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Principal Indecomposable Representations for Restricted Lie Algebras of Cartan Type

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Let L be any one of $W(n, \mathbf{1})$, $S(n, \mathbf{1})$, $H(n, \mathbf{1})$, and $K(n, \mathbf{1})$ over an algebraically closed field F of characteristic $p > 3$. In this paper, we extend the results concerning modular representations of classical Lie algebras and semisimple groups to the case of L and obtain some properties of principal indecomposable modules of $u(L)$ which parallel closely those of classical Lie algebras.

INTRODUCTION

Let G be a semisimple, simply connected algebraic group over an algebraically closed field F of characteristic $p > 0$, $\underline{g}$ its Lie algebra, and $u(\underline{g})$ the restricted universal enveloping algebra of $\underline{g}$. Let $\underline{h} = \langle h_1, ..., h_l \rangle$ be a Cartan subalgebra of $\underline{g}$, $\underline{b}$ is a Borel subalgebra such that $\underline{h} \subset \underline{b}$. Let Λ denote the collection of p^l restricted weights λ characterized by the conditions $0 \leqslant \lambda(h_i) < p$, $1 \leqslant i \leqslant l$. For each $\lambda \in \Lambda$, we can canonically obtain the one-dimensional $\underline{b}$-module which is denoted by F_λ. The induced module

$$Z(\lambda) = u(\underline{g}) \otimes_{u(\underline{b})} F_\lambda$$

is an indecomposable universal highest weight module. Let $V(\lambda)$ denote the restricted irreducible $\underline{g}$-module of highest weight λ and $Q_{\underline{g}}(\lambda)$ the projective cover (= injective hull) of $V(\lambda)$. In [6], Humphreys proved that the principal indecomposable module (PIM) $Q_{\underline{g}}(\lambda)$ of $u(\underline{g})$ has a filtration with quotients isomorphic to various $Z(\mu)$ and $Z(\mu)$ occurs as many times as $V(\lambda)$ occurs as a composition factor of $Z(\mu)$ (cf. [6, 12]).

Let W be the restricted generalized Jacobson–Witt algebra, S the restricted special algebra, H the restricted Hamiltonian algebra, and K the

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restricted contact algebra. Let L be any one of W, S, H, and K, and $u(L)$ the restricted universal enveloping algebra of L. In this paper, we extend these studies of [6, 12] to L and get the properties of PIMs for $u(L)$ which parallel closely those of $u(\underline{g})$.

According to the results of Shen Guangyu [17–19], any irreducible graded L-module is completely determined by its base space which is an irreducible $L_{[0]}$ $(=gl(n), sl(n)$, or $sp(n))$-module. This enables us to exploit certain techniques in the representation theory of reductive algebraic groups. In Section 2, we study certain induced $u(L)$-modules, denoted by $u(L)\otimes_{u(L_0)} Z(\lambda)$, which play an important role in the description of the PIMs $Q(\lambda)$ of $u(L)$. In Section 3, we study an artificial category of $u(L^e)$-T^e-modules, inspired by Jantzen's method in [12]. In Section 4, we discuss some properties of projective $u(L^e)$-T^e-modules. Finally, we show in Section 5 that $Q(\lambda)$ has a filtration with quotients isomorphic to various $u(L)\otimes_{u(L_0)} Z(\mu)$ and obtain a relation between the number of times of $u(L)\otimes_{u(L_0)} Z(\mu)$ occurring as a quotient and the multiplicity of $M(\lambda)$ (the top composition factor of $u(L)\otimes_{u(L_0)} Z(\lambda)$) as a composition factor of $u(L)\otimes_{u(L_0)} Z(\mu)$ (see Theorem 5.1). In particular, we know that in general, the former is greater than the latter.

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1. Preliminaries

Let F be an algebraically closed field, char $F=p>3$. All Lie algebras and modules treated in the present article are assumed to be finite-dimensional and restricted.

In the following our notations agree with those in [20, Chap. 4]. We write $W=W(n,\mathbf{1})$, $S=S(n,\mathbf{1})$, $H=H(n,\mathbf{1})$, and $K=K(n,\mathbf{1})$. If L is any one of W, S, and H, then $L=\bigoplus_{i\geqslant -1} L_{[i]}$ is a $\mathbb{Z}$-graded Lie algebra of depth 1 and under the linear map $x^{(\varepsilon_i)}D_j\mapsto E_{ij}$, $L_{[0]}$ is isomorphic to $gl(n)$, $sl(n)$, and $sp(n)$, respectively, where E_{ij} is the matrix whose (k,l) component is $\delta_{ik}\delta_{jl}$. Write $I:=\sum_{i=1}^n x^{(\varepsilon_i)}D_i$ and $I':=\sum_{i=1}^n x^{(\varepsilon_i)}D_i+x^{(\varepsilon_n)}D_n$. If $L=K$, let $K_{[i]}=A(n,\mathbf{1})_{[i]}$, then $L=\bigoplus_{i\geqslant -2} K_{[i]}$ is a $\mathbb{Z}$-graded Lie algebra of depth 2 and $L_{[0]}$ is isomorphic to $sp(2r)\oplus FI'$.

Let $u(\underline{a})$ be the restricted universal enveloping algebra of a Lie algebra $\underline{a}$. Then the notions of $\underline{a}$-module and $u(\underline{a})$-module are equivalent. We have

Proposition 1.1. [18]. *Every irreducible $u(L)$-module V is graded and the map $V(=\bigoplus_{i\geqslant 0} V_i)\mapsto V_0$ (base space) induces a bijection between the sets of isomorphism classes of irreducible $u(L)$-modules and irreducible $u(L_{[0]})$-modules, respectively.*

Let $L_i = \bigoplus_{j \geq i} L_{[j]}$ and $\sigma : L_0 \to F$ be the Lie algebra homomorphism given by $\sigma(x) := \mathrm{tr}(\mathrm{ad}_{L/L_0} x)$, $\forall x \in L_0$, and V an L_0-module. We introduce a twisted action on V by setting $x \cdot v := xv + \sigma(x)v$. The new L_0-module will be called V_σ. Note that if $L_{[0]} \cong sl(n)$ or $sp(n)$, then $\sigma = 0$. If V is an $L_{[0]}$-module, then we can extend the operations on V to L_0 by letting L_1 act trivially and regard it as an L_0-module. By [5, Corollary 1.6], there exists an isomorphism of $u(L)$-modules

$$u(L) \otimes_{u(L_0)} V_\sigma \cong \mathrm{Hom}_{u(L_0)}(u(L), V).$$

By [5, Proposition 1.5; 4, Proposition 2.4], $\mathrm{Hom}_{u(L_0)}(u(L), V)$ is a positively graded L-module whose base space is isomorphic to V. Hence we have

Proposition 1.2. *If V is an irreducible $u(L_{[0]})$-module, then the irreducible graded L-module with base space V is isomorphic to the (unique) minimum submodule of $u(L) \otimes_{u(L_0)} V_\sigma$, denoted by $(u(L) \otimes_{u(L_0)} V_\sigma)_{\min}$.*

2. The $u(L)$-Modules $u(L) \otimes_{u(L_0)} Z(\lambda)$

Let L be any one of W, S, H and K, $\underline{h}$ (resp. $\underline{h}(L_{[0]})$) the standard Cartan subalgebra of $\underline{g}(:= L_{[0]})$, $\underline{n}$ (or $\underline{n}^-$) the sum of positive (or negative) root spaces of $\underline{g}$, $\underline{b} = \underline{h} \oplus \underline{n}$ the Borel subalgebra of $\underline{g}$, $\underline{b}^- = \underline{h} \oplus \underline{n}^-$, $\mathcal{N} = \underline{n} \oplus \sum_{i \geq 1} L_{[i]}$, $\mathscr{B} = \underline{h} \oplus \mathcal{N}$, $\mathcal{N}^- = \underline{n}^- \oplus \sum_{i<0} L_{[i]}$, and $\mathscr{B}^- = \underline{h} \oplus \mathcal{N}^-$. Let $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_t\}$ be the bases of $\mathcal{N}$ and $\mathcal{N}^-$, respectively, such that $\{x_1, ..., x_m\}$ and $\{y_1, ..., y_m\}$ $(m < s, t)$ are the standard bases of $\underline{n}$ and $\underline{n}^-$, respectively, where $\dim \mathcal{N} = s$, $\dim \mathcal{N}^- = t$, and $\dim \underline{n} = m$. Let Λ_i $(i = 1, ..., n)$ be the linear functions on $\underline{h}(gl(n)) = \langle E_{11}, ..., E_{nn} \rangle$ such that

$$\Lambda_i(E_{jj}) = \delta_{ij}.$$

The restriction of Λ_i on every $\underline{h}(L_{[0]})$ will also be denoted by Λ_i. Let

$$\lambda_0 = 0, \qquad \lambda_i = \sum_{j=1}^{i} \Lambda_j, \qquad i = 1, ..., 1.$$

