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On the Resolutions of the Powers of the Pfaffian Ideal

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INTRODUCTION

Let R be any noetherian ring. Let n be a positive integer such that $n=2k+1$ $(k \geqslant 1)$. Let Z_{ij}, $1 \leqslant i<j \leqslant n$, be $\frac{1}{2}n(n-1)$ independent indeterminates. We let X stand for the $n \times n$ matrix (X_{ij}) such that

$$X_{ij}=\begin{cases} Z_{ij} & \text{if } i<j \\ -Z_{ij} & \text{if } i>j \\ 0 & \text{if } i=j \end{cases}$$

(X is a generic skew-symmetric matrix), and we let S stand for the polynomial ring $R[X]$ $(=R[Z_{ij}])$.

The pfaffians of the $2k$-order principal submatrices of X generate an ideal of S, the "pfaffian ideal," denoted by I. It is well known that I is a generically perfect Gorenstein prime ideal of grade 3, in fact a prototype of all the ideals of this kind (cf. [B-E, 3]). Hence, I has a finite free S-resolution of length 3, a resolution which looks the same regardless of the ring R.

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In our 1989 preprint "On the square of the pfaffian ideal" (largely reproduced in [B-S]), we described for the first time a finite free resolution, $\mathbb{A}$, of I^2 (we mean: of S/I^2). In particular, since $\mathbb{A}$ did not depend on R and its length was 3, I^2 appeared to be generically perfect, as had long been conjectured and was independently proved in [B–U], without any explicit construction of resolutions. (Incidentally, note that I^2 coincides with the ideal, say I_{n-1}, generated by the $(n-1)$-order minors of X, i.e., the "submaximal" minors.)

Our construction of the complex $\mathbb{A}$ was inspired by some heuristic considerations (cf. Section 2 below), which also indicated that a certain family of complexes might provide resolutions for all the powers I^m $(m \geqslant 1)$. (Interest in such a class of ideals was partially suggested to us by the desire of comparing its behavior with that of the family studied in [B-E, 2]).

In order to prove the exactness of our candidates, we first devised a new way of proving the exactness of $\mathbb{A}$ (based on the acyclicity lemma), which seemed more suitable for generalization than that of [B-S], based on the Buchsbaum–Eisenbud criterion [B-E, 1]. Then some more work enabled us to show the exactness of our conjectural resolutions. It was precisely when we were performing the latter step that Kustin and Ulrich came out with their long preprint [K-U], in which resolutions for the ideals I^m were obtained as a by-product (an unexpected one, the authors say in their introduction) of a more general construction related to residual intersections.

The approach and the techniques of [K-U], however, are different from ours. Indeed, as long as one concerns oneself only with the ideals I^m, our point of view is neater. Furthermore, we believe that methods like ours (making use of some universally free representations of the general linear group, and of their straightening laws) are of independent interest and can help in resolving the ideals of other significant classes. In a slightly different vein, they have already proved very effective: see for instance [A-B-W, 1] and [A-B-W, 2].

Section 1 contains some preliminaries. In Section 2, we give the construction of the complexes $\mathbb{C}_m$ later to be shown to resolve the ideals I^m $(m \geqslant 1)$, and some comments are made. Sections 3 and 4 carry the proof of the exactness of the complexes $\mathbb{C}_m$.

1. Preliminaries

1.1. Keeping in mind the notations of the Introduction, we first recall some facts.

Let F_0 be a free R-module of rank $n = 2k+1$, and take the symmetric algebra $A = S(\Lambda^2 F_0)$. Set $F = A \otimes_R F_0$. We define a degree 1 A-map $f: F \to F^*$ in the following way (F^* being the dual of F). For every r, f on $A_r \otimes_R F_0$ is the composite

$$A_r \otimes_R F_0 \xrightarrow{1 \otimes 1 \otimes C_{F_0}} A_r \otimes_R F_0 \otimes_R F_0 \otimes_R F_0^* \xrightarrow{1 \otimes m \otimes 1} A_r \otimes_R \Lambda^2 F_0 \otimes_R F_0^*$$

$$= A_r \otimes_R A_1 \otimes_R F_0^* \xrightarrow{m \otimes 1} A_{r+1} \otimes_R F_0^*,$$

where m denotes multiplication in the appropriate algebras and C_{F_0} stands for the element of $F_0 \otimes_R F_0^* \cong \mathrm{Hom}_R(F_0, F_0)$ corresponding to the identity on F_0.

Choosing dual bases $\{e_1, ..., e_n\}$ and $\{\varepsilon_1, ..., \varepsilon_n\}$ for F_0 and F_0^*, resp., $C_{F_0} = \sum_{j=1}^n e_j \otimes \varepsilon_j$ and it turns out that

$$F_0 \xrightarrow{1 \otimes C_{F_0}} F_0 \otimes_R F_0 \otimes_R F_0^* \xrightarrow{m \otimes 1} \Lambda^2 F_0 \otimes_R F_0^*$$

sends each e_i to $\sum_j (e_i \wedge e_j) \otimes \varepsilon_j$. Hence $(e_i \wedge e_j)$ is the matrix associated to f with respect to the induced bases $\{1 \otimes e_1, ..., 1 \otimes e_n\}$ and $\{1 \otimes \varepsilon_1, ..., 1 \otimes \varepsilon_n\}$ of F and F^*, respectively (we write our matrices row-wise). By means of the identification $e_i \wedge e_j \mapsto X_{ij}$, $(e_i \wedge e_j)$ coincides with the generic skew-symmetric matrix of the Introduction, and $A = S(\Lambda^2 F_0) \cong R[X] = S$. From now on, we use S to mean both $R[X]$ and $S(\Lambda^2 F_0)$, dropping the symbol A. Also, we denote $1 \otimes e_i$ and $1 \otimes \varepsilon_i$ simply by e_i and ε_i, resp.

f is called the generic alternating map. Using the isomorphism $\mathrm{Hom}_S(F, F^*) \cong F^* \otimes F^*$, f corresponds to an element $\alpha \in F^* \otimes F^*$. In fact, $\alpha \in \Lambda^2 F^*$, where $\Lambda^2 F^*$ is embedded into $F^* \otimes F^*$ by means of the diagonal map. Explicitly, $\alpha = \sum_{i<j} X_{ij} \varepsilon_i \wedge \varepsilon_j$. Since α is homogeneous of degree 2 in ΛF^*, there is a sequence of elements $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, ...$ called the divided powers of α. To hand these elements, we freely use the notation and properties established in [B–E, 3].

We are now in a position to describe the (minimal) free resolution of S/I contained in [B–E, 3]:

$$0 \longrightarrow S \xrightarrow{g^*} F \xrightarrow{f} F^* \xrightarrow{g} S. \qquad (\mathbb{E})$$

The morphism f is as above. As for g (whose dual g^* also occurs), choose an orientation $e \in \Lambda^n F$, i.e., an identification between $\Lambda^{2k} F$ and F^*, and let g be identified with

$$\omega_{2k}: \Lambda^{2k} F \to S$$
$$a \mapsto \beta(\alpha^{(k)} \otimes a),$$

where β stands for the natural pairing $\Lambda^{2k}F^* \otimes \Lambda^{2k}F \to S$ given by the ΛF^*-module structure of ΛF.

Concretely, g sends each element ε_i of the basis $\{\varepsilon_1, ..., \varepsilon_n\}$ of F^* to $(-1)^{i+1}\,\mathrm{Pf}_i(X)$, where $\mathrm{Pf}_i(X)$ is the pfaffian of the submatrix of X formed by deleting the ith row and the ith column.

Note that g^*, thought of as a map $S \to \Lambda^{2k}F^*$, is defined by means of $g^*(1) = \alpha^{(k)}$.

1.2. Throughout, we freely use the notion of Schur functor, as developed in [A–B–W, 2]. But we wish to point out a few things to the reader.

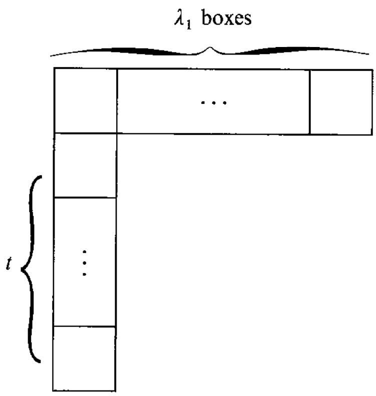
Given a partition $\lambda = (\lambda_1, \lambda_2, ...)$, $\lambda_1 \geqslant \lambda_2 \geqslant ...$, the Schur functor $L_\lambda F$ (F as before) is a free S-module and a $GL(F)$-representation. Furthermore, if $S \to S'$ is a morphism of rings, $L_\lambda F \otimes_S S' \cong L_\lambda(F \otimes_S S')$.

The ΛF^*-module structure of ΛF allows the construction of a natural isomorphism

$$L_\lambda F \otimes \underbrace{\Lambda^n F^* \otimes \cdots \otimes \Lambda^n F^*}_{q} \to L_{\lambda^*}F^*,$$

where q is the length of λ (i.e., the number of nonzero parts of λ) and $\lambda^* = (n - \lambda_q, ..., n - \lambda_1, 0, ...)$.

We say that λ is a hook if $\lambda_h \leqslant 1$ for all $h \geqslant 2$; i.e., $\lambda = (\lambda_1, \underbrace{1, ... 1}_{t}, 0, ...)$, also written $(\lambda_1, 1^t)$ and often identified with the diagram



Then $L_\lambda F$ coincides with

$$\mathrm{Coker}(\Lambda^{\lambda_1+1}F \otimes S_{t-1}F \xrightarrow{\Delta \otimes 1} \Lambda^{\lambda_1}F \otimes F \otimes S_{t-1}F \xrightarrow{1 \otimes m} \Lambda^{\lambda_1}F \otimes S_t F),$$

where Δ (resp., m) is diagonalization (resp., multiplication) in the algebra ΛF (resp., SF).

Consider the basis of $\Lambda^{\lambda_1}F \otimes S_t F$ induced by $\{e_1, ..., e_n\}$. Given an element $e_1 \wedge \cdots \wedge e_{\lambda_1} \otimes e_{j_1} \cdots e_{j_t}$ of such a basis, denote by the tableau

$$\begin{array}{|c|c|c|}\hline i_1 & \cdots & i_{\lambda_1} \\ \hline j_1 \\ \cline{1-1} \vdots \\ \cline{1-1} j_t \\ \cline{1-1} \end{array}$$

its image in $L_{(\lambda_1, 1^t)}F$.

The standard basis theorem says that:

(i) a basis of $L_{(\lambda_1, 1^t)}F$ is formed by all the tableaux such that the indices in the first row are strictly increasing and those in the first column are weakly increasing ("standard tableaux");

(ii) a tableau which is not standard is equal to a $\mathbb{Z}$-linear combination of standard tableaux ("straightening law").

