_\alpha$ in the weak sense.

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1. Introduction

In the past decade research on Radon transform has made considerable progress due to its wide applications to partial differential equations, X-ray technology, radio astronomy and so on. For the basic theory and further results of Radon transform we refer readers to the book [1] by S. Helgason and the references therein. The combination of Radon

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transform and wavelet transform has proved to be very useful both in pure mathematics and its applications. Recently, various authors deal with the inversion formula of Radon transform by using inverse wavelet transform (see [2–5]). When one considers the problems of radial function on the Heisenberg group, the fundamental manifold is the Laguerre hypergroup $K = [0, \infty) \times \mathbb{R}$. In [3], M.M. Nessibi and K. Trimèche defined the Radon transform $R_{\alpha}$ on $K$, and characterized a subspace $S_{\ast, 2}(K)$ of Schwartz space on which the Radon transform $R_{\alpha}$ is a bijection. Moreover, they obtained an inversion formula of the Radon transform on $S_{\ast, 2}(K)$ by use of the generalized wavelet transform. In this note we give another characterization which seems to be natural, we show that this characterization is equivalent to that in [3]. In addition, we obtain an inversion of the Radon transform on $K$ in the weak sense.

Let $K = [0, \infty) \times \mathbb{R}$, $\alpha \geq 0$, the Radon transform $R_{\alpha}$ on $K$ is given by

$$R_{\alpha}f(x, t) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_{0}^{\infty} T_{(x, t)}^{(\alpha)} f(y, 0) y^{2\alpha+1} dy,$$

where $T_{(x, t)}^{(\alpha)}$, $(x, t) \in K$, are the generalized translation operators on $K$ defined by

$$T_{(x, t)}^{(\alpha)} f(y, s) = \begin{cases} \frac{1}{2\pi} \int_{0}^{2\pi} f(\sqrt{x^2 + y^2 + 2xy \cos \theta}, s + t + xy \sin \theta) d\theta, & \text{if } \alpha = 0, \\ \alpha \pi \int_{0}^{2\pi} \int_{0}^{1} f(\sqrt{x^2 + y^2 + 2xyr \cos \theta}, s + t + xyr \sin \theta) \\ \times r(1-r^2)^{\alpha-1} dr d\theta, & \text{if } \alpha > 0. \end{cases}$$

Let $dm_{\alpha}(x, t)$ be the positive measure on $K$ given by $dm_{\alpha}(x, t) = \frac{1}{\pi \Gamma(\alpha+1)} x^{2\alpha+1} dx dt$. Then $L_{\alpha}^{2}(K)$ denotes the space of measurable functions on $K$ such that

$$\|f\|_{\alpha, 2} = \left( \int_{K} |f(x, t)|^2 dm_{\alpha}(x, t) \right)^{1/2} < +\infty.$$

The convolution product of $f$ and $g$ on $K$ is defined by

$$(f \ast g)(x, t) = \int_{K} T_{(x, t)}^{(\alpha)} f(y, s) g(y, -s) dm_{\alpha}(y, s).$$

For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, $(x, t) \in K$, we set $\varphi_{\lambda, m}(x, t) = e^{ix\lambda L_{m}^{(\alpha)}(|x|^2)}$, where $L_{m}^{(\alpha)}$ denotes the Laguerre function defined on $[0, +\infty)$ by $L_{m}^{(\alpha)}(x) = e^{-x/2} L_{m}^{(\alpha)}(x)/L_{m}^{(\alpha)}(0)$, $L_{m}^{(\alpha)}$ being the Laguerre polynomial of degree $m$ and order $\alpha$. The generalized Fourier transform on $K$ is defined by

$$\hat{f}(\lambda, m) = \int_{K} \varphi_{-\lambda, m}(x, t) f(x, t) dm_{\alpha}(x, t).$$

