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Heat equation with dynamical boundary conditions of reactive–diffusive type

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ABSTRACT

This paper deals with the heat equation posed in a bounded regular domain Ω of $\mathbb{R}^N$ $(N \geqslant 2)$ coupled with a dynamical boundary condition of reactive–diffusive type. In particular we study the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty) \times \Omega, \\ u_t = k u_\nu + l \Delta_\Gamma u & \text{on } (0,\infty) \times \Gamma, \\ u(0,x) = u_0(x) & \text{on } \Gamma, \end{cases}$$

where $u = u(t,x)$, $t \geqslant 0$, $x \in \Omega$, $\Gamma = \partial\Omega$, $\Delta = \Delta_x$ denotes the Laplacian operator with respect to the space variable, while Δ_Γ denotes the Laplace–Beltrami operator on Γ, ν is the outward normal to Ω, and k and l are given real constants, $l > 0$. Well-posedness is proved for data $u_0 \in H^1(\Omega)$ such that $u_{0|\Gamma} \in H^1(\Gamma)$. We also study higher regularity of the solution.

1. Introduction and main results

We deal with the evolution problem consisting in the standard heat equation posed in a bounded domain, supplied with a dynamical (or Wentzell) boundary condition. The precise problem is

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty) \times \Omega, \\ u_t = k u_\nu + l \Delta_\Gamma u & \text{on } (0,\infty) \times \Gamma, \\ u(0,x) = u_0(x) & \text{on } \Omega. \end{cases} \tag{1}$$

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Here $u = u(t, x)$, $t \geqslant 0$, $x \in \Omega$, where Ω is a C^∞ regular bounded domain of $\mathbb{R}^N$ ($N \geqslant 2$) and $\Gamma = \partial\Omega$. The first equation states the law of standard diffusion or heat conduction in Ω, and $\Delta = \Delta_x$ denotes the Laplacian operator with respect to the space variable. In the boundary equation $(1)_2$, the value of u is assumed to be the trace of the function u defined for $x \in \Omega$, Δ_Γ denotes the Laplace–Beltrami operator on Γ, ν is the outward normal to Ω, and $k \in \mathbb{R}$ and $l > 0$ are given constants; the term ku_ν represents the interaction domain–boundary, while $l\Delta_\Gamma u$ stands for a boundary diffusion.

A number of authors have studied parabolic problems with dynamical boundary conditions like $(1)_2$. Note that we can replace u_t by Δu in this boundary condition which leads to the form known as generalized Wentzell boundary condition. The problem has been mostly studied the case when there is no Laplacian term on the boundary condition, i.e., when $l = 0$. In particular, when $k \leqslant 0$ problem (1) is well-posed. See [1,12,15–19,24,25] in the case $k < 0$ which represents a dissipative interaction; the non-interactive case $k = 0$ is rather trivial. However, when $k > 0$ we are in the presence of a reactive interaction and problem (1) is ill-posed, as shown in the recent papers [3] and [33]. See also [2] and [32] for the related case $k = k(x)$.

The question we address in this paper is the following one: is the situation improved by adding to the dynamical boundary condition a Laplace–Beltrami correction term with $l > 0$? The interest of such a correction both for the modeling of parabolic and hyperbolic problems has been recently pointed out in [23]. In particular (1) describes (see [23, p. 465]) a heat conduction process in Ω with a heat source on the boundary which can depend on the heat flux around the boundary and on the heat flux across it. The case of dissipative interaction, $k < 0$, has been studied in [9,10,20] (see also [7,8] and [28]). It turns out from the quoted papers that problem (1) is well-posed in the framework of $L^p(\Omega) \times L^p(\Gamma)$, $1 \leqslant p \leqslant \infty$. This is to be expected since both terms in the right-hand side of the boundary condition have the "favorable sign". The aim of this paper is to solve the system in the reactive case $k > 0$, that is in the usually ill-posed case. The estimates of the quoted papers did not allow to cover this case.

A first step in this study has been performed by the authors of the present paper in [34], where we consider the Laplace equation instead of the heat equation as domain equation. The modified problem admits a simple functional framework; the paper helped the authors understand the dynamical boundary condition $(1)_2$ and allowed us to formulate the conjecture that turns out to be correct, but the arguments used there do not work for the heat equation. Indeed, a new estimate is needed to deal with problem (1), which cannot be obtained in the framework of $L^2(\Omega) \times L^2(\Gamma)$.