Explicitly, the key step of the straightening law is

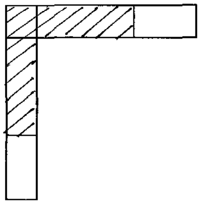
$$\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_{\lambda_1} \\ \hline j_1 \\ \cline{1-1} j_2 \\ \cline{1-1} \vdots \\ \cline{1-1} j_t \\ \cline{1-1} \end{array} \quad (j_1 < i_1 < \cdots < i_{\lambda_1}) \quad = \sum_{h=1}^{\lambda_1} (-1)^{h-1} \begin{array}{|c|c|c|c|c|c|}\hline j_1 & i_1 & \cdots & \hat{i}_h & \cdots & i_{\lambda_1} \\ \hline i_h \\ \cline{1-1} j_2 \\ \cline{1-1} \vdots \\ \cline{1-1} j_t \\ \cline{1-1} \end{array}$$

where $\hat{i}_h$ means i_h omitted.

Finally, assume that $F = F_1 \oplus F_2$ with $F_1 = \langle e_1, ..., e_h \rangle$ and $F_2 = \langle e_{h+1}, ..., e_n \rangle$ for some fixed h. One has an isomorphism of free S-modules

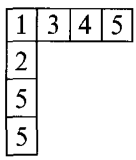
$$L(\lambda_1, 1^t)F \cong \left(\coprod_{(\mu_1, 1^u)} L_{(\mu_1, 1^u)} F_1 \otimes S_{t-u} F_2 \otimes \Lambda^{\lambda_1 - \mu_1} F_2 \right) \oplus L_{(\lambda_1, 1^t)} F_2,$$

where $(\mu_1, 1^u)$ ranges on all the hooks which are nested in $(\lambda_1, 1^t)$:

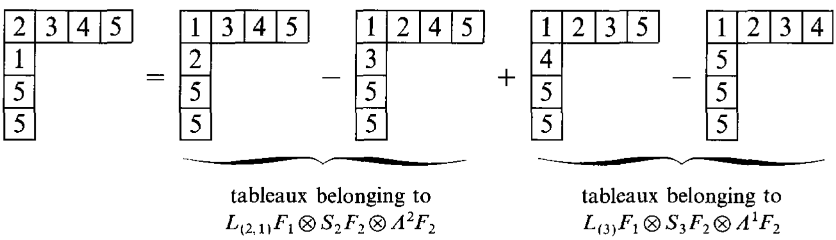


But as $GL(F_1) \times GL(F_2)$-modules, such an isomorphism usually holds only up to a filtration.

EXAMPLE. Let $n=5$, $h=3$, $\lambda_1=4$ and $t=3$. Acting on



(which belongs to $L_{(2,1)}F_1 \otimes S_2F_2 \otimes \Lambda^2F_2$) by the element of $GL(F)$ which exchanges e_1 and e_2 and fixes all the other basis elements, one gets



1.3. We end this section with some notations and properties about pfaffians.

Let W stand for any skew-symmetric matrix (w_{ij}) of order m, m either even or odd. (That is, $w_{ij}+w_{ji}=0$ whenever $i \neq j$, and $w_{ii}=0$ for every i.) We denote by $\mathrm{Pf}_{h_1,\dots,h_r}(W)$ the pfaffian of the skew-symmetric submatrix of W formed by deleting rows and columns indexed by $h_1, \dots, h_r$.

Remark. It is well known that for every fixed i_0,

$$\mathrm{Pf}(W) = \sum_{i=1}^{m} (-1)^{i+i_0+1} \gamma_{ii_0} w_{ii_0} \mathrm{Pf}_{ii_0}(W),$$

where

$$\gamma_{ii_0} = \begin{cases} 1 & \text{if } i < i_0 \\ 0 & \text{if } i = i_0. \\ -1 & \text{if } i > i_0 \end{cases}$$

In particular, setting $W = X$, $I^m \subseteq I = \mathrm{Pf}_{2k}(X) \subseteq \mathrm{Pf}_{2k-2}(X) \subseteq \cdots \subseteq \mathrm{Pf}_2(X) = (X_{ij})_{1 \leqslant i < j \leqslant n}$, where $\mathrm{Pf}_{2p}(X)$ denotes the ideal of S generated by the pfaffians of all $2p$-order principal submatrices of X, $1 \leqslant p \leqslant k$.

Another familiar fact is that for every fixed i_0 and j_0 such that $i_0 \neq j_0$,

$$\sum_{i=1}^{m} (-1)^{i+i_0+1} \gamma_{ii_0} w_{ij_0} \mathrm{Pf}_{ii_0}(W) = 0.$$

If this relation is applied to an augmented matrix obtained from W by duplicating a row of W and the corresponding column (and putting 0 at the intersection), one gets the following.

Formula. For every fixed i_0,

$$\sum_{i=1}^{m} w_{ii_0} T_i(W) = 0,$$

where $T_i(W)$ stands for $(-1)^{i+1} \mathrm{Pf}_i(W)$.

One should remark that with the notation above, the map g of 1.1 sends ε_i precisely to $T_i(X)$.

2. The Complexes $\mathbb{C}_m$

2.1. In this section we construct the complexes $\mathbb{C}_m$ $(m \geqslant 1)$ later to be shown to resolve the ideals I^m. We start by illustrating the heuristic considerations which guided us (and are in the spirit of [B]). For simplicity, we restrict to the case $m = 2$.

Having at hand the resolution $\mathbb{E}$ of I (cf. Subsection 1.1), it is natural to assume that the first map, ϑ_1, of a resolution of I^2 must coincide with the second symmetric power of the map $g: F^* \to S$ having $\mathrm{Im}(g) = I$.

For every r, ϑ_1 is defined by the composite

$$S_r \otimes S_2(\Lambda^{2k} F_0) \xrightarrow[1 \otimes S_2((\omega_{2k})_0)]{} S_r \otimes S_{2k}(\Lambda^2 F_0) \xrightarrow[m]{} S_{2k+r}(\Lambda^2 F_0),$$

where $(\omega_{2k})_0$ is the R-map $\Lambda^{2k} F_0 \to S_k(\Lambda^2 F_0)$ inducing ω_{2k} over S.

Since $S_2(\Lambda^{2k}F_0) \cong S_2F_0^* = L_{(2k+1,1^2)}F_0^* \cong L_{(2k,2k)}F_0$, the above composite is a map $S_r(\Lambda^2F_0)\otimes L_{(2k,2k)}F_0 \to S_{2k+r}(\Lambda^2F_0)$. We then study such a map, when r varies, in order to get a clue on $\mathrm{Ker}(\vartheta_1)$. And we do this by resorting to the fact that the given modules are universally free representations of the general linear group. (The word "universally" refers to the fact that $L_\lambda(F_0\otimes_R R') = (L_\lambda F_0)\otimes_R R'$ whenever a map $R\to R'$ is given.)

By universality, we may hope that all the necessary information on our complex is contained in the characteristic zero case, i.e., assuming R to be a field of characteristic zero. If we make such an extra assumption on the ground ring, the group $GL(F_0)$ is linearly reductive and the Schur functors provide a complete family of irreducibles. Then the irreducibles of $S_r(\Lambda^2F_0)\otimes L_{(2k,2k)}F_0$ either are mapped onto the corresponding irreducibles of $S_{2k+r}(\Lambda^2F_0)$, or must occur in the kernel.

For $r=0$, we have $S_0\otimes L_{(2k,2k)}F_0\to S_{2k}(\Lambda^2F_0)$, which is just the inclusion of $L_{(2k,2k)}F_0$ in $S_{2k}(\Lambda^2F_0)\cong \coprod_{|\lambda|=2k} L_{2\lambda}F_0$ (cf., e.g., [A-DF, Sect. 2]).

For $r=1$, we have

$$\Lambda^2F_0\otimes L_{(2k,2k)}F_0\to S_{2k+1}(\Lambda^2F_0).$$

By Pieri formula (cf., e.g., [A-B-W, 2, Corollary IV.2.6]), the domain is isomorphic to $L_{(2k+1,2k,1)}F_0\oplus L_{(2k,2k,2)}F_0$ (since $rkF_0=2k+1$).

As $S_{2k+1}(\Lambda^2F_0)\cong \coprod_{|\lambda|=2k+1} L_{2\lambda}F_0$, $L_{(2k+1,2k,1)}F_0$ must occur in the kernel.

For $r=2$, we thus have

$$\Lambda^2F_0\otimes L_{(2k+1,2k,1)}F_0\to S_2(\Lambda^2F_0)\otimes L_{(2k,2k)}F_0\to S_{2k+2}(\Lambda^2F_0);$$

since

$$\begin{aligned}\Lambda^2F_0&\otimes L_{(2k+1,2k,1)}F_0\\ &\cong L_{(2k+1,2k+1,2)}F_0\oplus L_{(2k+1,2k+1,1,1)}F_0\\ &\quad\oplus L_{(2k+1,2k,3)}F_0\oplus L_{(2k+1,2k,2,1)}F_0\end{aligned}$$

and

$$\begin{aligned}S_2(\Lambda^2F_0)&\otimes L_{(2k,2k)}F_0\\ &\cong (L_{(4)}F_0\oplus L_{(2,2)}F_0)\otimes L_{(2k,2k)}F_0\\ &\cong L_{(2k+1,2k,3)}F_0\oplus L_{(2k,2k,4)}F_0\\ &\quad\oplus L_{(2k+1,2k+1,1,1)}F_0\oplus L_{(2k+1,2k,2,1)}F_0\oplus L_{(2k,2k,2,2)}F_0\end{aligned}$$

(for $L_{(2,2)}F_0 \otimes L_{(2k,2k)}F_0$ use the Littlewood–Richardson rule [A-B-W, 2, Theorem IV.2.1]), and since again

$$S_{2k+2}(\Lambda^2 F_0) \cong \coprod_{|\lambda| = 2k+2} L_{2\lambda} F_0,$$

it follows that $L_{(2k+1,2k+1,2)}F_0$ must occur in the complex, in degree 3.

For $r=3$, however, we get

$$0 \to \Lambda^2 F_0 \otimes L_{(2k+1,2k+1,2)}F_0 \to S_2(\Lambda^2 F_0) \otimes L_{(2k+1,2k,1)}F_0$$
$$\to S_3(\Lambda^2 F_0) \otimes L_{(2k,2k)}F_0 \to S_{2k+3}(\Lambda^2 F_0);$$

that is, no new term is necessary in degree 4.

So in characteristic zero, one finds

$$0 \longrightarrow L_{(2k+1,2k+1,2)}F \xrightarrow{\vartheta_3} L_{(2k+1,2k,1)}F \xrightarrow{\vartheta_2} L_{(2k,2k)}F \xrightarrow{\vartheta_1} S.$$

Of course some extra terms could be necessary in characteristic free. Yet we take the above to be a reasonable candidate, and start looking for possible definitions of the morphisms (on which we have no hints).