Let $L_{\alpha}^{2}(\mathbb{R} \times \mathbb{N})$ be the space of measurable functions $g : \mathbb{R} \times \mathbb{N} \mapsto \mathbb{C}$ such that
\[ \|g\|_{L_2^\alpha} = \left( \int_{\mathbb{R} \times \mathbb{N}} |g(\lambda, m)|^2 \, d\gamma_\alpha(\lambda, m) \right)^{1/2} < +\infty, \quad (5) \]

where \( d\gamma_\alpha \) denotes the positive measure defined on \( \mathbb{R} \times \mathbb{N} \) by
\[
\int_{\mathbb{R} \times \mathbb{N}} g(\lambda, m) \, d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} g(\lambda, m) |\lambda|^\alpha+1 \, d\lambda.
\]

Let \( f, g \in L_2^\alpha(\mathbb{K}) \). The Parseval formula on \( \mathbb{K} \) [3, Corollary II.3, p. 350] is given by
\[
\int_{\mathbb{K}} f(x, t) g(x, t) \, dm_\alpha(x, t) = \int_{\mathbb{R} \times \mathbb{N}} \hat{f}(\lambda, m) \overline{\hat{g}(\lambda, m)} \, d\gamma_\alpha(\lambda, m). \quad (6)
\]

Specially, one has the Plancherel formula
\[
\|f\|_{L_2^{\alpha,2}}^2 = \sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} |\hat{f}(\lambda, m)|^2 |\lambda|^\alpha+1 \, d\lambda.
\]

And if \( f \in L_2^\alpha(\mathbb{K}) \), then inversion of the generalized Fourier transform holds
\[
f(x, t) = \int_{\mathbb{R} \times \mathbb{N}} \hat{f}(\lambda, m) \varphi_{\lambda, m}(x, t) \, d\gamma_\alpha(\lambda, m). \quad (7)
\]

Also, the identity (7) is valid with the convergence of the integral in the weak sense, i.e., taking the inner product of both sides of (7) with any \( g \in L_2^\alpha(\mathbb{K}) \), leads to a true formula.

Let \( \mathcal{S}(\mathbb{K}) \) denote the Schwartz function space on \( \mathbb{K} \), i.e., for all \( k, p, q \in \mathbb{N} \),
\[
\sup_{(x,t) \in \mathbb{K}} \left\{ (1 + x^2 + t^2)^k \left| \frac{\partial^{p+q}}{\partial x^p \partial t^q} f(x, t) \right| \right\} < +\infty.
\]

\( \mathcal{S}_{s,2}(\mathbb{K}) \) is the subspace of \( \mathcal{S}(\mathbb{K}) \) consisting of functions \( g \) such that
\[
\int_{\mathbb{R}} \int t^j f(x, t) \, dt = 0 \quad (8)
\]

for all \( j \in \mathbb{N} \) and for all \( x \in \mathbb{R} \). M.M. Nessibi and K. Trimèche proved that the Radon transform \( R_\alpha \) on \( \mathcal{S}_{s,2}(\mathbb{K}) \) is a bijection. By using wavelet inverse transform they obtained an inversion formula of the Radon transform on this space. Let \( l \in \mathbb{N} \), \( R_\alpha^l(f) = f \), \( R_\alpha^1(f) = R_\alpha(f) \), \( R_\alpha^2(f) = R_\alpha(R_\alpha(f)) \), \ldots, \( R_\alpha^{l+1}(f) = R_\alpha(R_\alpha^l(f)) \) and \( R_\alpha^{-l}(f) \) denotes the inverse operator of \( R_\alpha^l(f) \), i.e., \( R_\alpha^{-l}(f) = (R_\alpha^l)^{-1}(f) \). Now we suppose that \( \mathcal{S}_{R_\alpha}(\mathbb{K}) \) is a subspace of \( \mathcal{S}(\mathbb{K}) \) on which the Radon transform \( R_\alpha \) is also a bijection, then for any \( f \in \mathcal{S}_{R_\alpha}(\mathbb{K}) \), \( R_\alpha^j(f) \in \mathcal{S}_{R_\alpha}(\mathbb{K}) \) for all \( j \in \mathbf{I} \), where \( \mathbf{I} \) is the set of all integers. Of course, \( R_\alpha^j(f) \in L_2^\alpha(\mathbb{K}) \).