We want to show that problem (1) is well-posed in an appropriate setting. We propose to work in the space

$$H = \{(u, v) \in H^1(\Omega) \times H^1(\Gamma):\ u_{|\Gamma} = v\}, \tag{2}$$

where $u_{|\Gamma}$ denotes the trace of u on Γ, with the natural topology inherited by $H^1(\Omega) \times H^1(\Gamma)$. Here and in the sequel, we denote for any $s \in \mathbb{R}$, $H^s(\Omega)$ and $H^s(\Gamma)$ the Sobolev spaces of complex-valued distributions respectively on Ω and Γ (see [27] or [31]). For the sake of simplicity we shall identify, when useful, H with is isomorphic counterpart $\{u \in H^1(\Omega):\ u_{|\Gamma} \in H^1(\Gamma)\}$ through the identification $(u, u_{|\Gamma}) \mapsto u$, so we shall write, without further mention, $u \in H$ for functions defined on Ω.

Our main result is the following

Theorem 1. *For any $u_0 \in H$ problem (1) has a unique solution $u = u(u_0)$ such that*

$$\begin{aligned} u &\in C([0,\infty); H^1(\Omega)) \cap C^1((0,\infty); H^1(\Omega)) \cap C((0,\infty); H^3(\Omega)), \\ u_{|\Gamma} &\in C([0,\infty); H^1(\Gamma)) \cap C^1((0,\infty); H^1(\Gamma)) \cap C((0,\infty); H^3(\Gamma)). \end{aligned} \tag{3}$$

Moreover,

$$\begin{aligned}
&\|\nabla u(t)\|^2_{L^2(\Omega)} + \|d_\Gamma u_{|\Gamma}(t)\|^2_{L^2(\Gamma)} + \|u_{|\Gamma}(t)\|^2_{L^2(\Gamma)} \\
&\quad \leqslant e^{2\lambda_0 t}\big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|d_\Gamma u_{0|\Gamma}\|^2_{L^2(\Gamma)} + \|u_{0|\Gamma}\|^2_{L^2(\Gamma)}\big)
\end{aligned} \tag{4}$$

for all $t \geqslant 0$, where $\lambda_0 \geqslant 0$ is a constant depending on Ω. Finally, the family of maps $\{u_0 \mapsto u(u_0)(t),\ t \geqslant 0\}$ extends to an analytic quasi-contractive semigroup in H, and consequently

$$u \in C^\infty\big((0,\infty) \times \overline{\Omega}\big). \tag{5}$$

The solutions are in principle complex-valued but it is clear that for real-valued data the solution is likewise real-valued. As usual, more regular solutions are obtained for more regular initial data satisfying usual compatibility conditions. This is the content of the following regularity result.

Theorem 2. *If $u_0 \in H^{2n+1}(\Omega)$ and $u_{0|\Gamma} \in H^{2n+1}(\Gamma)$ for some $n \in \mathbb{N}$, and*

$$\big(\Delta^i u_0\big)_{|\Gamma} = k\big(\Delta^{i-1} u_0\big)_\nu + l\Delta_\Gamma\big(\big(\Delta^{i-1} u_0\big)_{|\Gamma}\big), \quad \textit{for all } i = 1,\dots,n, \tag{6}$$

then

$$\begin{aligned}
u &\in C\big([0,\infty); H^{2n+1}(\Omega)\big) \cap C^1\big([0,\infty); H^{2n-1}(\Omega)\big) \cap \dots \cap C^n\big([0,\infty); H^1(\Omega)\big), \\
u_{|\Gamma} &\in C\big([0,\infty); H^{2n+1}(\Gamma)\big) \cap C^1\big([0,\infty); H^{2n-1}(\Gamma)\big) \cap \dots \cap C^n\big([0,\infty); H^1(\Gamma)\big).
\end{aligned} \tag{7}$$

Finally, if $u_0 \in C^\infty(\overline{\Omega})$ and (6) hold for all $i \in \mathbb{N}$, then

$$u \in C^\infty\big([0,\infty) \times \overline{\Omega}\big). \tag{8}$$

The proofs of Theorems 1 and 2 rely on the study of the resolvent problem with eigenvalue-dependent boundary condition, that is

$$\begin{cases} -\Delta u + \lambda u = h & \text{in } \Omega, \\ -ku_\nu - l\Delta_\Gamma u + \lambda u = h & \text{on } \Gamma, \end{cases} \tag{9}$$

where $\lambda \in \mathbb{C}$ and $h \in H$. Such type of problems has been studied by some authors, starting from the classical papers (see [13,14]) to more recent ones (see [4] and the bibliography therein). Our result concerning problem (9) is Theorem 3 below. Finally, we study the limit behavior of the solution u when $l \to 0^+$ (vanishing boundary dissipation). See Theorem 6 below.