Then the sets of the fundamental weights of $L_{[0]}$ $(= gl(n), sl(n), sp(n), sp(n-1) \oplus FI')$ are $\{\lambda_1, ..., \lambda_n\}$, $\{\lambda_1, ..., \lambda_{n-1}\}$, $\{\lambda_1, ..., \lambda_{n/2}\}$, and $\{\lambda_1, ..., \lambda_{(n-1)/2}, \Lambda_n/2\}$, respectively. We denote the lattice of all weights of $\underline{h}(L_{[0]})$ by Λ. Each $\lambda \in \Lambda$ is a linear combination of the fundamental weights. We denote the canonical one-dimensional $\underline{b}$-module by F_λ and extend the operations on F_λ to $\mathscr{B}$ by letting L_1 act trivially which is also denoted by F_λ. Denote

$$Z(\lambda) = u(\underline{g}) \otimes_{u(\underline{b})} F_\lambda.$$

LEMMA 2.1. *If* $\lambda \in \Lambda$, *then*

$$u(L) \otimes_{u(L_0)} Z(\lambda) \cong u(L) \otimes_{u(\mathscr{B})} F_\lambda$$

and $\dim u(L) \otimes_{u(L_0)} Z(\lambda) = p^t$.

Proof. Let $L^- := \sum_{i<0} L_{[i]}$. Since the F-vector spaces $u(L) \otimes_{u(L_0)} Z(\lambda)$ and $u(L^-) \otimes_F u(\underline{n}^-)$ are isomorphic, we have $\dim_F u(L) \otimes_{u(L_0)} Z(\lambda) = p^t = \dim_F u(L) \otimes_{u(\mathscr{B})} F_\lambda$. The map $u(L) \times F_\lambda \to u(L) \otimes_{u(L_0)} Z(\lambda)$ that sends (u, α) onto $u \otimes 1 \otimes \alpha$ is $u(\mathscr{B})$ balanced and thus induces a a $u(L)$-linear map φ: $u(L) \otimes_{u(L_0)} Z(\lambda) \to u(L) \otimes_{u(\mathscr{B})} F_\lambda$. An application of the Poincaré–Birkhoff–Witt Theorem shows that φ is injective. Consequently, φ is an isomorphism. ∎

We refer to a nonzero vector v in an L-module as maximal (resp. minimal) if v is killed by all x_i, $i = 1, ..., s$ (resp. y_j, $j = 1, ..., t$). $u(L) \otimes_{u(L_0)} Z(\lambda)$ has a maximal vector v_M (resp. minimal vector v_m) corresponding to the coset of 1 (resp. $y_1^{p-1} \cdots y_t^{p-1}$). Obviously, $u(L) \otimes_{u(L_0)} Z(\lambda) \cong u(L)(1 \otimes 1)$ (i.e., is standard cyclic) and any L-module generated by a maximal vector of weight λ relative to $\underline{h}$ is a homomorphic image of $u(L) \otimes_{u(L_0)} Z(\lambda)$.

LEMMA 2.2. *Let* $\lambda \in \Lambda$. *Then* $u(L) \otimes_{u(L_0)} Z(\lambda)$ *is indecomposable.*

Proof. The remark of [4, p. 720] in conjunction with [5, (1.4) and (1.5)] implies that the functor $u(L) \otimes_{u(L_0)}$- sends indecomposables to indecomposables. Since $Z(\lambda)$ is as an indecomposable $L_{[0]}$-module, L_0-indecomposable our assertion follows. ∎

3. THE CATEGORY OF $u(L^e)$-T^e-MODULES

Let $L = W$, $A = A(n, \mathbf{1})$, and Aut W and Aut A be the automorphism group of W and the automorphism group of A respectively. By [11, or 16, Theorem 8], we have

$$\text{Aut } W \cong \text{Aut } A,$$

that is, if $\Phi \in \text{Aut } W$ then there is a unique $\varphi \in \text{Aut } A$ such that

$$\Phi(x) = \varphi \times \varphi^{-1}, \qquad \forall x \in W(= \text{Der}_F A). \tag{3.1}$$

Obviously, Aut A is a closed subgroup of $GL(A)$. Note that any $\varphi \in \text{Aut } A$ is uniquely determined by the action on $\{x^{(\varepsilon_1)}, ..., x^{(\varepsilon_n)}\}$. Clearly,

$$\{t \in \text{Aut } A \mid t(x^{(\varepsilon_i)}) = t_i x^{(\varepsilon_i)},\ t_i \in F^*,\ i = 1, ..., n\}$$

is both a Cartan subgroup and a maximal torus of the algebraic group Aut A, denoted by $T(W)$, which is isomorphic to

$$\{\text{diag}\,(t_1, ..., t_n) \mid t_i \in F^*, i = 1, ..., n\}$$

and whose Lie algebra is $\underline{h}(W_{[0]})$. For $a \in F^*$, we define $E_a \in \text{Aut}\, A$ by

$$\begin{cases} E_a(x^{(\varepsilon_i)}) = a x^{(\varepsilon_i)}, & i = 1, ..., n, \text{ if } L = W, S \text{ or } H, \\ E_a(x^{(\varepsilon_i)}) = a x^{(\varepsilon_i)}, & E_a(x^{(\varepsilon_n)}) = a^2 x^{(\varepsilon_n)}, i = 1, ..., n-1, \text{ if } L = K. \end{cases}$$

Write $T_1 := \{E_a \mid a \in F^*\}$. We set

$$T(L) := \begin{cases} \{t \in T(W) \mid t = \text{diag}(t_1, ..., t_n), \prod t_i = 1\}, \\ \qquad \text{if} \quad L = S, \\ \{t \in T(W) \mid t = \text{diag}(t_1, ..., t_{2r}), t_j t_{j+r} = 1, j = 1, ..., r\}, \\ \qquad \text{if} \quad L = H, \\ \{t \in T(W) \mid t = \text{diag}(t_1, ..., t_{2r}, 1), t_j t_{j+r} = 1, \\ \qquad j = 1, ..., r\} \times T_1, \qquad \text{if} \quad L = K, \end{cases}$$

whose Lie algebra is $\underline{h}(L_{[0]})$.

Let $L = W$, S, H, or K and $T = T(W)$, $T(S)$, $T(H)$, or $T(K)$. Let Δ be the set of simple roots of $L_{[0]}$, $X(T)$ the character group of T(i.e., the group of all homomorphisms $T \to F^*$) which may be identified with the lattice of all weights of T, $X(T)^+$ the set of dominant weights in $X(T)$, and $X_1(T) = \{\lambda \in X(T)^+ \mid 0 \leqslant \langle \lambda, \alpha \rangle < p, \text{ for all } \alpha \in \Delta\}$. More precisely we ought to replace $X_1(T)$ by $X(T)/pX(T)$. Then $X_1(T) = \Lambda$. Let

$$\Lambda_i(t) = t_i, \qquad i = 1, ..., n,$$

where $t \in T$ such that $t(x^{(\varepsilon_i)}) = t_i x^{(\varepsilon_i)}$. Then

$$X(T) = \begin{cases} \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n, \qquad \text{if} \quad L = W, \\ \left\{ \sum_{i=1}^n a_i \Lambda_i \,\middle|\, \sum_{j=1}^n \Lambda_j = 0, a_i \in \mathbb{Z}, i = 1, ..., n \right\} \qquad \text{if} \quad L = S, \\ \left\{ \sum_{i=1}^n a_i \Lambda_i \mid \Lambda_j + \Lambda_{j+r} = 0, a_i \in \mathbb{Z}, i = 1, ..., n, j = 1, ..., r \right\}, \\ \qquad \text{if} \quad L = H, \\ \left\{ \sum_{i=1}^{n-1} a_i \Lambda_i \mid \Lambda_j + \Lambda_{j+r} = 0, a_i \in \mathbb{Z}, i = 1, ..., n-1, \right. \\ \qquad \left. j = 1, ..., r \right\} \oplus \mathbb{Z}(\Lambda_n/2), \qquad \text{if} \quad L = K. \end{cases}$$