Using the formula (V.3) in [3],
\[
\hat{R_\alpha} f(\lambda, m) = (-1)^m |\lambda|^{-(\alpha+1)} (2\pi)^{\alpha+1} \hat{f}(\lambda, m),
\]
we then get
\[
\hat{R_\alpha^j} f(\lambda, m) = (-1)^{jm} |\lambda|^{-j(\alpha+1)} (-1)^{jm} (2\pi)^{j(\alpha+1)} \hat{f}(\lambda, m).
\]
It is natural to define
\[
S_{R_\alpha}(\mathbb{K}) = \left\{ f \in S(\mathbb{K}) : (2\pi)^{2j(\alpha+1)} \sum_{m=0}^{\infty} L_m^{(\alpha)}(0) \right. \\
\times \int_{\mathbb{R}} |\hat{f}(\lambda, m)|^2 |\lambda|^{-(2j-1)(\alpha+1)} \, d\lambda < +\infty \text{ for all } j \in \mathbb{N} \right\}. \tag{9}
\]

Now we are going to show that \( R_\alpha \) on \( S_{R_\alpha}(\mathbb{K}) \) is a bijection. In fact, let \( f \in S_{R_\alpha}(\mathbb{K}) \), obviously, \( R_\alpha(f) \in S_{R_\alpha}(\mathbb{K}) \). If \( R_\alpha(f) = R_\alpha(g) \), then for all \((\lambda, m) \in \mathbb{R} \times \mathbb{N}\)
\[
0 = R_\alpha(f - g)(\lambda, m) = |\lambda|^{-(\alpha+1)}(-1)^m (2\pi)^{\alpha+1}(\hat{f}(\lambda, m) - \hat{g}(\lambda, m)),
\]
which implies \( f = g \). Now let \( g \in S_{R_\alpha}(\mathbb{K}) \), we define a function \( f \) in terms of its Fourier transform by
\[
\hat{f}(\lambda, m) = |\lambda|^{\alpha+1}(-1)^m (2\pi)^{\alpha+1}\hat{g}(\lambda, m).
\]
Then we have \( R_\alpha(f) = g \). For \( f \in S(\mathbb{K}) \), by Proposition II.8 in [3] we can see that \( \lambda^\beta \hat{f}(\lambda, m) \in L^2_\alpha(\mathbb{R} \times \mathbb{N}) \) for all \( \beta > 0 \). Thus the space \( S_{R_\alpha}(\mathbb{K}) \) can be expressed by
\[
S_{R_\alpha}(\mathbb{K}) = \left\{ f \in S(\mathbb{K}) : (2\pi)^{2j(\alpha+1)} \sum_{m=0}^{\infty} L_m^{(\alpha)}(0) \right. \\
\times \int_{\mathbb{R}} |\hat{f}(\lambda, m)|^2 |\lambda|^{-(2j-1)(\alpha+1)} \, d\lambda < +\infty \text{ for all } j \in \mathbb{N} \right\}. \tag{10}
\]

We are now in a position to state our main results.

2. Main results and proofs

At the beginning of this section, we shall show that \( S_{R_\alpha}(\mathbb{K}) = S_{*,2}(\mathbb{K}) \). In order to do this we need following notation. Let \( j \in \mathbb{N}, \; D^j_2 f(x, 0) = \frac{\partial^j}{\partial t^j} f(x, t)|_{(x,0)}, \; D_2 f(x, t) = \frac{\partial}{\partial t} f(x, t)|_{(x,ty)} \).

**Theorem 1.** Let \( f \in S(\mathbb{K}) \). Then \( D^j_2 f(x, 0) = 0 \) for all \( j \in \mathbb{N} \) if and only if \( f(x, t) = tg(x, t) \) where \( g \in S(\mathbb{K}) \) and \( D^j_2 g(x, 0) = 0 \) for all \( j \in \mathbb{N} \). Furthermore, we have \( f(x, t) = t^k g_k(x, t) \), where \( k \in \mathbb{N}, \; g_k \in S(\mathbb{K}), \; D^j_2 g_k(x, 0) = 0 \) for all \( j \in \mathbb{N} \).