The paper is organized as follows. In Section 2 we recall some well-known facts and we state some preliminaries. In Section 3 we analyze the elliptic problem (9), while in Section 4 we apply the results obtained to problem (1). In Section 5 we analyze the limit behavior when $l \to 0^+$, while the final section contains some comments on future developments.

2. Preliminaries and functional setting

Notation. We simply denote by xy the duality product between vectors $x, y \in \mathbb{C}^N$, that is

$$xy = \sum_{i=1}^N x_i y_i \quad \text{when } x = (x_1,\dots,x_N),\ y = (y_1,\dots,y_N). \tag{10}$$

Moreover $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$, denotes the norm in $L^p(\Omega)$ and also the norm in $L^p(\Omega;\mathbb{R}^N)$ since no confusion is expected. We denote by $\|\cdot\|_{p,\Gamma}$ the norm in $L^p(\Gamma)$ and also, when $p=2$, the L^2 norm for square integrable 1-forms on Γ.

Laplace–Beltrami operator. We recall here, for the reader's convenience, some well-known facts on the Laplace–Beltrami operator Δ_Γ. We refer to [26] or [31] for more details and proofs. We start by fixing some notation. Clearly, Γ is a Riemannian manifold endowed with the natural metric inherited from $\mathbb{R}^N$, given in local coordinates by $(g_{ij})_{i,j=1,\dots,N-1}$. We denote by $d\sigma$ the natural volume element on Γ, given in local coordinates by $\sqrt{g}\,dy_1\dots dy_{N-1}$, where $g=\det(g_{ij})$. We denote by ∇_Γ the Riemannian gradient and by d_Γ the total differential on Γ. We use the notation $(\cdot,\cdot)$ for the Riemannian (complex) inner product of vectors while $(\cdot|\cdot)$ is used for the natural (complex) scalar product on 1-forms on Γ associated to the metric. Then, it is clear that $(d_\Gamma u|d_\Gamma v)=(\nabla_\Gamma u,\nabla_\Gamma v)$ for $u,v\in C^1(\Gamma)$, so the use of vectors or forms in the sequel is optional.

The Laplace–Beltrami operator Δ_Γ can be at first defined on $C^\infty(\Gamma)$ by the formula

$$-\int_\Gamma (\Delta_\Gamma u)\bar{v}\,d\sigma=\int_\Gamma (d_\Gamma u|d_\Gamma v)\,d\sigma \tag{11}$$

for any $u,v\in C^\infty(\Gamma)$, and it is given in local coordinates by

$$\Delta_\Gamma u=g^{-1/2}\sum_{i,j=1}^{N-1}\frac{\partial}{\partial y_i}\left(g^{ij}g^{1/2}\frac{\partial u}{\partial y_j}\right), \tag{12}$$

where $(g^{ij})=(g_{ij})^{-1}$ as usual. Clearly, by (12), Δ_Γ can be considered as a bounded operator from $H^{s+2}(\Gamma)$ to $H^s(\Gamma)$, for any $s\in\mathbb{R}$. Consequently, formula (11) extends by density to $u,v\in H^1(\Gamma)$, where the integral in the left-hand side has to be interpreted in the distributional sense, as $\Delta_\Gamma u\in H^{-1}(\Gamma)$.

Remark. In the sequel, the notation $d\sigma$ will be dropped from the boundary integrals; we hope that the reader will be able to put in the appropriate integration elements in all formulas.

Since $\Delta_\Gamma 1=0$ the operator is not injective, but by (11) we have

$$\int_\Gamma (-\Delta_\Gamma u+u)\bar{u}=\|d_\Gamma u\|^2_{L^2(\Gamma)}+\|u\|^2_{L^2(\Gamma)} \tag{13}$$

so that the operator $L:=-\Delta_\Gamma+1$ is a topological and algebraic isomorphism between $H^1(\Gamma)$ and $H^{-1}(\Gamma)$. Moreover, by elliptic regularity (see [31, p. 309]), $L^{-1}: H^{k-1}(\Gamma)\to H^{k+1}(\Gamma)$, $k=0,1,2,\dots$, is bounded, so $L: H^{k+1}(\Gamma)\to H^{k-1}(\Gamma)$ is an isomorphism. By interpolation, $L^{-1}: H^s(\Gamma)\to H^{s+2}(\Gamma)$ for all $s\in\mathbb{R}$, $s\geqslant -1$, giving the inverse of $L: H^{s+2}(\Gamma)\to H^s(\Gamma)$. By duality, this fact holds for all real s.