To define certain partial orderings of weights, we extend T. Let $T^e := TT_1$ and $\underline{h}^e := \underline{h} + \underline{h}_1$, where $\underline{h}_1$ is the Lie algebra of T_1. Note that if $L = W$ or K, then $T^e = T$ and $\underline{h} = \underline{h}^e$. If $L = S$ or H, then we define $\chi \in X(T^e)$ by means of

$$\chi(t) = \begin{cases} 1, & \text{if } t \in T, \\ \Lambda_n(t), & \text{if } t \in T_1. \end{cases}$$

Then

$$X(T_1) = \left\{ \sum_{i=1}^{n} a_i \Lambda_i \mid \Lambda_1 = \cdots = \Lambda_n, a_i \in \mathbb{Z}, i = 1, ..., n \right\} \cong \mathbb{Z}\chi.$$

For convenience, let $\chi = 0$ for $L = W$ or K. Then the character group of T^e is

$$X(T^e) \cong X(T) \oplus \mathbb{Z}\chi.$$

Note that for $L = S$, $X(T^e) = \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$.

By (3.1), the action of $t \in T^e$ on L ($= W, S, H$, or K) is conjugation by t, which is denoted by Ad t. For $t = \operatorname{diag}(t_1, ..., t_n) \in T^e$ and $h \in \underline{h}^e$, we have

$$\begin{aligned}
&(W)\begin{cases} \operatorname{Ad}(t)(x^{(\alpha)}D_j) = \left(\prod_i t_i^{\alpha_i}\right) t_j^{-1} x^{(\alpha)} D_j = \left(\sum_i \alpha_i \Lambda_i - \Lambda_j\right)(t)\, x^{(\alpha)} D_j, \\ [h, x^{(\alpha)}D_j] = \left(\sum_i \alpha_i \Lambda_i - \Lambda_j\right)(h)\, x^{(\alpha)} D_j, \end{cases} \\
&(S)\begin{cases} \operatorname{Ad}(t)(D_{i,j}(x^{(\alpha)})) = \left(\sum_{k=1}^{n} \alpha_k \Lambda_k - \Lambda_i - \Lambda_j\right)(t)\, D_{i,j}(x^{(\alpha)}), \\ [h, D_{i,j}(x^{(\alpha)})] = \left(\sum_{k=1}^{n} \alpha_k \Lambda_k - \Lambda_i - \Lambda_j\right)(h)\, D_{i,j}(x^{(\alpha)}), \end{cases} \\
&(H)\begin{cases} \operatorname{Ad}(t)(D_H(x^{(\alpha)})) = \left(\sum_{k=1}^{n} \alpha_k \Lambda_k - 2\chi\right)(t)\, D_H(x^{(\alpha)}), \\ [h, D_H(x^{(\alpha)})] = \left(\sum_{k=1}^{n} \alpha_k \Lambda_k - 2\chi\right)(h)\, D_H(x^{(\alpha)}), \end{cases} \\
&(K)\begin{cases} \operatorname{Ad}(t)(D_K(x^{(\alpha)})) = \left(\sum_{k=1}^{n} \alpha_k \Lambda_k + (\alpha_n - 2)/2\Lambda_n\right)(t)\, D_K(x^{(\alpha)}), \\ [h, D_K(x^{(\alpha)})] = \left(\sum_{k=1}^{n} \alpha_k \Lambda_k + (\alpha_n - 2)/2\Lambda_n\right)(h)\, D_K(x^{(\alpha)}), \end{cases}
\end{aligned} \tag{3.2}$$

i.e., the action of T^e coincides with that of $\underline{h}^e$.

Let $L^c := L + \underline{h}_1$, $\mathscr{B}^c := \mathscr{B} + \underline{h}_1$, and $L_0^c := L_0 + \underline{h}_1$. By (3.2), the adjoint L^c-module L^c is also a T^c-module and z ($= x^{(\alpha)}D_j$, $D_{i,j}(x^{(\alpha)})$, $D_H(x^{(\alpha)})$, or $D_K(x^{(\alpha)})$) is not only a weight vector relative to $\underline{h}^c$ but also a weight vector relative to T^c. Let $u = z_1 \cdots z_k \in u(L^c)$, we define

$$\mathrm{Ad}(t)(u) = \mathrm{Ad}(t)(z_1) \cdots \mathrm{Ad}(t)(z_k) = tut^{-1}, \qquad \text{for} \quad t \in T^c.$$

Then $u(L^c)$ is also a T^c-module.

DEFINITION 3.1. A finite dimensional vector space V is called a $u(L^c)$-T^c-module (for convenience, we just call it a $\hat{u}(L)$-module), if V is both $u(L^c)$-module and T^c-module and satisfies:

(a) The actions of $\underline{h}^c$ coming from L^c and from T^c coincide;

(b) $t \cdot (u \cdot v) = (\mathrm{Ad}(t)\, u) \cdot (t \cdot v)$, for $v \in V$, $t \in T^c$, $u \in u(L^c)$.

Let $V = \bigoplus_\lambda V^\lambda$ be the weight space decomposition (relative to T^c). Then (a) means that for $h \in \underline{h}^c$, $v \in V^\lambda$, $h \cdot v = \lambda(h)\, v$, where $\lambda \in X(T^c)$ induces the weight $\lambda \in X(T^c)/pX(T^c)$ (relative to $\underline{h}^c$), while (b) means that $u \cdot V^\lambda \subseteq V^{\lambda + \mu}$ for $\lambda \in X(T^c)$ and $u \in u(L^c)^\mu$. Obviously, L^c and $u(L^c)$ are $\hat{u}(L)$-modules.

Now we define a partial ordering on $X(T^c)$. Let $\mathbb{Z}^n$ be the set of n-tuples of integers, which is ordered lexicographically. For $\underline{a} = (a_1, ..., a_n) \in \mathbb{Z}^n$, write $|\underline{a}| := \sum_{i=1}^n a_i$. We define a partial ordering on $X(T^c)$: (a) For $L = W$ or S, $\Sigma a_i A_i < \Sigma b_i A_i$ if and only if $|(a_1, ..., a_n)| < |(b_1, ..., b_n)|$ or $|(a_1, ..., a_n)| = |(b_1, ..., b_n)|$ and $(a_1, ..., a_n) < (b_1, ..., b_n)$. (b) For $L = H$ or K, let λ, $\mu \in X(T^c)$ and $\lambda|_T = \sum_{i=1}^r a_i A_i$, $\mu|_T = \sum_{i=1}^r b_i A_i$, then $\lambda < \mu$ if and only if $\lambda(E_2) < \mu(E_2)$ or $\lambda(E_2) = \mu(E_2)$ and $(a_1, ..., a_r) < (b_1, ..., b_r)$.

Let the T^c-weight if $x_i \in \mathscr{N}$, $i = 1, ..., s$ (resp. $y_j \in \mathscr{N}^-$, $j = 1, ..., t$) be $\mu(x_i)$ (resp. $\mu(y_j)$). By (3.2), we have

LEMMA 3.1. (a) $0 < \mu(x_i)$, $i = 1, ..., s$; $\mu(y_j) < 0$, $j = 1, ..., t$.

(b) *Let* $V = \bigoplus V^\lambda$ *be a* $\hat{u}(L)$*-module and* $v \in V^\lambda$. *Then the* T^c*-weight* $\lambda + \mu(x_i)$ (*resp.* $\lambda + \mu(y_j)$) of $x_i v$ (*resp.* $y_j v$) *is greater* (*resp. less*) *than* λ.