**Proof.** Using the formula \( f(x, y) = f(x, 0) + y \int_0^1 D_2 f(x, ty) \, dt \) together with \( f(x, 0) = 0 \), we immediately obtain \( f(x, y) = yg_1(x, y) \) where
\[
g_1(x, y) = \int_0^1 D_2 f(x, ty) \, dt. \tag{11}
\]
Notice that
\[
\begin{align*}
\int_0^1 D_2 f(x, ty) dt &= \frac{1}{y} \int_0^y D_2 f(x, y) dy \\
&= \frac{1}{y} \left[ \int_0^{+\infty} D_2 f(x, y) dy - \int_y^{+\infty} D_2 f(x, y) dy \right] \\
&= -\frac{1}{y} \int_y^{+\infty} D_2 f(x, y) dy = O(y^{-k})
\end{align*}
\]
as \( |y| \to +\infty \), for all \( k \in \mathbb{N} \). Thus we can see that \( g_1 \in S(\mathbb{K}) \). By \( D_j^2 f(x, 0) = 0 \) and (11) we get \( D_j^2 g_1(x, 0) = 0 \) for all \( j \in \mathbb{N} \). This proves the necessary condition with \( g = g_1 \). The sufficiency is obvious.

By induction we can obtain that \( f(x, t) = t^k g_k(x, t) \), where \( k \in \mathbb{N} \), \( g_k \in S(\mathbb{K}) \), \( D_j^2 g_k(x, 0) = 0 \) for all \( j \in \mathbb{N} \). \( \Box \)

**Theorem 2.**

\[ S_{*,2}(\mathbb{K}) = S_{R_\alpha}(\mathbb{K}). \]  

**Proof.** Let \( f \in S_{*,2}(\mathbb{K}) \), then \( \int_{\mathbb{R}} t^j f(x, t) dt = 0 \) for all \( j \in \mathbb{N} \). This yields \( D_j^2 \tilde{f}(x, \lambda) = 0 \) at \( \lambda = 0 \), where

\[
\tilde{f}(x, \lambda) = \int_{\mathbb{R}} f(x, t) \exp(-i\lambda t) dt.
\]

Clearly, \( \tilde{f}(x, \lambda) \in S(\mathbb{K}) \). For a given \( \alpha \geq 0 \), we choose a positive integer \( m \) such that \( m - 1 < \alpha \leq m \). By Theorem 1, we have

\[
\tilde{f}(x, \lambda) = \lambda^{j(m+1)} \tilde{g}_{j(m+1)}(x, \lambda)
\]
for all \( j \in \mathbb{N} \). Taking the generalized Fourier transform for \( f(x, t) \) on \( \mathbb{K} \), we then get

\[
\tilde{f}(\lambda, k) = \lambda^{j(m+1)} \tilde{g}_{j(m+1)}(\lambda, k).
\]

By the Plancherel formula we have

\[
\sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\tilde{f}(\lambda, k)|^2 |\lambda|^{-2j(m+1)} |\lambda|^\alpha+1 d\lambda
\]

\[
= \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\tilde{g}_{j(m+1)}(\lambda, k)|^2 |\lambda|^\alpha+1 d\lambda
\]

\[
= \|g_{j(m+1)}\|_{L_2^2}^2 < +\infty
\]
for all \( j \in \mathbb{N} \). Note that

\[
(2\pi)^{2j(\alpha+1)} \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\hat{f}(\lambda, k)|^2 |\lambda|^{-(2j-1)(\alpha+1)} d\lambda
\]

\[
= (2\pi)^{2j(\alpha+1)} \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2j(\alpha+1)} |\lambda|^\alpha d\lambda
\]

\[
= (2\pi)^{2j(\alpha+1)} \left( \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{|\lambda| \leq 1} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2j(\alpha+1)} |\lambda|^\alpha d\lambda \right)
\]

\[
\leq (2\pi)^{2j(\alpha+1)} \left( \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{|\lambda| \leq 1} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2j(m+1)} |\lambda|^\alpha d\lambda \right)
\]

\[
+ \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{|\lambda| > 1} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2jm} |\lambda|^\alpha d\lambda
\]

\[
\leq (2\pi)^{2j(\alpha+1)} \left( \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2j(m+1)} |\lambda|^\alpha d\lambda \right)
\]

\[
+ \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2jm} |\lambda|^\alpha d\lambda
\]

\[
< +\infty
\]

for all \( j \in \mathbb{N} \). It follows that \( f \in S_{R_\alpha}(K) \). On the other hand, if \( f \in S_{R_\alpha}(K) \), then for all \( j \in \mathbb{N} \),

\[
(2\pi)^{2j(\alpha+1)} \sum_{k=0}^{\infty} L_k^{(\alpha)}(0) \int_{\mathbb{R}} |\hat{f}(\lambda, k)|^2 |\lambda|^{-2j(m+1)} |\lambda|^\alpha d\lambda < +\infty.
\]