Note that our candidate can also be expressed in terms of F^*:

$$\begin{array}{ccccccccc} 0 \longrightarrow & L_{(2k-1)}F^* & \xrightarrow{\vartheta_3} & L_{(2k,1)}F^* & \xrightarrow{\vartheta_2} & L_{(2k+1,1,1)}F^* & \xrightarrow{\vartheta_1} & S. \\ & \| & & & & \| & & \\ & \Lambda^{2k-1}F^* & & & & S_2F^* & & \end{array}$$

Since the map $f\colon F \to F^*$ of $\mathbb{E}$ can easily be identified with

$$L_{(2k)}F^* \to L_{(2k+1,1)}F^*, \qquad u \mapsto \sum_\delta p_{(2k+1,1)}(\alpha'_{\delta 1} \wedge u \otimes \alpha_{\delta 1}),$$

where $\sum_\delta \alpha'_{\delta 1} \otimes \alpha_{\delta 1} = \Delta(\alpha)$, Δ the diagonal map $\Lambda^2 F^* \to F^* \otimes F^*$, and $p_{(2k+1,1)}$ is the projection $\Lambda^{2k+1}F^* \otimes S_1 F^* \to L_{(2k+1,1)}F^*$, one conjectures that ϑ_2 and ϑ_3 are induced by

$$\Lambda^{2k}F^* \otimes S_1 F^* \to \Lambda^{2k+1}F^* \otimes S_2 F^*, \qquad u \otimes v \mapsto \sum_\delta (\alpha'_{\delta 1} \wedge u \otimes v\alpha_{\delta 1}),$$

and

$$\Lambda^{2k-1}F^* \to \Lambda^{2k}F^* \otimes S_1 F^*, \qquad u \mapsto \sum_\delta (\alpha'_{\delta 1} \wedge u \otimes \alpha_{\delta 1}),$$

respectively. One checks that ϑ_2 and ϑ_3 are actually well defined, by invoking the following more general result.

LEMMA. *Let $\tilde{\varphi}$ be the map $\Lambda^a F^* \otimes S_b F^* \to \Lambda^{a+1} F^* \otimes S_{b+1} F^*$, $u \otimes v \mapsto \sum_\delta (\alpha'_{\delta 1} \wedge u \otimes v \alpha_{\delta 1})$. Then $\tilde{\varphi}$ induces a morphism*

$$\varphi: L_{(a, 1^b)} F^* \to L_{(a+1, 1^{b+1})} F^*.$$

Proof. Since

$$L_{(a,1^b)} F^* = \operatorname{Coker}(\Lambda^{a+1} F^* \otimes S_{b-1} F^* \xrightarrow{\Delta \otimes 1} \Delta^a F^* \otimes F^* \otimes S_{b-1} F^* \xrightarrow{1 \otimes m} \Lambda^a F^* \otimes S_b F^*),$$

it suffices to show that the composition

$$\Lambda^{a+1} F^* \otimes S_{b-1} F^* \xrightarrow{\Delta \otimes 1} \Lambda^a F^* \otimes F^* \otimes S_{b-1} F^* \xrightarrow{1 \otimes m} \Lambda^a F^* \otimes S_b F^* \xrightarrow{\tilde{\varphi}} \Lambda^{a+1} F^* \otimes S_{b+1} F^* \xrightarrow{p_{(a+1, 1^{b+1})}} L_{(a+1, 1^{b+1})} F^*$$

is zero.

Given a basis element $\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{a+1}} \otimes \varepsilon_{j_1} \cdots \varepsilon_{j_{b-1}} \in \Lambda^{a+1} F^* \otimes S_{b-1} F^*$, one gets

$$\tilde{\varphi}\left(\sum_{h=1}^{a+1} (-1)^{h-1} \varepsilon_{i_1} \wedge \cdots \wedge \hat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_{a+1}} \otimes \varepsilon_{i_h} \cdot \varepsilon_{j_1} \cdots \varepsilon_{j_{b-1}}\right)$$
$$= \sum_{h=1}^{a+1} (-1)^{h-1} \left[\sum_{i<j} X_{ij} (\varepsilon_i \wedge \varepsilon_{i_1} \wedge \cdots \wedge \hat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_{a+1}} \otimes \varepsilon_j \varepsilon_{i_h} \varepsilon_{j_1} \cdots \varepsilon_{j_{b-1}} - \varepsilon_j \wedge \varepsilon_{i_1} \wedge \cdots \wedge \hat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_{a+1}} \otimes \varepsilon_i \varepsilon_{i_h} \varepsilon_{j_1} \cdots \varepsilon_{j_{b-1}})\right].$$

But this belongs to the image of the composite map $\Lambda^{a+2} F^* \otimes S_b F^* \to \Lambda^{a+1} F^* \otimes F^* \otimes S_b F^* \to \Lambda^{a+1} F^* \otimes S_{b+1} F^*$ (whose cokernel is $L_{(a+1, 1^{b+1})} F^*$): just take in $\Lambda^{a+2} F^* \otimes S_b F^*$ the element

$$-\sum_{i<j} X_{ij} (\varepsilon_i \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{a+1}} \otimes \varepsilon_j \varepsilon_{j_1} \cdots \varepsilon_{j_{b-1}} - \varepsilon_j \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{a+1}} \otimes \varepsilon_i \varepsilon_{j_1} \cdots \varepsilon_{j_{b-1}}).$$

This concludes the heuristic considerations. We now begin the formal construction of the complexes $\mathbb{C}_m$.

2.2. DEFINITION. (i) If $m \geq 2k$, $\mathbb{C}_m$ is the sequence

$$0 \longrightarrow L_{(1^{m-n+2})} F^* \xrightarrow{\varphi} L_{(2, 1^{m-n+2})} F^* \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} L_{(n-1, 1^{m-1})} F^* \xrightarrow{\varphi} L_{(n, 1^m)} F^* \xrightarrow{\psi} S.$$

(ii) If $m=2r$, $1 \leqslant r \leqslant k-1$, $\mathbb{C}_m$ is the sequence

$$0 \longrightarrow L_{(n-m)}F^* \xrightarrow{\varphi} L_{(n-m+1,1)}F^*$$

$$\xrightarrow{\varphi} \cdots \xrightarrow{\varphi} L_{(n-1,1^{m-1})}F^* \xrightarrow{\varphi} L_{(n,1^m)}F^* \xrightarrow{\psi} S.$$

(iii) If $m=2r+1$, $0 \leqslant r \leqslant k-1$, $\mathbb{C}_m$ is the sequence

$$0 \longrightarrow S \xrightarrow{\chi} L_{(n-m)}F^* \xrightarrow{\varphi} L_{(n-m+1,1)}F^*$$

$$\xrightarrow{\varphi} \cdots \xrightarrow{\varphi} L_{(n-1,1^{m-1})}F^* \xrightarrow{\varphi} L_{(n,1^m)}F^* \xrightarrow{\psi} S.$$

In all cases, φ stands for the map induced by the approprate $\tilde{\varphi}: \Lambda^a F^* \otimes S_b F^* \to \Lambda^{a+1} F^* \otimes S_{b+1} F^*$, $u \otimes v \mapsto \sum_\delta \alpha'_{\delta 1} \wedge u \otimes v\alpha_{\delta 1}$, and ψ is the mth symmetric power of $g: F^* \to S$ defined in Subsection 1.1 (explicitly: $\psi(\varepsilon_{i_1} \cdots \varepsilon_{i_m}) = T_{i_1}(X) \cdots T_{i_m}(X)$).

As for χ in case (iii), it is defined by $1 \mapsto \alpha^{(k-r)}$.

Remarks. (a) The definition rests on Lemma 2.1.

(b) $\mathbb{C}_m$ specializes to $\mathbb{E}$ and $\mathbb{A}$ (the resolutions of [B-E, 3] and [B-S], resp.) when $m=1$ and 2, resp. Moreover, if $n=3$, each $\mathbb{C}_m$ coincides with the well known resolution of the mth power of an ideal generated by a regular sequence (having three elements); cf. e.g., [B-E, 2, Sect. 5].

(c) The sequences $\mathbb{C}_m$ could also be expressed in terms of F, but the Schur functors involved would no longer be associated to hooks. (By the way, also in [B-E, 2], hooks were the only needed partitions. In fact, the functors $L_{(\lambda_1, 1^t)}F^*$ were introduced there for the first time.)

2.3. Proposition. *$\mathbb{C}_m$ is a complex, for every $m \geqslant 1$.*

Proof. First of all, let us show that any composition $\tilde{\varphi} \circ \tilde{\varphi}$: $\Lambda^a F^* \otimes S_b F^* \to \Lambda^{a+1} F^* \otimes S_{b+1} F^* \to \Lambda^{a+2} F^* \otimes S_{b+2} F^*$ is zero. One has

$$(\tilde{\varphi} \circ \tilde{\varphi})(u \otimes v) = \tilde{\varphi}\left[\sum_{i<j} X_{ij}\,(\varepsilon_i \wedge u \otimes \varepsilon_j v - \varepsilon_j \wedge u \otimes \varepsilon_i v)\right]$$

$$= \sum_{i<j} X_{ij} \sum_{p<q} X_{pq}(\varepsilon_p \wedge \varepsilon_i \wedge u \otimes \varepsilon_q \varepsilon_j v - \varepsilon_p \wedge \varepsilon_j \wedge u \otimes \varepsilon_q \varepsilon_i v$$
$$- \varepsilon_q \wedge \varepsilon_i \wedge u \otimes \varepsilon_p \varepsilon_j v + \varepsilon_q \wedge \varepsilon_j \wedge u \otimes \varepsilon_p \varepsilon_i v).$$

But if one performs the interchanges $j \leftrightarrow q$ and $i \leftrightarrow p$, the coefficient $X_{ij} X_{pq}$ is left fixed, while each of $\varepsilon_p \wedge \varepsilon_i \wedge u \otimes \varepsilon_q \varepsilon_j v$, $\varepsilon_q \wedge \varepsilon_j \wedge u \otimes \varepsilon_p \varepsilon_i v$, and $-(\varepsilon_p \wedge \varepsilon_j \wedge u \otimes \varepsilon_q \varepsilon_i v + \varepsilon_q \wedge \varepsilon_i \wedge u \otimes \varepsilon_p \varepsilon_j v)$ is changed to its opposite. This amounts to saying that all summands cancel out, and we are done.

Next, we prove that any composition

$$L_{(2k,1^{m-1})}F^* \xrightarrow{\varphi} L_{(2k+1,1^m)}F^* \xrightarrow{\psi} S$$

is zero. Let ε^i stand for $\varepsilon_1 \wedge \cdots \wedge \hat{\varepsilon}_i \wedge \cdots \wedge \varepsilon_n$. Then

$$\begin{aligned}
&(\psi \circ \varphi)(p_{(2k,1^{m-1})}(\varepsilon^i \otimes \varepsilon_{j_1} \cdots \varepsilon_{j_{m-1}}))\\
&\quad = \sum_{p \neq q} X_{pq} \psi(p_{(2k+1,1^m)}(\varepsilon_p \wedge \varepsilon^i \otimes \varepsilon_{j_1} \cdots \varepsilon_{j_{m-1}} \varepsilon_q))\\
&\quad = \sum_q X_{iq}(-1)^{i-1} T_{j_1}(X) \cdots T_{j_{m-1}}(X) \cdot T_q(X)\\
&\quad = (-1)^{i-1} T_{j_1}(X) \cdots T_{j_{m-1}}(X) \left(\sum_q X_{iq} T_q(X) \right)\\
&\quad = 0.
\end{aligned}$$

because of Fomula 1.3.