We can canonically define the category of $\hat{u}(L)$-modules and easily obtain

LEMMA 3.2. (a) *The kernel and image of* $\hat{u}(L)$*-homomorphisms are* $\hat{u}(L)$*-modules.*

(b) *Given a* $\hat{u}(L)$*-submodule* V' *of a* $\hat{u}(L)$*-module* V, *the quotient* V/V' *has a canonical structure of* $\hat{u}(L)$*-module for which the map* $V \to V/V'$ *is a* $\hat{u}(L)$*-homomorphism.*

Similar to [12, Sect. 2.4], other standard constructions can be done in the category of $\hat{u}(L)$-modules, e.g., dual modules and tensor products. Thus $\mathrm{Hom}_F(V_1, V_2) \cong V_1^* \otimes V_2$ has a $\hat{u}(L)$-module structure if V_1, V_2 do.

LEMMA 3.3. *Let V, V_1, and V_2 be $\hat{u}(L)$-modules. Then*

(a) $V^{u(L)} = \{v \in V \mid x \cdot v = 0,$ *for all* $x \in L\}$ *and* $\operatorname{Hom}_{u(L)}(V_1, V_2)$ *are* $\hat{u}(L)$*-modules.*

(b) *The set of T^e-weights of* $\operatorname{Hom}_{u(L)}(V_1, V_2)$ *is contained in* $pX(T) + \mathbb{Z}\chi$ *and if* $\lambda \in pX(T) + \mathbb{Z}\chi$, *then*

$$\operatorname{Hom}_{u(L)}(V_1, V_2)^{\lambda} \cong \operatorname{Hom}_{\hat{u}(L)}(V_1, V_2 \otimes F_{-\lambda}). \tag{3.3}$$

Proof. (a) Clearly, for $L = S$ or H, we have

$$x(I \cdot v) = [x, I]v + I(x \cdot v) = 0, \qquad \text{for all } x \in L \text{ and } v \in V^{u(L)}.$$

Let $x = x^{(\alpha)}D_j$, $D_{i,j}(x^{(\alpha)})$, $D_H(x^{(\alpha)})$, or $D_K(x^{(\alpha)})$ and $t \in T^e$. Then

$$\operatorname{Ad}(t)\, x = cx, \qquad \text{for some } c \in F^*.$$

Thus we have

$$x(t \cdot v) = c^{-1} \operatorname{Ad}(t)\, x(t \cdot v) = c^{-1} t(x \cdot v) = 0, \qquad \text{for } v \in V^{u(L)}.$$

Hence $V^{u(L)}$ is a $\hat{u}(L)$-module and so is $\operatorname{Hom}_{u(L)}(V_1, V_2) \cong \operatorname{Hom}_F(V_1, V_2)^{u(L)}$.

(b) Since the $\hat{u}(L)$-module $\operatorname{Hom}_{u(L)}(V_1, V_2)$ is a trivial $u(L)$-module, any T^e-weight of $\operatorname{Hom}_{u(L)}(V_1, V_2)$ on restriction to $\underline{h}$ is trivial, whereas it is contained in $pX(T) + \mathbb{Z}\chi$. In particular, $\operatorname{Hom}_{\hat{u}(L)}(V_1, V_2)$ is just the 0-weight space of $\operatorname{Hom}_{u(L)}(V_1, V_2)$.

If $\lambda \in pX(T) + \mathbb{Z}\chi$, then

$$V_2 \cong V_2 \otimes F_{-\lambda} \qquad (\text{as } u(L)\text{-modules})$$

and

$$\operatorname{Hom}_{u(L)}(V_1, V_2) \cong \operatorname{Hom}_{u(L)}(V_1, V_2 \otimes F_{-\lambda}) \qquad (\text{as vector spaces}).$$

Also $\operatorname{Hom}_{u(L)}(V_1, V_2)^{\lambda}$ consists of the maps $\varphi \in \operatorname{Hom}_{u(L)}(V_1, V_2)$ such that φ maps $(V_1)^{\mu}$ into $(V_2)^{\mu+\lambda}$ for all $\mu \in X(T^e)$, so its image in $\operatorname{Hom}_{u(L)}(V_1, V_2 \otimes F_{-\lambda})$ consists of the maps $\psi \in \operatorname{Hom}_{u(L)}(V_1, V_2 \otimes F_{-\lambda})$ such that ψ maps $(V_1)^{\mu}$ into $(V_2)^{\mu+\lambda} \otimes F_{-\lambda} = (V_2 \otimes F_{-\lambda})^{\mu}$, that is, ψ is a $\hat{u}(L)$-homomorphism. This concludes the proof. ∎

One further construction is as follows. Take a subalgebra A of $u(L^e)$ containing $u(\underline{h}^e)$ and stable under $\operatorname{Ad}(T^e)$ (such as $u(\underline{h}^e)$, $u(\mathscr{B}^e)$, $u(L_0^e)$). Let M be an $\hat{A}$-module (defined similar to a $\hat{u}(L)$-module), and we consider the "induced" module $u(L^e) \otimes_A M$, where $u(L^e)$ acts on the left factor via multiplication, and T^e acts on the left factor via $\operatorname{Ad}(T^e)$ and on the right factor by the given action. This is easily seen to be a $\hat{u}(L)$-module. Moreover, for all $\hat{u}(L)$-modules V we get a canonical vector space isomorphism.

$$\operatorname{Hom}_{\hat{u}(L)}(u(L)\otimes_A M, V)\cong \operatorname{Hom}_{\hat{A}}(M, V). \tag{3.4}$$

An arbitrary $\lambda\in X(T^c)$ (resp. $X(T)$), viewed as a homomorphism λ: $u(\underline{h}^c)$ (resp. $u(\underline{h}))\to F$, can be extended to a homomorphism λ: $u(\mathscr{B}^c)$ (resp. $u(\mathscr{B}))\to F$ by setting $\lambda(x_i)=0$ for $i=1, ..., s$. So via λ we can give F the structure of 1-dimensional $u(\mathscr{B}^c)$ (resp. $u(\mathscr{B})$)-module F_λ. Define $\hat{Z}(\lambda)=u(L^c)\otimes_{u(\mathscr{B}^c)}F_\lambda$. (In the case $\lambda\in X_1(T)$, its restriction to $u(L)$ is essentially the same as the previous $u(L)\otimes_{u(L_0)}Z(\lambda)\cong u(L)\otimes_{u(\mathscr{B})}F_\lambda$. But here λ can be arbitrary in $X(T^c)$.) Then $\hat{Z}(\lambda)$ is a $\hat{u}(L)$-module of highest weight λ and has an obvious basis consisting of weight vectors $y_1^{i_1}\cdots y_t^{i_t}\otimes 1$ $(0\leqslant i_1, ..., i_t<p)$. Moreover, the λ-weight vector $1\otimes 1$ generates $\hat{Z}(\lambda)$. Clearly each proper $\hat{u}(L)$-submodule of $\hat{Z}(\lambda)$ lies in the sum of weight spaces for weights $\neq\lambda$, so there is a unique maximal $\hat{u}(L)$-submodule and a unique irreducible quotient which is denoted by $\hat{M}(\lambda)$.

On the other hand, let V be an arbitrary irreducible $\hat{u}(L)$-module. Its finite set of weights has at least one maximal element λ. Choose a nonzero element $v\in V^\lambda$, since $x_i\cdot v=0$, $i=1, ..., s$, Fv is stable under $u(\mathscr{B}^c)$ and T^c. Set $V'=u(L)\,v=u(\mathscr{N}^-)\,v$. Then V' is a $\hat{u}(L)$-submodule of V. Hence $V'=V$ and we have

$$\mu\leqslant\lambda, \qquad \text{for any weight } \mu \text{ of } V,$$

that is, λ is the highest weight of V. Obviously, $\dim V^\lambda=1$, that is, V^λ is the unique stable line under $u(\mathscr{B}^c)$ and T^c. Since $\operatorname{Hom}_{\hat{u}(L)}(\hat{Z}(\lambda), V)\cong \operatorname{Hom}_{\hat{u}(\mathscr{B})}(F_\lambda, V)\neq 0$, V must be isomorphic to a quotient of $\hat{Z}(\lambda)$, hence to $\hat{M}(\lambda)$. Note finally that, because their highest weights differ, the modules $\hat{M}(\lambda)$, $\lambda\in X(T^c)$, are non-isomorphic to each other.