Because of \( f \in S(K) \), we can see that for all \( j \in \mathbb{N} \), \( D^j_\lambda \hat{f}(\lambda, k) = 0 \) at \((0, k)\), which implies

\[
\int_{\mathbb{R}} t^j f(x, t) dt = 0
\]

for all \( j \in \mathbb{N} \) and for all \( x \in \mathbb{R} \). That is, \( f \in S_{*2}(K) \). This concludes the proof of Theorem 2. \( \square \)
Let
\[ L^2_{R_a}(\mathbb{K}) = \left\{ f \in L^2_{\alpha}(\mathbb{K}) : (2\pi)^{2j(\alpha+1)} \sum_{m=0}^{\infty} L_m^{(\alpha)}(0) \times \int_{\mathbb{R}} \left| \hat{f}(\lambda, m) \right|^2 \left| \lambda \right|^{-(2j-1)(\alpha+1)} d\lambda < +\infty \text{ for all } j \in \mathbb{N} \right\}. \] (13)

Clearly, the Radon transform \( R_\alpha \) on this space is also a bijection. For \( \alpha \geq 0 \), we denote by \( L_\alpha \) the operator defined on \( \mathcal{S}_{s,2}(\mathbb{K}) \) by
\[ \hat{L}_\alpha(\hat{f})(\lambda, m) = |\lambda|^{\alpha+1} \hat{f}(\lambda, m). \]

It is easy to see that if \( g \in \mathcal{S}_{s,2}(\mathbb{K}) \), then
\[ L_\alpha R_\alpha L_\alpha(g)(\lambda, m) = (-1)^m (2\pi)^{\alpha+1} |\lambda|^{\alpha+1} \hat{g}(\lambda, m). \]

Let \( g \in L^2_{\alpha}(\mathbb{K}) \). Then \( g \) is called a generalized wavelet on \( \mathbb{K} \) if there is a constant \( C_g \in (0, +\infty) \) such that for all \( m \in \mathbb{N} \) and \( \lambda \neq 0 \),
\[ C_g = \int_{\mathbb{R}} \left| \hat{g}(a\lambda, m) \right|^2 |a|^{2} \frac{da}{|a|} \] (14)
(see [3, p. 359]). Let \( g \in \mathcal{S}_{s,2}(\mathbb{K}) \) be a generalized wavelet on \( \mathbb{K} \). For \( 0 \neq a \in \mathbb{R} \), the functions \( g_a \) and \( g_{a,(x,t)} \) are respectively defined by
\[ g_a(y, s) = \frac{1}{|a|^{\alpha+2}} g\left( \frac{y}{\sqrt{|a|}}, \frac{s}{|a|} \right), \quad g_{a,(x,t)}(y, s) = |a|^{1+\alpha/2} T^{(\alpha)}_{(x,t)} g_a(y, s). \]

Then we find
\[ \hat{g}_a(\lambda, m) = \hat{g}(a\lambda, m), \quad \hat{g}_{a,(x,t)}(\lambda, m) = |a|^{1+\alpha/2} \varphi_{\lambda,m}(x, t) \hat{g}(a\lambda, m). \] (15)

It is not difficult to check that \( (L_\alpha R_\alpha L_\alpha(g))_a = |a|^{\alpha+1} L_\alpha R_\alpha L_\alpha(g_a) \). In fact, this identity can be proved by taking the generalized Fourier transform on both sides. Now we define an operator \( \tilde{\Phi}_g \) on \( L^2_{R_a}(\mathbb{K}) \) by
\[ \tilde{\Phi}_g(f)(x, t) = (f \ast \tilde{g})(x, t). \]