Finally, we check that if $m = 2r+1$, $0 \leqslant r \leqslant k-1$, the composition $S \xrightarrow{\chi} L_{(n-m)}F^* \xrightarrow{\varphi} L_{(n-m+1,1)}F^*$ is zero. I.e., $\varphi(\chi(1)) = 0$. Since $\alpha^{(k-r)} = \sum_{1 \leqslant i_1 < \cdots < i_{n-m} \leqslant n} \mathrm{Pf}_{j_1,\ldots,j_m}(X)\, \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{n-m}}$, where $\{j_1, \ldots, j_m\}$ is the complement of $\{i_1, \ldots, i_{n-m}\}$ in $\{1, 2, \ldots, n\}$ (cf., e.g., [B-E, 3, p. 460]),

$$\begin{aligned}
\varphi(\chi(1)) &= \sum_{1 \leqslant i_1 < \cdots < i_{n-m} \leqslant n} \mathrm{Pf}_{j_1,\ldots,j_m}(X)\, \varphi(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{n-m}})\\
&= \sum \mathrm{Pf}_{j_1,\ldots,j_m}(X) \left(\sum_{p \neq q} X_{pq}\, p_{(n-m+1,1)}(\varepsilon_p \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{n-m}} \otimes \varepsilon_q) \right).
\end{aligned}$$

In the latter combination, we find two kinds of terms:

(a) terms in which $q \in \{i_1, \ldots, i_{n-m}\}$,

(b) terms in which $q \notin \{i_1, \ldots, i_{n-m}\}$.

Let us fix a term of type (a), say

$X_{pq}\, \mathrm{Pf}_{j_1,\ldots,j_m}(X)$

p	i_1	$\cdots$	i_{n-m}
q			

If $p \in \{i_1, \ldots, i_{n-m}\}$, the tableau is 0. Hence we assume that $p \in \{j_1, \ldots, j_m\}$, say $p = j_t$. Let us rearrange in increasing order the first row of the tableau:

if there are $u-1$ terms of type i_h which are smaller than $p=j_t$, the above term is equal to

$$(-1)^{u-1} X_{pq} \operatorname{Pf}_{j_1,\dots,j_m}(X) \begin{array}{|c|c|c|c|c|c|}\hline i_1 & i_2 & \cdots & p & \cdots & i_{n-m} \\ \hline q & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array}, \qquad (*)$$

and the indicated tableau, say T, is standard, since $q=i_{h_0} \geqslant i_1$.

But the same standard tableau T can be found in other terms (after rearranging in the first row, we mean), namely the terms

$$(-1)^{h-1} X_{i_h q} \operatorname{Pf}_{i_h, j_1, \dots, \hat{j}_t, \dots, j_m}(X) \cdot T, \qquad 1 \leqslant h \leqslant u-1,$$

and

$$(-1)^{h} X_{i_h q} \operatorname{Pf}_{i_h, j_1, \dots, \hat{j}_t, \dots, j_m}(X) \cdot T, \qquad u \leqslant h \leqslant n-m.$$

The sum of all these terms, including (*), is precisely equal to $\sum_{l=1}^{n-m+1} Y_{l l_0} T_l(Y)$, where Y is the skew-symmetric submatrix of X obtained by erasing all the rows and columns indexed by $j_1, \dots, \hat{j}_t, \dots, j_m$, and l_0 is the index which labels the row (column) of Y corresponding to the row (column) of X indexed by q. (Note that Y has odd order, $n-m+1$.)

Owing to Formula 1.3, $\sum Y_{l l_0} T_l(Y)=0$, and we have verified that all the terms of type (a) cancel out.

Next, let us fix a nonzero term of type (b), again denoted by

$$X_{pq} \operatorname{Pf}_{j_1,\dots,j_m}(X) \begin{array}{|c|c|c|c|}\hline p & i_1 & \cdots & i_{n-m} \\ \hline q & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array},$$

but with $p=j_t$ and $q=j_s$. Suppose $q<p$. A reordering of the first row of the tableau yields the term (*). And it is not hard to see (by means of Remark 1.3) that the sum of all the terms containing (after rearrangements in their first rows) the standard tableau T of (*) is exactly equal to $(-1)^v \operatorname{Pf}(Z) \cdot T$, where v is the place of q in the increasing string of indices

$$i_1, \dots, i_{v-1}, q, i_v, \dots, i_{u-1}, p, i_u, \dots, i_{n-m},$$

and Z is the skew-symmetric submatrix of X obtained by erasing all the rows and columns indexed by $j_1, \dots, \hat{j}_s, \dots, \hat{j}_t, \dots, j_m$. (Note that Z has even order, $n-m+2$).

But the definition of Schur functor says that

$$\Delta(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{v-1}} \wedge \varepsilon_q \wedge \varepsilon_{i_v} \wedge \cdots \wedge \varepsilon_{i_{u-1}} \wedge \varepsilon_p \wedge \varepsilon_{i_u} \wedge \cdots \wedge \varepsilon_{i_{n-m}})$$

belongs to the kernel of $p_{(n-m+1,1)}$; i.e., that it is identically zero the following linear combination:

$$\begin{array}{|c|c|c|c|c|c|}\hline i_2 & \cdots & q & \cdots & p & \cdots \\ \hline i_1 & \multicolumn{5}{c}{} \\ \cline{1-1}\end{array}$$

$$+\sum_{h=2}^{v-1} (-1)^{h-1} \begin{array}{|c|c|c|c|c|c|c|c|}\hline i_1 & \cdots & \hat{i}_h & \cdots & q & \cdots & p & \cdots \\ \hline i_h & \multicolumn{7}{c}{} \\ \cline{1-1}\end{array}$$

$$+(-1)^{v-1} T + \sum_{h=v}^{u-1} (-1)^{h-1} \begin{array}{|c|c|c|c|c|c|c|c|}\hline i_1 & \cdots & q & \cdots & \hat{i}_h & \cdots & p & \cdots \\ \hline i_h & \multicolumn{7}{c}{} \\ \cline{1-1}\end{array}$$

$$+(-1)^u \begin{array}{|c|c|c|c|c|c|}\hline i_1 & \cdots & q & \cdots & \hat{p} & \cdots \\ \hline p & \multicolumn{5}{c}{} \\ \cline{1-1}\end{array}$$

$$+\sum_{h=u}^{n-m} (-1)^{h+1} \begin{array}{|c|c|c|c|c|c|c|c|}\hline i_1 & \cdots & q & \cdots & p & \cdots & \hat{i}_h & \cdots \\ \hline i_h & \multicolumn{7}{c}{} \\ \cline{1-1}\end{array} \qquad \text{(LC)}$$

It is straightforward to check that each tableau of (LC) occurs in several terms of type (b) (after rearrangements in their first rows), and the total coefficient belonging to it is precisely $\pm \mathrm{Pf}(Z)$. Here Z is the same as before, and the sign $\pm$ coincides with the opposite of that attributed to the tableau inside (LC).

Therefore, the terms of kind (b) can be grouped to get a sum of expressions of type $-(\mathrm{LC}) \cdot \mathrm{Pf}(Z)$, each of which is $\equiv 0$.

This completes the proof of $\varphi \circ \chi = 0$ (and of the proposition).

It should be observed that (LC) $\equiv 0$ is an instance of the straightening law mentioned in Subsection 1.2. It just says that the nonstandard tableau

i_2	$\cdots$	q	$\cdots$	p	$\cdots$
i_1					

$(i_1 < i_2)$ is equal to a linear combination of the remaining ones, which are all standard.

2.4. *Remark.* In Definition 2.2, the description of $\mathbb{C}_m$ is threefold for the sake of clarity. But one could give a more concise statement (cf. Subsection 2.6 below). Also, one could adjoin the case $m = 2k$ to part (ii), thereby stressing that something different happens when m is smaller than the rank of F^*.

2.5. We are now ready to state the central result of this paper.

THEOREM. *For every $m \geqslant 1$, the complex $\mathbb{C}_m$ provides a resolution of S/I^m.*

The proof is deferred to the next two sections. Here are some comments.

2.6. The resolution $\mathbb{C}_m$ is minimal; that is, for every $i \geqslant 1$, the image of the ith morphism of $\mathbb{C}_m$ is included in $J(\mathbb{C}_m)_{i-1}$, where J denotes the ideal of S generated by the indeterminates occurring in X.

$\mathbb{C}_m$ is also generic (or "universal"), in the sense that it is defined over $\mathbb{Z}[X]$ and then carried over to every other $R[X]$, R noetherian. In particular, the Betti numbers $\beta_i(I^m)$ (i.e., the ranks of the modules $(\mathbb{C}_m)_i$) do not depend on char(R). Explicitly:

$$B_i(I^m) = \operatorname{rk} L_{(n-i+1,1^{m-i+1})}F^* = \binom{n+m-i+1}{n+m-2i+2}\binom{n+m-2i+1}{m-i+1} \quad \text{if} \quad 1 \leqslant i \leqslant \min\{n, m+1\}$$

$$\beta_{1+\min\{n,m+1\}}(I^m) = \begin{cases} 1 & \text{if } m \text{ is odd and } < n \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_i(I^m) = 0 \qquad \text{if} \quad i \geqslant 2 + \min\{n, m+1\}.$$

(We are using $\operatorname{rk} L_{(\lambda_1, 1^t)}F^* = \binom{n+t}{\lambda_1+t}\binom{\lambda_1+t-1}{t}$, which follows from the definition of $L_{(\lambda_1, 1^t)}F^*$: cf. [B-E, 2, Proposition 2.5]).

The existence of a generic minimal free resolution for S/I^m is all the more remarkable since it is known that $S/\mathrm{Pf}_{2p}(X)$, $2 \leqslant p \leqslant k-1$, may not admit such a finite free resolution: cf. [K], where the Betti numbers of $S/\mathrm{Pf}_{2p}(X)$, in one case, are shown to depend on char(R). Thus the ideals $\mathrm{Pf}_{2p}(X)$ seem to behave pretty much in the same way as the determinantal ideals of a generic $n_1 \times n_2$ matrix (cf. [H]).