For any arbitrary $\hat{u}(L)$-module V, let $[V:\hat{M}(\lambda)]$ be the number of times of $\hat{M}(\lambda)$ occurring as a composition factor of V. The results in [12, Sect. 2.8] can be applied to the case of $\hat{u}(L)$-modules as follows.

PROPOSITION 3.1. *If $\lambda\in X(T^c)$ and $\mu\in pX(T)+\mathbb{Z}\chi$, then*

(a) *$\hat{M}(\mu)$ is 1-dimensional, with trivial $u(L)$-action.*

(b) $\hat{M}(\lambda+\mu)\cong\hat{M}(\lambda)\otimes_F\hat{M}(\mu)$.

(c) $\hat{Z}(\lambda+\mu)\cong\hat{Z}(\lambda)\otimes_F\hat{M}(\mu)$.

(d) $[V\otimes\hat{M}(\mu):\hat{M}(\lambda)]=[V:\hat{M}(\lambda-\mu)]$.

It is useful to attach a formal charactor $\operatorname{ch}(V)$ to a $\hat{u}(L)$-module V. Let $\mathbb{Z}[X(T^c)]$ be the group ring of $X(T^c)$ with basis consisting of symbols $e(\lambda)$

in $1-1$ correspondence with the elements of $X(T^c)$, and with multiplication determined by the rule $e(\lambda)e(\mu)=e(\lambda+\mu)$. Let $m_V(\lambda)$ be the multiplicity of λ as a T^c-weight of V(the dimension of the corresponding T^c-weight space V^λ), and set $\mathrm{ch}(V)=\sum_{\lambda\in X(T^c)} m_V(\lambda)\, e(\lambda)\in \mathbb{Z}[X(T^c)]$. The sum is of course finite.

4. Projective $\hat{u}(L)$-Modules

For $\lambda\in X(T^c)$, we view F_λ as a $\hat{u}(\underline{h}^c)$-module and form an induced $\hat{u}(L)$-module $I(\lambda)=u(L^c)\otimes_{u(\underline{h}^c)} F_\lambda$ with basis consisting of all $y_1^{i_1}\cdots y_t^{i_t}x_1^{j_1}\cdots x_s^{j_s}\otimes 1$ $(0\leqslant i_1,...,i_t,\ j_1,...,j_s<p)$, hence $\dim I(\lambda)=p^{s+t}$. Note that let $\lambda_1=\lambda|_T$, then $I(\lambda)\cong u(L)\otimes_{u(h)} F_{\lambda_1}$ (regarded as $u(L)$-modules).

Lemma 4.1. *$I(\lambda)$ is a projective $\hat{u}(L)$-module (resp. projective $u(L)$-module).*

Proof. For any $\hat{u}(L)$-module (resp. $u(L)$-module) V, by (3.4), we have

$$\begin{aligned}\mathrm{Hom}_{\hat{u}(L)}(I(\lambda), V) &\cong \mathrm{Hom}_{\hat{u}(h)}(F_\lambda, V|_{\hat{u}(h)})\\ &\cong \mathrm{Hom}_{T^c}(F_\lambda, V|_{T^c}) \end{aligned}\tag{4.1}$$

(resp. $\mathrm{Hom}_{u(L)}(I(\lambda), V)\cong \mathrm{Hom}_{u(h)}(F_{\lambda_1}, V|_{u(h)})$). Since any $\hat{u}(\underline{h})$-module (resp. $u(\underline{h})$-module) is completely reducible, the functor $\mathrm{Hom}_{\hat{u}(h)}(F_\lambda, \quad)$ (resp. $\mathrm{Hom}_{u(h)}(F_{\lambda_1}, \text{---})$) is exact. Hence $\mathrm{Hom}_{\hat{u}(L)}(I(\lambda), \quad)$ (resp. $\mathrm{Hom}_{u(L)}(I(\lambda), \text{---})$) is also exact. This completes the proof. ∎

Now for an arbitrary $\hat{u}(L)$-module V, we get an epimorphism $\bigoplus_\lambda I(\lambda)^{\dim V^\lambda}\to V$ (here we take the homomorphism $I(\lambda)\to V$ corresponding to elements of a basis for V^λ). So the $\hat{u}(L)$-module V is the quotient of a projective module $\bigoplus_\lambda I(\lambda)^{\dim V^\lambda}$, hence the category of $\hat{u}(L)$-modules has enough projectives. Since all modules in the category have a finite composition series, standard arguments show that each projective $\hat{u}(L)$-module is a direct sum of indecomposable projectives, each irreducible $\hat{u}(L)$-module $\hat{M}(\lambda)$ has an indecomposable projective cover $\hat{Q}(\lambda)$ with $\hat{M}(\lambda)$ as its unique irreducible quotient and each indecomposable projective $\hat{u}(L)$-module is isomorphic to some $\hat{Q}(\lambda)$. Moreover, for any $\lambda\in X(T^c)$ and an arbitrary $\hat{u}(L)$-module V, we have

$$[V:\hat{M}(\lambda)]=\dim \mathrm{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda), V).\tag{4.2}$$

Lemma 4.2. *$\hat{Q}(\lambda+\mu)\cong\hat{Q}(\lambda)\otimes_F\hat{M}(\mu)$, for all $\lambda\in X(T^c)$ and $\mu\in pX(T)+\mathbb{Z}\chi$.*

Proof. For an arbitrary $\hat{u}(L)$-module V, there is a natural isomorphism of vector spaces

$$\mathrm{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda)\otimes_F \hat{M}(\mu), V)\cong \mathrm{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda), V\otimes_F \hat{M}(-\mu)).$$

It impies that the functor $\mathrm{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda)\otimes_F \hat{M}(\mu), -)$ is exact. Hence $\hat{Q}(\lambda)\otimes_F \hat{M}(\mu)$ is a projective $\hat{u}(L)$-module, which is indecomposable (on account of Proposition 3.1(a)), with $\hat{M}(\lambda)\otimes_F \hat{M}(\mu)\cong \hat{M}(\lambda+\mu)$ as quotient. Hence $\hat{Q}(\lambda)\otimes_F \hat{M}(\mu)\cong(\lambda+\mu)$. ∎

DEFINITION 4.1. A $\hat{u}(L)$-module V is a $\hat{Z}$-filtered module, if there is a filtration

$$0=V_0\subset V_1\subset \cdots \subset V_r=V$$

such that the filtration quotients $V_i/V_{i-1}\cong \hat{Z}(\mu_i)$ for some $\mu_i\in X(T^c)$, $i=1, ..., r$. The above filtration is called a $\hat{Z}$-filtration.

Obviously, $\mathrm{ch}\, V=\sum_{i=1}^r \mathrm{ch}\, \hat{Z}(\mu_i)$ and the various $\mathrm{ch}\, \hat{Z}(\mu)$ are linearly independent in $\mathbb{Z}[X(T^c)]$, since each involves a distinct highest weigtht. So the number $(V: \hat{Z}(\mu))$ of indices with $\mu_i=\mu$ is well determined by V, independent of which $\hat{Z}$-filtration we choose.