We thus get
\[ \tilde{\Phi}_{(L_\alpha R_\alpha L_\alpha(g))_a}(R_\alpha(f))(\lambda, m) = R_\alpha(f)(\lambda, m)(L_\alpha R_\alpha L_\alpha(g))_a(\lambda, m) \]
\[ = |a|^{\alpha+1} (2\pi)^{2(\alpha+1)} \hat{f}(\lambda, m) \hat{\varphi}_{\lambda,m}(\lambda, m). \] (16)

Therefore, we have

**Theorem 3.** Let \( g \in \mathcal{S}_{s,2}(\mathbb{K}) \) be a generalized wavelet on \( \mathbb{K} \). Then for all \( f \) in \( L^2_{R_a}(\mathbb{K}) \) we have
\[
R_{\alpha}^{-1}(f)(x,t) = \frac{(2\pi)^{-2\alpha-2}}{C_g} \times \int_{\mathbb{R} \times \mathbb{K}} \hat{\Phi}(L_{\alpha} R_{\alpha} L_{\alpha}(g))_{\alpha}(y,s) g_{\alpha}(y,s)(x,-t) \, dm_{\alpha}(y,s) \, \frac{da}{|a|^{3(\alpha/2+1)}}.
\]

Also, the formula (17) holds in the weak sense.

**Proof.** The proof is similar to that of Theorem VI.1 in [3]. We first should notice the following facts:

\[
\begin{align*}
(f * \hat{g}_{\alpha})(\lambda, m) &= \hat{f}(\lambda, m) \hat{g}_{\alpha}(-\lambda, m), \\
(T^{(\alpha)}_{(x,-t)} g_{\alpha})(-\lambda, m) &= \varphi_{\lambda, m}(x,t) \hat{g}_{\alpha}(-\lambda, m).
\end{align*}
\]

By the Parseval formula (6) together with (15), (16) we deduce

\[
\int_{\mathbb{K}} \hat{\Phi}(L_{\alpha} R_{\alpha} L_{\alpha}(g))_{\alpha}(R_{\alpha}(f))(y,s) g_{\alpha}(y,s)(x,-t) \, dm_{\alpha}(y,s)
\]

\[
= |a|^{\alpha/2+1} \int_{\mathbb{R} \times \mathbb{N}} (R_{\alpha}(f) * \hat{(L_{\alpha} R_{\alpha} L_{\alpha}(g))}_{\alpha})(\lambda, m) \hat{(T^{(\alpha)}_{(x,-t)} g_{\alpha})}(-\lambda, m) \, d\gamma_{\alpha}(\lambda, m)
\]

\[
= |a|^{3\alpha/2+2} \int_{\mathbb{R} \times \mathbb{N}} (R_{\alpha}(f) * \hat{(L_{\alpha} R_{\alpha} L_{\alpha}(g))}_{\alpha})(\lambda, m) \hat{(T^{(\alpha)}_{(x,-t)} g_{\alpha})}(-\lambda, m) \, d\gamma_{\alpha}(\lambda, m)
\]

\[
= |a|^{3\alpha/2+2} (2\pi)^{2(\alpha+1)} \int_{\mathbb{R} \times \mathbb{N}} \hat{f}(\lambda, m) \hat{g}(-a\lambda, m) \varphi_{\lambda, m}(x,t) \, d\gamma_{\alpha}(\lambda, m).
\]

Thus we have

\[
\frac{(2\pi)^{-2\alpha-2}}{C_g} \int_{\mathbb{R} \times \mathbb{K}} \hat{\Phi}(L_{\alpha} R_{\alpha} L_{\alpha}(g))_{\alpha}(R_{\alpha}(f))(y,s) g_{\alpha}(y,s)(x,-t) \, dm_{\alpha}(y,s) \, \frac{da}{|a|^{3(\alpha/2+1)}}
\]

\[
= \int_{\mathbb{R} \times \mathbb{N}} \hat{f}(\lambda, m) \varphi_{\lambda, m}(x,t) \, d\gamma_{\alpha}(\lambda, m) = f(x,t).
\]

Taking the inverse Radon transform \(R_{\alpha}^{-1}\) on \(f\), we obtain our desired result. \(\Box\)

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