2.7. It follows from Theorem 2.5 that the projective dimension of S/I^m satisfies

$$\text{proj. dim}(S/I^m) = \begin{cases} n & \text{if } m \geqslant 2k-1 \\ m+1 & \text{if } m = 2r,\ 1 \leqslant r \leqslant k-1 \\ m+2 & \text{if } m = 2r+1,\ 0 \leqslant r \leqslant k-2. \end{cases}$$

Therefore, since grade$(I^m) =$ grade$(I) = 3$, I^m is perfect—hence generically perfect—when $m = 1, 2$ (no matter what n one has) and whenever $n = 3$ (in this case, I is generated by a regular sequence). In all the other cases, I^m is not perfect. However, there is a regular pattern for proj. dim(S/I^m), namely

$$\text{proj. dim}(S/I^{2r+1}) = \text{proj. dim}(S/I^{2r+2}) = 2r+3, \qquad \text{if } 0 \leqslant r \leqslant k-2;$$
$$\text{proj. dim}(S/I^m) = 2k+1, \qquad \text{if } m \geqslant 2k-1.$$

This regularity can be viewed as a special case of a more general phenomen related to almost alternating maps: cf. [K-U, Sect. 5].

2.8. As observed in the Introduction, $I^2 = I_{n-1}$ (cf. [C; He, Relation (2.24)]). Hence $I^s_{n-1} = I^{2s}$ and $\mathbb{C}_m$, with m ranging over the even positive numbers, gives generic minimal finite free resolutions for all the powers of the ideal I_{n-1}. In particular,

$$\text{proj. dim}(S/I^s_{n-1}) = \begin{cases} 2s+1 & \text{if } 1 \leqslant s \leqslant k-1 \\ 2k+1 & \text{if } s \geqslant k. \end{cases}$$

3. Proof of Theorem 2.5: First Part

3.1. We prove Theorem 2.5 by induction on $k = (n-1)/2$.

When $k = 1$, that is, $n = 3$, we have already noticed that $\mathbb{C}_m$ resolves I^m (cf. Remark 2.2(b)). So we prove the statement for $k \geqslant 2$, assuming it true for $k-1$, $k-2$, etc.

Since we already know that $\mathrm{Im}(\psi) = I^m$, it is enough to show that

$H_i(\mathbb{C}_m) = 0$ for every $i \geq 1$. To this aim, we make use of the following form of the Peskine–Szpiro acyclicity lemma [P–S].

ACYCLICITY LEMMA. *Let S be a noetherian ring. Let*

$$0 \longrightarrow F_t \xrightarrow{f_t} F_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \qquad (\mathbb{F})$$

be a complex of finitely generated free S-modules. Then $(\mathbb{F})$ is exact if, and only if, $\mathbb{F} \otimes_S S_p$ is exact for all primes P such that $\operatorname{grade}(PS_p) < t$.

3.2. Let $P \in \operatorname{Spec}(S)$ be such that

$$\operatorname{grade}(PS_p) < \operatorname{length}(\mathbb{C}_m) = \begin{cases} n & \text{if } m \geq 2k \\ m+1 & \text{if } m = 2r,\ 1 \leq r \leq k-1 \\ m+2 & \text{if } m = 2r+1,\ 0 \leq r \leq k-1. \end{cases}$$

The ideal J, generated in S by the indeterminates occurring in X, has grade $\frac{1}{2}(n-1)n \geq n$. Thus $\operatorname{grade}(JS_p) > \operatorname{grade}(PS_p)$, no matter what m one has, and one of the variables X_{ij} is invertible over S_p, say X_{12}. But then the exactness of $\mathbb{C}_m \otimes S_p$ follows, if we show that $\mathbb{C}_m$ is exact after localization at the powers of X_{12}.

The idea is that if X_{12} can be assumed invertible, one can find new dual bases such that the corresponding matrix associated to the generic alternating map is of the form

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline & 0 & X' \end{array}\right),$$

X' is a generic skew-symmetric matrix of order $n' = n-2 = 2(k-1)+1$, and the inductive hypothesis applies.

3.3. Let us now be explicit (cf. the proof of Theorem 2.3 in [J-P]).

Let R' and S' be the localizations at the powers of X_{12} of $R[X_{12}, X_{13}, ..., X_{1n}, X_{23}, X_{24}, ..., X_{2n}]$ and S, resp. (we keep denoting by F, F^*, e_i, ε_i and $X = (X_{ij})$ the corresponding objects over S'). Let us choose new dual bases $\{\varepsilon'_1, ..., \varepsilon'_n\}$ and $\{e'_1, ..., e'_n\}$ for F^* and F, resp., by setting $(e_1, ..., e_n) = (e'_1, ..., e'_n)C$ and $(\varepsilon_1, ..., \varepsilon_n) = (\varepsilon'_1, ..., \varepsilon'_n)C^t$, with

$$C = \left(\begin{array}{cc|ccc} X_{12}^{-1} & 0 & X_{23}X_{12}^{-1} & \cdots & X_{2n}X_{12}^{-1} \\ 0 & 1 & -X_{13}X_{12}^{-1} & \cdots & -X_{1n}X_{12}^{-1} \\ \hline & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{array}\right)$$

It follows (cf. [J-P, Lemma 1.2]) that:

(i) the new matrix associated to $f \otimes_s S'$ is

$$C^t X C = \left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline & 0 & X' \end{array}\right),$$

(ii) X' is skew-symmetric and $X'_{ij} = X_{ij} + (X_{1j} X_{2i} - X_{1i} X_{2j}) X_{12}^{-1}$, $3 \leqslant i, j \leqslant n$.

(iii) $IS' = I'$, where I' is the ideal $\mathrm{Pf}_{2(k-1)}(X')$ of S'.

Furthermore, $S' = R'[X'_{ij}]$, and the elements X'_{ij} $(3 \leqslant i, j \leqslant n)$ are algebraically independent (cf. [J–P, Lemma 2.4]).

Finally $\psi \otimes_s S'$ sends ε'_1 and ε'_2 to 0, and sends ε'_i $(3 \leqslant i \leqslant n)$ to $X_{12} \cdot T_{i-2}(X')$.

3.4. Another consequence of the change of dual bases described above is that we can write the element of $\Lambda^2 F^*$ associated to $f \otimes_s S'$ as

$$\varepsilon'_1 \wedge \varepsilon'_2 + \sum_{1 \leqslant i < j \leqslant n'} X'_{ij} \varepsilon'_{i+2} \wedge \varepsilon'_{j+2}$$

$(n' = 2k - 1$, as above).

Accordingly, let us decompose F^* as $H^* \oplus G^*$, where $H^* = \langle \varepsilon'_3 \cdots \varepsilon'_n \rangle$ and $G^* = \langle \varepsilon'_1, \varepsilon'_2 \rangle$ (so that $\varepsilon'_1 \wedge \varepsilon'_2$ can be denoted by α_{G^*} and $\sum X'_{ij} \varepsilon'_{i+2} \wedge \varepsilon'_{j+2}$ by α_{H^*}). Following Subsection 1.2, one has

$$(\mathbb{C}_m \otimes_s S')_i = L_{(a, 1^b)}(H^* \oplus G^*)$$

$$\cong \left(\coprod L_{(\mu_1, 1^u)} H^* \otimes S_{b-u} G^* \otimes \Lambda^{a-\mu_1} G^* \right) \oplus L_{(a, 1^b)} G^*,$$

where $a=n+1-i$, $b=m+1-i$, and some summands may be 0, if $\operatorname{rk} H^*=n'$ is not large enough.

It is our intention to use the decompositions of the modules $L_{(a,1^b)}(H^*\oplus G^*)$ to filter the complex $\mathbb{C}_m\otimes_s S'$. The exactness of $\mathbb{C}_m\otimes_s S'$ will thus be reduced to that of the factors of the filtration.

In order to follow this strategy easily, it is convenient to discuss separately the cases $m\geqslant 2k$, $m=2r$ $(1\leqslant r\leqslant k-1)$ and $m=2r+1$ $(0\leqslant r\leqslant k-1)$. We deal with the first case in here, and defer the others to Section 4. So $m\geqslant 2k$ *until further notice* (and $\mathbb{C}_m$ is of the kind described in Definition 2.2(i)).

To further simplify matters, let us distinguish between m even and m odd.

Situation when $m\geqslant 2k$ is Even $(m=2p)$

3.5. The H^*-content of a summand of $L_{(a,1^b)}(H^*\oplus G^*)$ is defined as the number μ_1+u, denoted by $|H^*|$. Hence $L_{(a,1^b)}(H^*\oplus G^*)$ can be decomposed into the direct sum of two modules: the one, $M_0(a,1^b)$, comprising all summands with $|H^*|$ even, and the other, $M_1(a,1^b)$, comprising all summands with $|H^*|$ odd.

If a morphism $\varphi\otimes_s S'$ is applied to $L_{(a,1^b)}(H^*\oplus G^*)$, $M_0(a,1^b)$ is mapped to $M_0(a+1,1^{b+1})$ and $M_1(a,1^b)$ is mapped to $M_1(a+1,1^{b+1})$. If $\psi\otimes_s S'$ is applied to $L_{(a,1^b)}(H^*\oplus G^*)$, it means that $a=n$ and $b=m$, whence necessarily $\mu_1=n'$; recalling the end of Subsection 3.3, $\psi\otimes_s S'$ is zero on all summands, except for $L_{(\mu_1,1^m)}H^*\otimes\Lambda^2G^*\subseteq M_1(n,1^m)$ (here we use m even). Therefore $\mathbb{C}_m\otimes_s S'$ is the direct sum of two subcomplexes M_0 and M_1, respectively given by the terms $M_0(a,1^b)$, and by the terms $M_1(a,1^b)$ together with S'. We show that both of them are exact, by filtering them separately.

3.6. The filtration $\{X_t\}$ of M_1 is described as follows (recall that $m=2p$). For each fixed $\bar{t}\in\{0,1,...,p-1\}$, in every $L_{(a,1^b)}(H^*\oplus G^*)$ we assign to $X_{\bar{t}}$ all the summands having

$$\mu_1=n'-j \qquad \text{and} \qquad u=2t-j$$

for some $t\leqslant\bar{t}$ and for some nonnegative integer j. To X_p we assign S' and all the summands of $L_{(a,1^b)}(H^*\oplus G^*)$ having $\mu_1=n'-j$ and $u=2t-j$ for some $t\leqslant p$ and some nonnegative integer j.

PROPOSITION. $X_p=M_1$.

Proof. Let $L_{(\mu_1,1^u)}H^*\otimes S_{b-u}G^*\otimes\Lambda^{a-\mu_1}G^*$, μ_1+u odd, be a fixed summand of a module $L_{(a,1^b)}(H^*\oplus G^*)$. The linear system $\mu_1=n'-j$,

$u = 2t - j$ (with unknowns j and t) has the unique solution $j = n' - \mu_1$, $t = \frac{1}{2}(u + n' - j)$. We claim that $\frac{1}{2}(u + n' - j) \leqslant p$.