LEMMA 4.3. *Each $\hat{u}(L)$-module $I(\lambda)$ has a $\hat{Z}$-filtration. Moreover*

$$(I(\lambda):\hat{Z}(\mu))=m_{u(\mathcal{N})}(\mu-\lambda), \qquad \textit{for all} \quad \mu\in X(T^c).$$

Proof. First we arrange the monomials $x_1^{i_1}\cdots x_s^{i_s}$ $(0\leqslant i_1, ..., i_s<p)$ in $u(\mathcal{N})$ in a certain order $X_1, ..., X_{p^s}$ such that the T^c-weight μ_i of X_i is maximal in $\{\mu_i, \mu_{i+1}, ..., \mu_{p^s}\}$. Let

$$I_i=\sum_{j=1}^{i} u(L)(X_j\otimes 1)\subseteq I(\lambda).$$

Then

$$0=I_0\subset I_1\subset \cdots \subset I_{p^s}=I(\lambda),$$

each I_i is a $\hat{u}(L)$-submodule and I_i/I_{i-1} is generated just by the coset of $X_i\otimes 1$. By Lemma 3.1, x_kX_i equals a linear combination of certain X_j of higher weight than X_i for $k=1, ..., s$, so $j<i$. Thus we get $x_k(X_i\otimes 1)\in I_{i-1}$, x_k annihilates the coset of $X_i\otimes 1$ in I_i/I_{i-1}. Hence I_i/I_{i-1} is a quotient of $\hat{Z}(\lambda+\mu_i)$ and

$$\dim(I_i/I_{i-1})\leqslant \dim \hat{Z}(\lambda+\mu_i)=p^t.$$

Since

$$p^{r+s} = \dim I(\lambda) = \sum_{i=1}^{p^s} \dim(I_i/I_{i-1}) \leqslant p^{r+s},$$

we have $\dim(I_i/I_{i-1}) = p^r$ and

$$I_i/I_{i-1} \cong \hat{Z}(\lambda + \mu_1), \qquad i = 1, ..., p^s.$$

Thus $I(\lambda)$ has a $\hat{Z}$-filtration, and by construction the multiplicities $(I(\lambda) : \hat{Z}(\mu))$ are as claimed. ∎

To show that all projective $\hat{u}(L)$-modules have $\hat{Z}$-filtrations, we must see how to handle direct summands.

LEMMA 4.4. (a) *Let V have a $\hat{Z}$-filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$. Let λ be a maximal weight of V, $v \in V^\lambda$, and $v \neq 0$. Then $u(L) \cdot v \cong \hat{Z}(\lambda)$ and $V/u(L) \cdot v$ has a $\hat{Z}$-filtration.*

(b) *Suppose a direct sum $V_1 \oplus V_2$ of two $\hat{u}(L)$-modules has a $\hat{Z}$-filtration. Then V_1, V_2 also do.*

Proof. This follows from the same argument as [12, Lemma 3.5 and 3.6]. ∎

COROLLARY 4.1. *Every projective $\hat{u}(L)$-module has a $\hat{Z}$-filtration.*

Proof. It is already proved for $I(\lambda)$. By (4.1),

$$\mathrm{Hom}_{\hat{u}(L)}(I(\lambda), \hat{M}(\lambda)) \cong \hat{M}(\lambda)^\lambda \neq 0,$$

so $\hat{M}(\lambda)$ is a quotient of $I(\lambda)$, forcing $\hat{Q}(\lambda)$ to be a direct summand of $I(\lambda)$. By Lemmas 4.3 and 4.4(b), $\hat{Q}(\lambda)$ has a $\hat{Z}$-filtration. The same is true for direct sums, i.e., for all projective $\hat{u}(L)$-modules. ∎

Now we prove the reciprocity theorem for the category of $\hat{u}(L)$-modules.

THEOREM 4.1. *Let λ, $\mu \in X(T^c)$. Then*

$$m_{u(L^{s})}(\lambda - \mu)(\hat{Q}(\lambda) : \hat{Z}(\mu)) = m_{u(L^{s})}(\mu - \lambda)[\hat{Z}(\mu) : \hat{M}(\lambda)].$$

Proof. By (4.2), we have

$$[\hat{Z}(\mu) : \hat{M}(\lambda)] = \dim \mathrm{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda), \hat{Z}(\mu)).$$

We must show

$$m_{u(L^{-})}(\lambda-\mu)(\hat{Q}(\lambda):\hat{Z}(\mu)) \\ = m_{u(L^{+})}(\mu-\lambda)\dim \operatorname{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda), \hat{Z}(\mu)), \\ \text{for all} \quad \lambda, \mu \in X(T^c). \tag{4.3}$$

This follows from the more general statement: for arbitrary projective P,

$$m_{u(L^{-})}(\lambda-\mu)(P:\hat{Z}(\mu)) \\ = m_{u(L^{+})}(\mu-\lambda)\dim \operatorname{Hom}_{\hat{u}(L)}(P, \hat{Z}(\mu)). \tag{4.4}$$

Note that both sides are additive in the first variable. First we verify (4.4) for $P=I(\lambda)$. Note that

$$\dim \operatorname{Hom}_{\hat{u}(L)}(I(\lambda), \hat{Z}(\mu)) = \dim \hat{Z}(\mu)^{\lambda} = m_{u(L^{-})}(\lambda-\mu).$$

But $(I(\lambda): \hat{Z}(\mu)) = m_{u(L^{+})}(\mu-\lambda)$. Hence (4.4) holds for $P=I(\lambda)$. Next we note that the number of times that $\hat{Q}(\tau)$ appears in a direct sum decomposition of $I(\lambda)$ into indecomposable projective modules equals $\dim \operatorname{Hom}_{\hat{u}(L)}(I(\lambda), \hat{M}(\tau)) = \dim \hat{M}(\tau)^{\lambda}$, $\lambda \leqslant \tau$. If $\lambda \nleqslant \mu$, then

$$(\hat{Q}(\lambda):\hat{Z}(\mu)) \leqslant (I(\lambda):\hat{Z}(\mu)) = 0,$$

since $m_{u(L^{+})}(\mu-\lambda)=0$. At the same time, if $\lambda \nleqslant \mu$, then

$$\dim \operatorname{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda), \hat{Z}(\mu)) = [\hat{Z}(\mu):\hat{M}(\lambda)] = 0.$$

So for $\lambda \nleqslant \mu$, we get (4.3).

Now we must still prove (4.3) when $\lambda \leqslant \mu$. Fix μ and we use induction on λ from above. If $\lambda=\mu$, since $I(\lambda) = \hat{Q}(\lambda) \oplus \bigoplus_{\tau<\lambda} \hat{Q}(\tau)^{m(\tau)}$ where $m(\tau) = \dim \hat{M}(\tau)^{\lambda}$, then $\tau > \lambda = \mu$ and (4.4) holds for $\hat{Q}(\lambda)$, using (4.4) for $I(\lambda)$ along with the fact that (4.4) is additive in the first variable P. If $\lambda < \mu$, by induction (4.4) holds for $P = \bigoplus_{\tau>\lambda} \hat{Q}(\tau)^{m(\tau)}$, since (4.4) is now known for $I(\lambda)$, the additivity of (4.4) yields the same conclusion for $\hat{Q}(\lambda)$. ∎

Let $\Pi(V)$ be the set of all T^c-weights of a $\hat{u}(L)$-module V. By the proof of Theorem 4.1, we have

Corollary 4.2. *Let $\lambda, \mu \in X(T^c)$. Then we have*

(a) $m_{u(L^{+})}(\mu-\lambda) \geqslant m_{u(L^{-})}(\lambda-\mu) > 0$, *if* $\lambda-\mu \in \Pi(u(L^{-}))$.

(b) $[\hat{Z}(\mu):\hat{M}(\lambda)] = 0$, if $\lambda-\mu \notin \Pi(u(L^{-}))$ *and* $\mu-\lambda \in \Pi(u(L^{+}))$.

(c) $(\hat{Q}(\lambda):\hat{Z}(\mu)) = [\hat{Z}(\mu):\hat{M}(\lambda)] = 0$, *if* $\lambda-\mu \notin \Pi(u(L^{-})) \cup \Pi(u(L^{+}))$.

5. Projective $u(L)$-Modules

For $\lambda \in X_1(T)$, $\hat{Z}(\lambda)$ is essentially $u(L) \otimes_{u(L_0)} Z(\lambda)$. Its restriction to $u(L)$ is just $u(L) \otimes_{u(L_0)} Z(\lambda)$. For the irreducible quotient $\hat{M}(\lambda)$ of $\hat{Z}(\lambda)$, we denote its restriction to $u(L)$ by $\hat{M}(\lambda)|_{u(L)}$. To show the irreducibility of $\hat{M}(\lambda)|_{u(L)}$, we introduce the notion of $\hat{u}(L)$-gradation.

Definition 5.1. A $\hat{u}(L)$-module V is called $\hat{u}(L)$-graded if $V = \bigoplus_{i \geq 0} V_i$ (direct sum of subspaces) is $u(L)$-graded and $T^e V_i \subseteq V_i$.

Let $L^- = \bigoplus_{i<0} L_{[i]}$, $\mathrm{Ann}_V L^- = \{v \in V \mid xv = 0, \forall x \in L^-\}$ and $L^e_{[0]} = L_{[0]} + \underline{h}_1$.

Lemma 5.1. (a) *If $V = \bigoplus_{i \geq 0} V_i$ is an irreducible $\hat{u}(L)$-graded module, then V_0 is an irreducible $\hat{u}(L_{[0]})$-module (i.e., $u(L^e_{[0]}) - T^e$-module).*

(b) *If a $\hat{u}(L)$-graded module $V = \bigoplus_{i \geq 0} V_i$ is transitive (i.e., $\mathrm{Ann}_V L^- = V_0$) and V_0 is an irreducible $\hat{u}(L_{[0]})$-module, then $u(L)\, V_0$ is the unique irreducible $\hat{u}(L)$-submodule of V.*

Proof. (a) If V_0' is a proper $\hat{u}(L_{[0]})$-module of V_0, then $u(L)\, V_0'$ is obviously a proper $\hat{u}(L)$-submodule of V.

(b) Since V is transitive, every nonzero $\hat{u}(L)$-submodule of V has a nonzero intersection with V_0. Hence it contains V_0 and therefore contains $u(L)\, V_0$. ∎

Let $u(L^-) = \bigoplus_{i \geq 0} u(L^-)_{-i}$ whose gradation is derived from that of L^-. Let $V = \bigoplus_{i \geq 0} V_i$ be a $\hat{u}(L)$-graded module, $R_i = \langle v \in V_i \mid u(L^-)_{-i} v = 0 \rangle$, and $R = \bigoplus_{i \geq 0} R_i$ which is called the radical of V.

Lemma 5.2. *If V_0 is a $\hat{u}(L_{[0]})$-module and $V = u(L)\, V_0$, then* (a) *V is $\hat{u}(L)$-graded;* (b) *the radical R is a homogeneous $\hat{u}(L)$-submodule of V.*

Proof. (a) This is clear. (b) By [18, Proposition 1.1], R is a homogeneous $u(L)$-submodule of V. Clearly, we have $T^e R_i \subseteq R_i$. The proof is finished. ∎

The following lemma can now be obtained by Lemma 5.2 and adopting the arguments of [18, Corollary 1.5] mutatis mutandis.

Lemma 5.3. *Every irreducible $\hat{u}(L)$-module is isomorphic to a $\hat{u}(L)$-graded module.*

Proposition 5.1. *If every irreducible $\hat{u}(L_{[0]})$-module V_0 is $L_{[0]}$-irreducible, then every irreducible $\hat{u}(L)$-module V is $u(L)$-irreducible.*

Proof. By Lemma 5.3, V is $\hat{u}(L)$-graded. Set $V = \bigoplus_{i \geq 0} V_i$. By Lemma 5.1(a), V_0 is $\hat{u}(L_{[0]})$-irreducible and therefore $L_{[0]}$-irreducible. This implies that the $L_{[0]}$-submodule $\mathrm{Ann}_{V} L_{-} = V_0$, that is, V is transitive. Let V' be an irreducible $u(L)$-submodule. Using the argument of [18, Corollary 1.2(2)], we can easily show that $V' = u(L) V_0$. By Lemma 5.1(b), we have $V = V'$. Hence V is $u(L)$-irreducible. ∎

Note that $L_{[0]}$ has a canonical graduation such that $L_{[0]} = \bigoplus_{i=-r}^{r} g_i$ with $g_0 = \underline{h}$. By the same argument of Proposition 5.1, we can show that if every irreducible $\hat{u}(h)$-module V_0 is $\underline{h}$-irreducible, then every irreducible $\hat{u}(L_{[0]})$-module is $L_{[0]}$-irreducible. We know that the notions of $\hat{u}(\underline{h})$-module and T-module are equivalent and any irreducible T-module is one-dimensional which must be $\underline{h}$-irreducible. Hence we have

COROLLARY 5.1. *If V_0 is an irreducible $\hat{u}(L_{[0]})$-module, then V_0 is $L_{[0]}$-irreducible.*

Remark 5.1. Corollary 5.1 is a result of the representation theory of algebraic groups (cf. [12, Sect. 4.1]). Now we obtain this result by direct proof.

By Proposition 5.1 and Corollary 5.1, we obtain

COROLLARY 5.2. *Every irreducible $\hat{u}(L)$-module is $u(L)$-irreducible. In particular, $\hat{M}(\lambda)|_{u(L)}$ is $u(L)$-irreducible, for $\lambda \in X_1(T)$.*

For convenience, we write $M(\lambda) = \hat{M}(\lambda)|_{u(L)}$, for arbitrary $\lambda \in X(T^c)$. For $\lambda \in X_1(T)$ and $\mu \in pX(T) + \mathbb{Z}\chi$, by Proposition 3.1, we have

$$\begin{cases} M(\lambda + \mu) \cong M(\lambda), \\ \hat{Z}(\lambda + \mu)|_{u(L)} \cong u(L) \otimes_{u(L_0)} Z(\lambda) \end{cases} \qquad (\text{as } u(L)\text{-}modules). \qquad (5.1)$$

It follows that each composition series of a $\hat{u}(L)$-module V is also a composition series of the $u(L)$-module V, with multiplicity of $M(\lambda)$ as $u(L)$-composition factor given by

$$[V: M(\lambda)] = \sum_{\mu \in pX(T) + \mathbb{Z}\chi} [V: \hat{M}(\lambda + \mu)], \qquad \text{for} \quad \lambda \in X_1(T). \qquad (5.2)$$

As for $\hat{u}(L)$, the category of $u(L)$-modules has enough projective modules. For $\lambda \in X_1(T)$, let $Q(\lambda)$ be the PIM corresponding to $M(\lambda)$. Then $Q(\lambda)$ is an indecomposable projective $u(L)$-module with quotient $M(\lambda)$ (i.e., $Q(\lambda)$ is a projective cover of $M(\lambda)$). As in (4.2), we have

$$\dim \mathrm{Hom}_{u(L)}(Q(\lambda), V) = [V: M(\lambda)].$$

LEMMA 5.4. *Let $\lambda \in X_1(T)$. Then $\hat{Q}(\lambda) \cong Q(\lambda)$ as $u(L)$-modules.*

Proof. Since $\hat{Q}(\lambda)$ is a $\hat{u}(L)$-summand of $I(\lambda)$, $\hat{Q}(\lambda)$ is also a $u(L)$-summand of $I(\lambda)$. By Lemma 4.1, $\hat{Q}(\lambda)|_{u(L)}$ is projective. Let $\mu \in X_1(T)$. Then

$$\begin{aligned} &\dim \operatorname{Hom}_{u(L)}(\hat{Q}(\lambda)|_{u(L)}, M(\mu)) \\ &= \sum_{\tau \in pX(T) + \mathbb{Z}\chi} \dim \operatorname{Hom}_{u(L)}(\hat{Q}(\lambda)|_{u(L)}, M(\mu))^{\tau} \\ &\quad \text{(by Lemma 3.3(b))} \\ &= \sum_{\tau \in pX(T) + \mathbb{Z}\chi} \dim \operatorname{Hom}_{\hat{u}(L)}(\hat{Q}(\lambda), \hat{M}(\mu - \tau)) \\ &\quad \text{(by (3.3) and Proposition 3.1)} \\ &= \sum_{\tau \in pX(T) + \mathbb{Z}\chi} \delta_{\lambda\mu}\delta_{\tau o} = \delta_{\lambda\mu}. \end{aligned}$$

Thus regarded as a $u(L)$-module, $\hat{Q}(\lambda)$ has a unique irreducible quotient $M(\lambda)$. This implies that

$$\hat{Q}(\lambda)|_{u(L)} \cong Q(\lambda). \quad \blacksquare$$

THEOREM 5.1. *Let $\lambda \in X_1(T)$. Then*

(a) *There is a filtration $0 = Q_0 \subset Q_1 \subset \cdots \subset Q_r = Q(\lambda)$ with $Q_i/Q_{i-1} \cong u(L) \otimes_{u(L_0)} Z(\mu_i)$, for some $\mu_i \in X_1(T)$.*