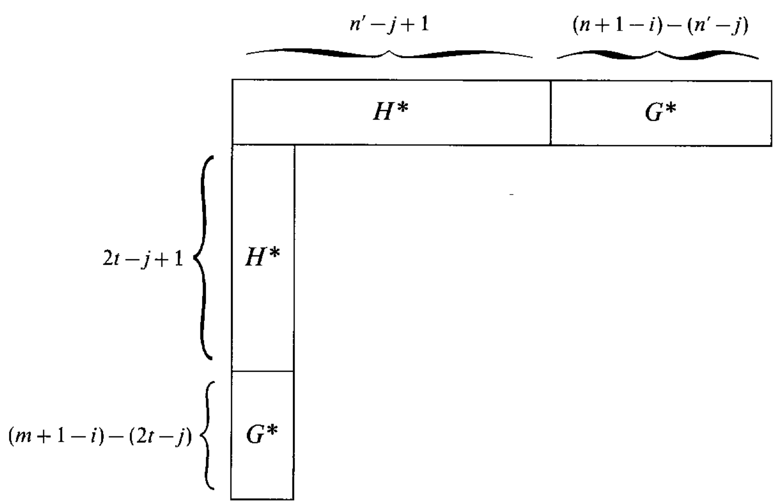
Let us recall that $a = n + 1 - i$ and $b = m + 1 - i$. Therefore $a - \mu_1 = n + 1 - i - (n' - j) = 3 - (i - j)$. Since $rkG^* = 2$, $0 \leqslant a - \mu_1 \leqslant 2$ and $1 \leqslant i - j \leqslant 3$. But $b - u = m + 1 - i - (2t - j) = m + 1 - 2t - (i - j)$ then implies $m\text{-}2t - 2 \leqslant b - u \leqslant m - 2t$. Since $b - u$ cannot be negative, $m - 2t$ cannot be either, i.e., $2t \leqslant m = 2p$.

3.7. PROPOSITION. *Each X_t, $0 \leqslant t \leqslant p$, is indeed a complex.*

Proof. In a fixed $(\mathbb{C}_m \otimes_s S')_i = L_{(a, 1^b)}(H^* \oplus G^*)$, let us take a standard tableau T (*relative to the ordering* $\varepsilon_3' < \cdots < \varepsilon_n' < \varepsilon_1' < \varepsilon_2'$) which belongs to $L_{(\mu_1, 1^u)} H^* \otimes S_{b-u} G^* \otimes \Lambda^{a - \mu_1} G^*$, where $\mu_1 = n' - j$ and $u = 2t - j$.

In order to compute $(\varphi \otimes_s S')(T)$, we apply α_{G^*} and α_{H^*} separately to T. α_{G^*} gives an element of $(\mathbb{C}_m \otimes_s S')_{i-1}$ which belongs to $L_{(\mu_1, 1^u)} H^* \otimes S_{b-u+1} G^* \otimes \Lambda^{a - \mu_1 + 1} G^*$; since $a - \mu_1 + 1 = (n + 1 - i) - (n' - j) + 1$ and $b - u + 1 = (m + 1 - i) - (2t - j) + 1$ can be written as $[n + 1 - (i - 1)] - (n' - j)$ and $[m + 1 - (i - 1)] - (2t - j)$, resp., we have in fact remained inside X_t.

As for α_{H^*}, it gives in $(\mathbb{C}_m \otimes_s S')_{i-1}$ a linear combination of tableaux of type



not necessarily standard. Those which are standard (possibly after trivial reorderings in the arm and the leg of the hook) belong to

$$N = L_{(n' - (j-1), 1^{2t - (j-1)})} H^* \otimes S_{m + 1 - (i-1) - 2t + (j-1)} G^* \\ \otimes \Lambda^{n + 1 - (i-1) - n' + j - 1} G^*$$

which pertains to X_t. Those which are not standard (after all trivial reorderings in the arm and the leg of the hook) have a violation of H^*-standardness in the corner of the hook. Removing such a violation (cf. Example 1.2), one obtains some further standard tableaux belonging to the N above, and some standard tableaux belonging to

$$L_{(n'-(j-2),1^{2(t-1)-(j-2)})}H^* \otimes S_{m+1-(i-1)-2(t-1)+(j-2)}G^* \otimes \Lambda^{n+1-(i-1)-n'+j-2}G^*,$$

which pertains to $X_{t-1} \subseteq X_t$.

This completes the proof.

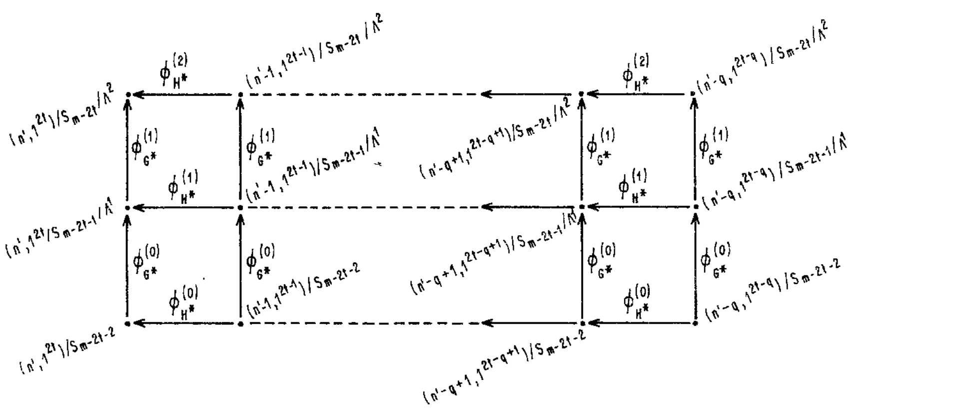
3.8. We now describe the factors X_t/X_{t-1}, $0 \leqslant t \leqslant p-1$ (X_{-1} is assumed to be zero), and prove their exactness.

The modules occurring in X_t/X_{t-1} are given by $\mu_1 = n'-j$ and $u = 2t-j$, with j ranging between 0 and q, where

$$q = \begin{cases} 2t & \text{if } 2t < n' \\ n'-1 & \text{if } 2t > n'. \end{cases}$$

It follows that $a-\mu_1 = 3-(i-j)$ and $b-u = m+1-2t-(i-j)$. Since $\operatorname{rk} G^* = 2$ implies $0 \leqslant a-\mu_1 \leqslant 2$, $i-j$ can only be 1, 2, 3, whence, $b-u = m-2t$, $m-2t-1$, $m-2t-2$, resp. Thus, no matter what j is, one finds $L_{(n'-j,2t-j)}H^* \otimes S_{m-2t-2}G^*$, $L_{(n'-j,2t-j)}H^* \otimes S_{m-2t-1}G^* \otimes \Lambda^1 G^*$, and $L_{(n'-j,2t-j)}H^* \otimes S_{m-2t}G^* \otimes \Lambda^2 G^*$.

If one remembers the way α_{G^*} and α_{H^*} operate (and recalls, from the previous subsection, that factoring X_{t-1} takes care of the straightening sometimes required in the H^*-part), it is easy to check that X_t/X_{t-1} is the total complex of the following bicomplex, D_t:



In the diagram, $(a, 1^b)$, S_h, Λ^h, and / stand for $L_{(a,1^b)}H^*$, S_hG^*, Λ^hG^*, and $\otimes$, resp. Moreover, for each $w = 0, 1, 2$, one defines

$$\varphi_{H^*}^{(w)}: L_{(\mu_1,1^u)}H^* \otimes S_vG^* \otimes \Lambda^wG^* \to L_{(\mu_1+1,1^{u+1})}H^* \otimes S_vG^* \otimes \Lambda^wG^*$$

by

$$p_{(\mu_1,1^u)}(x \otimes y) \otimes z_1 \otimes z_2$$
$$\mapsto \sum_\delta p_{(\mu_1+1,1^{u+1})}((\alpha_{H^*})'_{\delta_1} \wedge x \otimes (\alpha_{H^*})_{\delta_1} \cdot y)\, z_1 \otimes z_2,$$

and for each $w = 0, 1$, one defines

$$\varphi_{G^*}^{(w)}: L_{(\mu_1,1^u)}H^* \otimes S_vG^* \otimes \Lambda^wG^* \to L_{(\mu_1,1^u)}H^* \otimes S_{v+1}G^* \otimes \Lambda^{w+1}G^*$$

by

$$x \otimes y \otimes z \mapsto (-1)^{\mu_1} \sum_\delta x \otimes (\alpha_{G^*})_{\delta_1}\, y \otimes (\alpha_{G^*})'_{\delta_1} \wedge z$$

(as usual, $\sum_\delta (\alpha_{H^*})'_{\delta_1} \otimes (\alpha_{H^*})_{\delta_1} = \Delta(\alpha_{H^*})$, and similarly for α_{G^*}).

We remark that each line of D_t is a complex essentially because $\varphi \circ \varphi = 0$. The anticommutativity of the boxes is straightforward from the definitions, particularly from the sign $(-1)^{\mu_1}$ introduced in $\varphi_{G^*}^{(w)}$, $w = 0, 1$.

We also notice that each column of D_t is isomorphic to a short sequence

$$0 \to S_l G^* \to S_{l+1}G^* \otimes \Lambda^1G^* \to S_{l+2}G^* \otimes \Lambda^2G^* \to 0$$

which is isomorphic ($\Lambda^2G^* \cong S'$) to a graded component of a suitable Koszul complex resolving the ideal generated by two indeterminates; hence it is exact. But the exactness of the columns of D_t implies that $\text{Tot}(D_t) = X_t/X_{t-1}$ is exact, too (cf., e.g., [R, Example 11.17, p. 331]).

3.9. We finally show that X_p/X_{p-1} is exact, so that the whole M_1 is so. The modules occurring in X_p/X_{p-1} are S' and those associated to $\mu_1 = n' - j$ and $u = m - j$, with j ranging between 0 and $n' - 1$ (since $m \geqslant 2k$ gives $2p > n'$, and $q = n' - 1$). Then $a - \mu_1 = 3 - (i - j)$ and $b - u = 1 - (i - j)$. Since $0 \leqslant a - \mu_1 \leqslant 2$ and $b - u \geqslant 0$, it must be that $a - \mu_1 = 2$ and $b - u = 0$. It follows that X_p/X_{p-1} is isomorphic ($\Lambda^2G^* \cong S'$) to the complex

$$0 \longrightarrow L_{(1^{m-n'+2})}H^* \xrightarrow{\varphi_{H^*}^{(2)}} L_{(2,1^{m-n'+2})}H^*$$
$$\xrightarrow{\varphi_{H^*}^{(2)}} \cdots \xrightarrow{\varphi_{H^*}^{(2)}} L_{(n'-1,1^{m-1})}H^* \xrightarrow{\varphi_{H^*}^{(2)}} L_{(n',1^m)}H^* \to S', \qquad (\mathbb{C}'_m)$$

where $\varphi^{(2)}_{H^*}$ is as in the previous subsection, and $L_{(n',1^m)}H^* \to S'$ is the only nonzero component of $\psi \otimes_s S'$, namely $(\Lambda^{n'}H^* \cong S')$, the morphism defined by

$$\underbrace{(\varepsilon_3')^{b_3} \cdot \cdots \cdot (\varepsilon_n')^{b_n}}_{b_3 + \cdots + b_n = m} \mapsto T_3(X')^{b_3} \cdot \cdots \cdot T_n(X')^{b_n}.$$

But $\mathbb{C}'_m$ is a complex of the same type of $\mathbb{C}_m$, relative to the generic matrix X' of order $2(k-1)+1$. Therefore it is exact by inductive hypothesis.