(b) *The set $\{u(L) \otimes_{u(L_0)} Z(\mu_i) \mid i = 1, ..., r\}$ of the filtration quotients (counted with multiplicity) in* (a) *is uniquely determined by $Q(\lambda)$.*

(c) *Let $(Q(\lambda) : u(L) \otimes_{u(L_0)} Z(\mu))$ denote the multiplicity of $u(L) \otimes_{u(L_0)} Z(\mu)$ $(\mu \in X_1(T))$ as filtration quotient of the filtration of $Q(\lambda)$. Then*

$$\begin{aligned} (Q(\lambda) : u(L) \otimes_{u(L_0)} Z(\mu)) &= \sum_{\tau \in pX(T) + \mathbb{Z}\chi} (\hat{Q}(\lambda) : \hat{Z}(\mu + \tau)) \\ &\geqslant [u(L) \otimes_{u(L_0)} Z(\mu) : M(\lambda)]. \end{aligned}$$

Proof. (a) We start with a $\hat{Z}$-filtration of $\hat{Q}(\lambda)$ having quotients $\hat{Z}(\mu)$. It gives on restriction to $u(L)$ a filtration with quotients $u(L) \otimes_{u(L_0)} Z(\mu)$, so we get (a).

(b) By the argument of [12, Proposition 4.2], we can easily get (b).

(c) For the filtration in (a), we use Lemma 5.4 and (5.1) to compute multiplicity of $u(L) \otimes_{u(L_0)} Z(\mu)$, which is equal to

$$
\begin{aligned}
&(Q(\lambda) : u(L) \otimes_{u(L_0)} Z(\mu)) \\
&\quad = \sum_{\tau \in pX(T) + \mathbb{Z}\chi} (\hat{Q}(\lambda) : \hat{Z}(\mu + \tau)) \\
&\quad \geqslant \sum_{\tau \in pX(T) + \mathbb{Z}\chi} [\hat{Z}(\mu + \tau) : \hat{M}(\tau)] \qquad \text{(by Corollary 4.2)} \\
&\quad = \sum_{\tau \in pX(T) + \mathbb{Z}\chi} [\hat{Z}(\mu) : \hat{M}(\lambda - \tau)] \qquad \text{(by Proposition 3.1)} \\
&\quad = [u(L) \otimes_{u(L_0)} Z(\mu) : M(\lambda)] \qquad \text{(by (5.2)).} \quad \blacksquare
\end{aligned}
$$

Let λ, $\mu \in X_1(T)$. Write $b_{\lambda\mu} = (Q(\lambda) : u(L) \otimes_{u(L_0)} Z(\mu))$, $c_{\lambda\mu} = [Q(\lambda) : M(\mu)]$ (called the Cartan invariants of $u(L)$) and $d_{\mu\lambda} = [u(L) \otimes_{u(L_0)} Z(\mu) : M(\lambda)]$. Let B, C, and D be the corresponding $p^1 \times p^1$ matrices of integers (C and D are called the Cartan matrix and the decomposition matrix of $u(L)$, respectively). By Theorem 5.1, we have

COROLLARY 5.3. *$C = BD$.*

Since $u(L)$ is a symmetric algebra, C is symmetric and $Q(\lambda)$ ($\lambda \in X_1(T)$) is the injective envelope of $M(\lambda)$, that is, the socle of $Q(\lambda)$ is isomorphic to the unique highest composition factor $M(\lambda)$.

We finally generalize the linkage principle in [8] to the cases of $L = W$, S, or H, using the following results of [19].

If $\lambda \in \Lambda$, then we denote the irreducible module of $u(L_{[0]})$ with highest weight λ by $V(\lambda)$ and write $\tilde{M}(\lambda) = (u(L) \otimes_{u(L_0)} V(\lambda)_\sigma)_{\min}$.

By [4, Corollary 2.6; 19, Theorems 2.1, 2.2, and 2.3], we have

LEMMA 5.5. *Let L be any one of W, S, and H.* (a) *If V_0 is $u(L_{[0]})$-irreducible, then $(u(L) \otimes_{u(L_0)} (V_0)_\sigma)$ is $u(L)$-irreducible unless V_0 is trivial or a highest weight module with a fundamental weight as its highest weight.*

(b) *For $L = W$, the composition factors of $u(L) \otimes_{u(L_0)} V(\lambda_i)_\sigma$ are $\tilde{M}(\lambda_i)$ (socle), $\tilde{M}(\lambda_{i+1})$ (top), and $F(C^n_i - \delta_{i0}$ times), $i = 0, 1, ..., n$, where $\tilde{M}(\lambda_0) = F$ and $\tilde{M}(\lambda_{n+1}) = 0$ (the top composition factor of $u(L) \otimes_{u(L_0)} V(\lambda_n)_\sigma$ is F). For $L = S$, the composition factor of $u(L) \otimes_{u(L_0)} V(\lambda_i)_\sigma$ are $\tilde{M}(\lambda_i)$ (socle), $\tilde{M}(\lambda_{i+1})$ (top), and $F(C^n_i + \delta_{i1}$ times), $i = 0, 1, ..., n-1$, where $\tilde{M}(\lambda_0) = F$ and $\tilde{M}(\lambda_n)$ has composition factors F and $\tilde{M}(\lambda_1)$ (the top composition factor of $u(L) \otimes_{u(L_0)} V(\lambda_{n-1})_\sigma$ is F). For $L = H(n = 2r)$, the composition factors of $u(L) \otimes_{u(L_0)} V(\lambda_i)_\sigma$ are $\tilde{M}(\lambda_{i-1})(1 - \delta_{i1}$ times), $\tilde{M}(\lambda_i)$ (socle), $\tilde{M}(\lambda_i)$ (top), $\tilde{M}(\lambda_{i+1})$ and $F(C^n_i + C^{n+1}_i - 2\delta_{i0} + \delta_{i1}$ times), $i = 0, 1, ..., r$, where $\tilde{M}(\lambda_{-1}) = 0$, $\tilde{M}(\lambda_0) = F$, and $\tilde{M}(\lambda_{r+1}) = \tilde{M}(\lambda_{r-1})$.*

Remark 5.2. By Lemma 5.5 and Corollary 5.2, we have

$$M(\lambda) \cong \begin{cases} \tilde{M}(\lambda), & \text{if } \lambda \neq \lambda_0, \lambda_1, ..., \lambda_1 \text{ or } L = H, \\ \tilde{M}(\lambda_{i+1}), & \text{if } L = W (\text{or } S) \text{ and } \lambda = \lambda_i, i = 0, ..., 1, \end{cases}$$

where $M(\lambda_{l+1}) = F$.

Let $\mathscr{W}$ be the Weyl group of $L_{[0]}$, w_0 the longest element in $\mathscr{W}$, and δ = half the sum of positive roots. We denote $w \cdot \lambda = w(\lambda + \delta) - \delta$, for $w \in \mathscr{W}$ and $\lambda \in \Lambda$ and say that two weights λ, $\mu \in \Lambda$ are linked and write $\lambda \sim \mu$ if $w \cdot \lambda = \mu$.

By the lankage principle (cf. [8, Theorem 3.2]) and Lemma 5.5, we can easily obtain

THEOREM 5.2. *Let L be any one of W, S, and H.* (a) *If* $\tilde{M}(\mu)$ *is a composition factor of* $u(L) \otimes_{u(L_0)} Z(\lambda + \sigma|_h)$ $(\lambda, \mu \in \Lambda)$, *then one of the following statements hold.* (1) $\lambda \sim \mu$. (2) $\mu = 0$ *and* $\lambda \sim \lambda_j$ $(j = 0, 1, ..., 1)$. (3) $\mu = \lambda_i$ $(i > 0)$ *and* $\lambda \sim \lambda_{i-1}$.

(b) $u(L) \otimes_{u(L_0)} Z(\lambda + \sigma|_h)$ *and* $u(L) \otimes_{u(L_0)} Z(\lambda + \sigma|_h)$ *share a composition factor if and only if* $\lambda \sim \mu$ *or* $\lambda \sim \lambda_i$ *and* $\mu \sim \lambda_j$ *for some i, j.*

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