3.10. It remains to prove the exactness of M_0, by means of a suitable filtration $\{Y_t\}$. We omit some details, when the situation is very close to that of M_1. For each $\bar{t} \in \{0, 1, ..., p-1\}$, in every $L_{(a,1^b)}(H^* \oplus G^*)$ we assign to $Y_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t + 1 - j$ for some $t \leqslant \bar{t}$ and some nonnegative integer j.

PROPOSITION. *$Y_{p-1} = M_0$ and each Y_t, $0 \leqslant t \leqslant p-1$, is indeed a complex.*

Proof. Mimic what has been done in Subsections 3.6 and 3.7. (Also cf. Remark 3.11 below.)

3.11. *Remark.* An original feature of M_0 is that it contains terms with $|H^*| = 0$. They belong to $Y_{k-1} \subseteq Y_k \subseteq \cdots$, because if $\mu_1 = 0 = u$, then $n' = j = 2t + 1$ and $t = k - 1$ ($\leqslant p - 1$, since $m = 2p \geqslant 2k$).

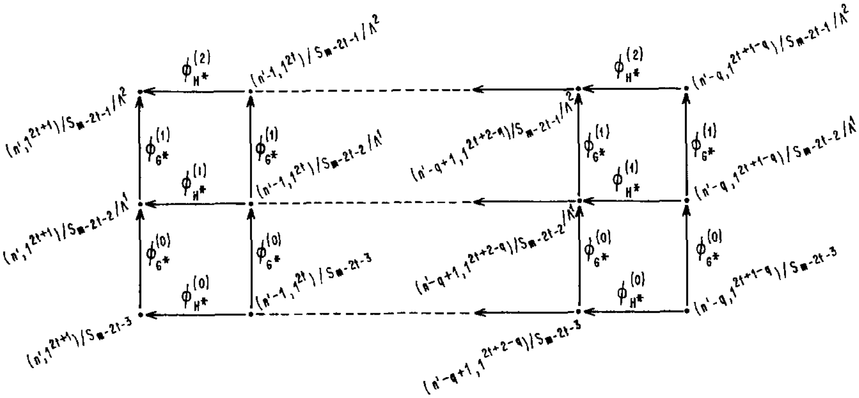
Explicitly, these terms are of type $L_{(n+1-i, 1^{m+1-i})}H^*$, where $n + 1 - i$ can be either 1 or 2 (it cannot be 0, for a hook with nonzero leg cannot have a zero arm). But $n + 1 - i = 1, 2$ implies $i = n, n-1$, resp.; thus $m + 1 - i = m - n + 1$, $m - n + 2$, resp., and one finds $L_{(1^{m-n+2})}G^*(i = n)$ and $L_{(2,1^{m-n+2})}G^*(i = n-1)$.

When one applies $\varphi \otimes_s S'$ to $L_{(1^{m-n+2})}G^*$, α_{G^*} produces an element of $L_{(2,1^{m-n+2})}G^*$ acting as the identity $(L_{(2,1^{m-n+2})}G^* \cong L_{(1^{m-n+2})}G^*)$, while α_{H^*} yields an element of $L_{(2)}H^* \otimes S_{m-n+2}G^*$ by means of $x \mapsto \sum X'_{ij} (\varepsilon'_{i+2} \wedge \varepsilon'_{j+2} \otimes x)$. Note that $L_{(2)}H^* \otimes S_{m-n+2}G^*$ pertains to $Y_{k-2} \subseteq Y_{k-1} \subseteq \cdots$.

When one applies $\varphi_{n-1} \otimes_s S'$ to $L_{(2,1^{m-n+2})}G^*$, α_{G^*} acts as zero, α_{H^*} produces an element of $L_{(2)}H^* \otimes S_{m-n+3}G^* \otimes \Lambda^1 G^*$ by means of $x \mapsto \sum X'_{ij} (\varepsilon'_{i+2} \wedge \varepsilon'_{j+2} \otimes x)$ (recall that $L_{(2,1^{m-n+2})}G^* \subseteq S_{m-n+3}G^* \otimes G^*$). Again, note that $L_{(2)}H^* \otimes S_{m-n+3}G^* \otimes \Lambda^1 G^*$ pertains to $Y_{k-2} \subseteq Y_{k-1} \subseteq \cdots$.

3.12. Let us describe the factors Y_t/Y_{t-1}, $0 \leqslant t \leqslant p-1$ (we mean that $Y_{-1} = 0$), and show their exactness.

Reasoning as in Subsection 3.8, it turns out that for $t \neq k-1$ and $t \neq p-1$, Y_t/Y_{t-1} is isomorphic to the total complex of the bicomplex P_t,



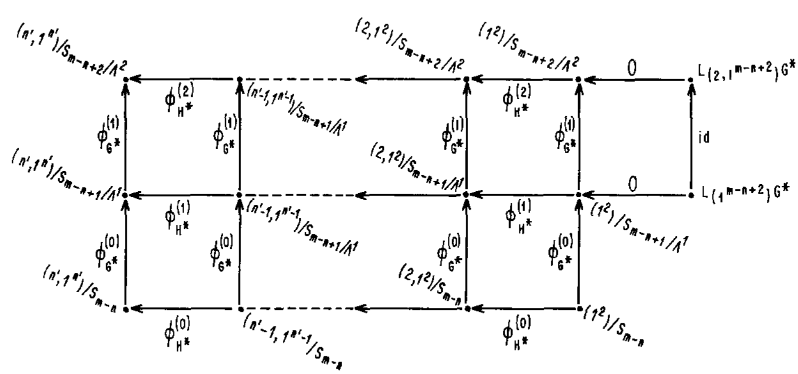
where

$$q = \begin{cases} 2t+1 & \text{if } 2t+1 < n' \\ n'-1 & \text{if } 2t+1 > n' \end{cases}$$

and the other notations are as in Subsection 3.8.

When $t = p-1$ (with $p \neq k$), one obtains a bicomplex as before, but with the bottom row missing (since $p-1 = (m-2)/2$, and $m-2t-3 = m-(m-2)-3 < 0$). And $\varphi_{G^*}^{(1)}$ essentially is $\Lambda^1 G^* \to S_1 G^* \otimes \Lambda^2 G^*$, $x \mapsto x \otimes \varepsilon_1' \wedge \varepsilon_2'$.

When $t = k-1$, one obtains a bicomplex as in one of the two cases before, but with an extra box,



(remark that if $k = p$, then $m - n < 0$ and the bottom row is missing).

Since in all cases the columns of the bicomplex are exact, Y_t/Y_{t-1} is exact for each $t \in \{0, 1, ..., p-1\}$ and M_0 is exact as well.

This completes the proof of Theorem 2.5 in the situation $m \geqslant 2k$, m even.

Situation when $m \geqslant 2k$ is Odd ($m=2p+1$)

3.13. Also in this situation, $\mathbb{C}_m \otimes_s S'$ is the direct sum of two subcomplexes M_0 and M_1, but S' belongs to M_0. Thus we start by filtering M_0.

For each $\bar{t} \in \{0, 1, ..., p-1\}$, in every $L_{(a, 1^b)}(H^* \oplus G^*)$ we assign to $X_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t + 1 - j$ for some $t \leqslant \bar{t}$ and some nonnegative integer j. To X_p we assign S' and all the summands having $\mu_1 = n' - j$ and $u = 2t - j$ for $t \leqslant p$ and $j \geqslant 0$.

Again one checks that X_t is indeed a subcomplex and that $X_p = M_0$.

Furthermore, $X_{k-1} \subseteq X_k \subseteq \cdots$ contain two terms in which H^* does not occur, namely $L_{(1^{m-n+2})} G^*(i=n)$ and $L_{(2, 1^{m-n+2})} G^*(i=n-1)$. (Note that $k-1 \neq p$, since $m = 2p+1 \geqslant 2k$ implies $p \geqslant k$).

For $t \neq k-1$ and $t \neq p$, $X_t / X_{t-1} = \mathrm{Tot}(P_t)$, where P_t is as in the previous subsection. When $t = k-1$, X_{k-1}/X_{k-2} is the total complex of a bicomplex like P_t, but with an extra box added as in the previous subsection. When $t = p$, X_p / X_{p-1} is isomorphic to a complex which looks like $\mathbb{C}'_m$ of Subsection 3.9.

Thus M_0 turns out to be exact.

3.14. Let us now deal with M_1.

For each $\bar{t} \in \{0, 1, ..., p\}$, in every $L_{(a, 1^b)}(H^* \oplus G^*)$ we assign to $Y_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t - j$ for some $t \leqslant \bar{t}$ and some $j \geqslant 0$.

Y_t is a subcomplex for every t, and $Y_p = M_1$.

For $t \neq p$, $Y_{t-1} = \mathrm{Tot}(D_t)$, where D_t is as in Subsection 3.8. When $t = p$, one obtains a bicomplex of type D_t, but with the bottom row missing. It then follows as usual that M_1 is exact.

4. End of the Proof of Theorem 2.5

4.1. In this section we complete the proof of Theorem 2.5, when either $m = 2r$ ($1 \leqslant r \leqslant k-1$), or $m = 2r+1$ ($0 \leqslant r \leqslant k-1$). Accordingly, $\mathbb{C}_m$ is of the type described in Definition 2.2(ii) and (iii).

We still dwell on Subsections 3.1 to 3.4.

Case $m = 2r$, $1 \leqslant r \leqslant k-1$

4.2. $\mathbb{C}'_m \otimes_s S'$ is again the direct sum of two subcomplexes M_0 and M_1, characterized by $|H^*|$ even and odd, resp. Furthermore, S' pertains to M_1, because the mth power of I' is covered by $L_{(n', 1^m)} H^* \otimes \Lambda^2 G^*$, and $n' + m$ is odd. We filter M_1 and M_0 separately.

For M_1, let $\{X_t\}$ be defined as follows (recall that $m = 2r$). For each $\bar{t} \in \{0, 1, ..., r-1\}$, in every $L_{(a, 1^b)}(H^* \oplus G^*)$ we assign to $X_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t - j$ for some $t \leqslant \bar{t}$ and some $j \geqslant 0$. To X_r we assign S' and all the summands having $\mu_1 = n' - j$ and $u = 2t - j$ for $t \leqslant r$ and $j \geqslant 0$.

It is easy to check that every X_t is a subcomplex and that $X_r = M_1$.

As for X_t/X_{t-1}, $0 \leqslant t < r$, it is exact because it coincides with $\mathrm{Tot}(D_t)$, where D_t is as in Subsection 3.8. (Note, however, that q is always $2t$, because $t < r$ implies $2t < 2r \leqslant 2k - 2 < n'$).

As for X_r/X_{r-1}, one obtains $\mu_1 = n' - j$, $u = 2r - j$, $a - \mu_1 = 2$ and $b - u = 0$, with j ranging between 0 and $2r = m$ (cf. Subsection 3.9). That is, one has the complex ($\Lambda^2 G^* \cong S'$ and $m < n'$):

$$0 \to L_{(n'-m)} H^* \to L_{(n'-m+1,1)} H^*$$
$$\to \cdots \to L_{(n'-1,1^{m-1})} H^* \to L_{(n',1^m)} H^* \to S',$$

which is exact by the inductive hypothesis.

We remark that if $m \leqslant 2k - 4$ (i.e., $r \leqslant k - 2$), the above complex in H^* is still of the kind described in Definition 2.2(ii), as $\mathbb{C}_m$. But if $m = 2k - 2$ (i.e., $r = k - 1$), the above complex in H^* is of the type described in Definition 2.2(i) and discussed in Section 3.

We have thus finished the proof of the exactness of M_1.

4.3. For the filtration $\{Y_t\}$ of M_0, for each $\bar{t} \in \{0, 1, ..., r-1\}$, in every $L_{(a,1^b)}(H^* \oplus G^*)$ we assign to $Y_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t + 1 - j$ for some $t \leqslant \bar{t}$ and $j \geqslant 0$.

One checks as usual that every Y_t is a subcomplex and $M_0 = Y_{r-1}$. Then one also realizes that (unlike what we remarked in Subsection 3.11) M_0 contains no case $\mu_1 = 0 = u$, for this would imply $t = k - 1$, while $t \leqslant r - 1 \leqslant k - 2$.

For $t \in \{0, 1, ..., r-1\}$, $Y_t/Y_{t-1} = \mathrm{Tot}(P_t)$, where P_t is as in Subsection 3.12, but with $q = 2t + 1$, since $2t + 1 \leqslant 2r - 1 < n'$ (for $r < k$). Of course, when $t = r - 1$, the bottom row of P_t is missing.

Therefore every Y_t/Y_{t-1} is exact, M_0 is too, and we have completely proven Theorem 2.5 also in the case $m = 2r$, $1 \leqslant r \leqslant k - 1$.

Case $m = 2r + 1$, $0 \leqslant r \leqslant k - 1$

4.4. The complex $\mathbb{C}_m \otimes_s S'$ we are dealing with now is of the type described in Definition 2.2(iii). Hence it contains two copies of the ring S'. Let us focus our attention on the copy in position $i = m + 2$.

Since $\chi: S \to L_{(n-m)} F^*$ is defined by $1 \mapsto \alpha^{(k-r)}$, it is not hard to check that $\chi \otimes_s S' = \rho + \sigma$, where $\rho: S' \to L_{(n'-m)} H^* \otimes \Lambda^2 G^*$ and $\sigma: S' \to L_{(n'-m+2)} H^*$ are defined by $\rho(1) = \alpha_{H^*}^{(k-r-1)} \otimes \alpha_{G^*}$ and $\sigma(1) = \alpha_{H^*}^{(k-r)}$, resp. (observe that $k - r = (n' - m)/2 + 1$). This suggests that we decompose $\mathbb{C}_m \otimes_s S'$ into the direct sum of two subcomplexes M_0 and M_1, characterized by $|H^*|$ even and odd, resp., and with M_0 containing both the copies of S'. We are going to show the exactness of M_0 and M_1 by suitable filtrations, as in all previous cases.

Let us start with a filtration $\{X_t\}$ of M_0. For each $\bar{t} \in \{0, 1, ..., r-1\}$, in every $L_{(a,1^b)}(H^* \oplus G^*)$ we assign to $X_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t + 1 - j$ for $t \leqslant \bar{t}$ and $j \geqslant 0$. To X_r we assign both copies of S', and all the summands having $\mu_1 = n' - j$ and $u = 2t + 1 - j$ for $t \leqslant r$ and $j \geqslant 0$.

That $M_0 = X_r$ and each X_t is a subcomplex is verified as usual (no problem is posed by the extra copy of S').

One should remark that if $r < k - 1$, no term with $|H^*| = 0$ occurs in M_0, for $\mu_1 = 0 = u$ implies $t = k - 1$; but if $r = k - 1$, i.e., $m = n'$, there is one such term, namely $\Lambda^2 G^*$, for which $i = m + 1$.

For $t \in \{0, 1, ..., r-1\}$, $X_t / X_{t-1} = \mathrm{Tot}(P_t)$, where P_t is as in Subsection 3.12, but with $q = 2t + 1$, since $2t + 1 \leqslant 2r - 1 < n'$ (for $r < k - 1$). Thus X_t / X_{t-1} is exact.

As for X_r / X_{r-1}, one has $\mu_1 = n' - j$, $u = 2r + 1 - j$, $a - \mu_1 = 2$ and $b - u = 0$, with j ranging between 0 and $2r + 1$. Hence one obtains the complex

$$0 \to S' \xrightarrow{\rho} L_{(n'-m)} H^* \otimes \Lambda^2 G^* \to L_{(n'-m+1,1)} H^* \otimes \Lambda^2 G^*$$
$$\to \cdots \to L_{(n'-1,1^{m-1})} H^* \otimes \Lambda^2 G^* \to L_{(n',1^m)} H^* \otimes \Lambda^2 G^* \to S',$$

where ρ is the same map described before.

If $r < k - 1$, the above complex is isomorphic ($\Lambda^2 G^* \cong S'$) to one which is still of the kind described in Definition 2.2(iii), and is exact by inductive hypothesis.

If $r = k - 1$, the above complex deserves a closer inspection. Since $r = k - 1$ implies $m = n'$, we find the expected $\Lambda^2 G^*$ for $i = m + 1$. But it is not hard to see that $(\varphi \otimes_s S')(\Lambda^2 G^*) \subseteq L_{(2)} H^* \otimes S_1 G^* \otimes \Lambda^1 G^*$; hence the morphism $\Lambda^2 G^* \to L_{(n'-m+1,1)} H^* \otimes \Lambda^2 G^*$ is zero. Furthermore, ρ is the identity on $S' \cong \Lambda^2 G^*$. Thus the complex in object is exact if and only if one can show the exactness of

$$0 \to L_{(n'-m+1,1)} H^* \otimes \Lambda^2 G^*$$
$$\to \cdots \to L_{(n'-1,1^{m-1})} H^* \otimes \Lambda^2 G^* \to L_{(n',1^m)} H^* \otimes \Lambda^2 G^* \to S'.$$

Up to $\Lambda^2 G^* \cong S'$, the latter complex is of the type described in Definition 2.2(i), hence it is exact by the inductive hypothesis.

This ends the proof of the exactness of M_0.

4.5. Let us turn our attention to a filtration $\{Y_t\}$ for M_1. For each $\bar{t} \in \{0, 1, ..., r\}$, in every $L_{(a,1^b)}(H^* \oplus G^*)$ we assign to $Y_{\bar{t}}$ all the summands having $\mu_1 = n' - j$ and $u = 2t - j$ for $t \leqslant \bar{t}$ and $j \geqslant 0$.

$Y_r = M_1$ and each Y_t is a subcomplex.

For every t, $Y_t/Y_{t-1} = \mathrm{Tot}(\mathbb{D}_t)$, where $\mathbb{D}_t$ is as in Subsection 3.8 (with $q = 2t$ in all cases). Of course, when $t = r$, the bottom row of $\mathbb{D}_t$ is missing.

Thus every Y_t/Y_{t-1} is exact, as well as M_1, and we have proven Theorem 2.5 in all cases.

4.6. *Final Remark.* In an earlier version of this paper, another proof of Theorem 2.5 was given. Given $P \in \mathrm{Spec}(S)$ with $\mathrm{grade}(PS_p) < \mathrm{length}(\mathbb{C}_m)$, it was shown that $\mathbb{C}_m \otimes S_p$ was the total complex of an appropriate tricomplex T (pictured as a family of modules in the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ cartesian space). The exactness of $\mathbb{C}_m \otimes S_p$ was then verified by showing that each bicomplex T_h, obtained as the intersection of T and the plane $z = h$, had exact total complex. In fact, the bicomplexes T_h looked precisely like the factors X_t/X_{t-1} and Y_t/Y_{t-1} discussed in this version.

Acknowledgment

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References

[A-B-W, 1] K. Akin, D. A. Buchsbaum, and J. Weyman, Resolution of determinantal ideals: The submaximal minors, *Adv. Math.* **39** (1981), 1–30.

[A-B-W, 2] K. Akin, D. A. Buchsbaum, and J. Weyman, Schur functors and Schur complexes, *Adv. Math.* **44** (1982), 207–278.

[A-DF] S. Abeasis and A. Del Fra, Young diagrams and ideals of Pfaffians, *Adv. Math.* **35** (1980), 158–178.

[B] D. A. Buchsbaum, Resolutions and representations of *GL(n)*, *in* "Advanced Studies in Pure Mathematics 11, Commutative Algebra and Combinatorics" (M. Nagata and H. Matsumura, Eds.), pp. 21–28, Kinokuniya, Tokyo, and North Holland, Amsterdam/New York/Oxford, 1987.

[B-E, 1] D. A. Buchbaum and D. Eisenbud, What makes a complex exact? *J. Algebra* **25** (1973), 259–268.

[B-E, 2] D. A. Buchsbaum and D. Eisenbud, Generic free resolutions and a family of generically perfect ideals, *Adv. Math.* **18** (1975), 245–301.

[B-E, 3] D. A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99** (1977), 447–485.

[B-S] G. Boffi and R. Sánchez, Some classical formulas and a determinantal ideal, *in* "Seminari di Geometria 1989" (S. Coen, Ed.), Dipartimento di Matematica dell'Università di Bologna (Tecnoprint), Bologna, in press.

[B-U] R. O. Buchweitz and B. Ulrich, Homological properties which are invariant under linkage, preprint.

[C] A. Cayley, Sur les déterminants gauches, *J. Reine Angew. Math.* **38** (1849), 93–96.

[C-E] H. CARTAN AND S. EILENBERG, "Homological Algebra," Princeton Univ. Press, Princeton, 1956.

[H] M. HASHIMOTO, Determinantal ideals without generic minimal free resolutions, *Nagoya J. Math.*, in press.

[He] P. HEYMANS, Pfaffians and skew-symmetric matrices, *Proc. Lond. Math. Soc. (3)* **19** (1969), 730–768.

[J-P] T. JÓZEFIAK AND P. PRAGACZ, Ideals generated by Pfaffians, *J. Algebra* **61** (1979), 189–198.

[K] K. KURANO, Relations on pfaffians. II. A counterexample, preprint.

[K-U] A. R. KUSTIN AND B. ULRICH, A family of complexes associated to an almost alternating map, with applications to residual intersections, preprint.

[P-S] C. PESKINE ND L. SZPIRO, Dimension projective finie et cohomologie locale, *Inst. Hautes Etudes Sci. Publ. Math.* **42** (1973), 47–119.

[R] J. J. ROTMAN, "An Introduction to Homological Algebra," Academic Press, New York/San Francisco/London, 1979.