Dirichlet-to-Neumann operator. We will also need some well-known facts about this operator that will be used at some technical points. We refer to [27] for details and proofs. For any $u\in H^s(\Gamma)$, $s\in\mathbb{R}$, the non-homogeneous Dirichlet problem

$$\begin{cases}\Delta v=0 & \text{in } \Omega,\\ v=u & \text{on } \Gamma,\end{cases} \tag{14}$$

has a unique solution $v \in H^{s+1/2}(\Omega)$, here denoted by $v = \mathbb{D}u$. Moreover $\mathbb{D}$ is a bounded operator from $H^s(\Gamma)$ to $H^{s+1/2}(\Omega)$ for all real s, and v has a normal derivative $v_\nu \in H^{s-1}(\Gamma)$. The operator $u \mapsto v_\nu$, known as the Dirichlet-to-Neumann operator, is bounded from $H^s(\Gamma)$ to $H^{s-1}(\Gamma)$, and it will be denoted in the sequel by $\mathcal{A}$. For all $u, v \in C^\infty(\Gamma)$, integrating by parts twice we have

$$\int_\Gamma \mathcal{A}u\bar{v} = \int_\Omega \nabla(\mathbb{D}u)\nabla(\mathbb{D}\bar{v}) = \int_\Gamma u\mathcal{A}\bar{v} \tag{15}$$

which, by density, holds for all $u, v \in H^1(\Gamma)$.

Functional setting. In the sequel we equip $H^1(\Gamma)$ with the equivalent norm in (13), so we denote

$$(u, v)_{H^1(\Gamma)} = \int_\Gamma u\bar{v} + \int_\Gamma (d_\Gamma u|d_\Gamma v), \qquad \|u\|^2_{H^1(\Gamma)} = (u, u)_{H^1(\Gamma)} \tag{16}$$

for all $u, v \in H^1(\Gamma)$. Moreover, since $-\Delta_\Gamma + 1 : H^2(\Gamma) \to L^2(\Gamma)$ is an isomorphism we can equip $H^2(\Gamma)$ with the equivalent norm

$$(u, v)_{H^2(\Gamma)} = \int_\Gamma u\bar{v} + \int_\Gamma \Delta_\Gamma u \Delta_\Gamma \bar{v}, \qquad \|u\|^2_{H^2(\Gamma)} = (u, u)_{H^2(\Gamma)} \tag{17}$$

for all $u, v \in H^2(\Gamma)$. Moreover, we denote as usual

$$\|u\|^2_{H^1(\Omega)} = \|u\|_2^2 + \|\nabla u\|_2^2. \tag{18}$$

The space *H*. We now introduce, as anticipated in the introduction, the space H given in (2), which by the Trace Theorem is a closed subset of $H^1(\Omega) \times H^1(\Gamma)$, hence a Hilbert space with respect to the scalar product inherited from $H^1(\Omega) \times H^1(\Gamma)$. For the sake of simplicity, we shall drop the notation $u_{|\Gamma}$, when clear, so we shall write $\|u\|_{2,\Gamma}$, $\int_\Gamma u$, and so on, for elements of H, through the already mentioned identification $(u, u_{|\Gamma}) \mapsto u$. We equip H with an equivalent norm which simplifies our calculations. This is the content of the following

Lemma 1. *We set, for any $u, v \in H$,*

$$(u, v)_H = \int_\Omega \nabla u\nabla\bar{v} + \int_\Gamma (d_\Gamma u|d_\Gamma v) + \int_\Gamma u\bar{v}, \qquad \|u\|_H^2 = (u, u)_H. \tag{19}$$

Then $\|\cdot\|_H$ is equivalent in H to the standard norm inherited by $H^1(\Omega) \times H^1(\Gamma)$.

Proof. We just have to show that if we drop $\|\cdot\|_2$ in the standard norm of $H^1(\Omega) \times H^1(\Gamma)$ we get an equivalent norm. This follows by a Poincaré-type inequality which says (see [35, Theorem 4.4.6] in the real-valued case, the extension to the complex-valued one being trivial) that

$$\left\| u - \int_\Gamma u \right\|_{2^*} \leqslant C_1 \|\nabla u\|_2 \quad \text{for all } u \in H^1(\Omega),$$

where $C_1 = C_1(N, \Omega) > 0$, 2^* is the Sobolev critical exponent, i.e. $2^* = 2N/(N-2)$ when $N \geqslant 3$, $1 \leqslant 2^* < \infty$ when $N = 2$. Consequently, since Ω is bounded and Γ is compact, we get

$$\|u\|_2 \leqslant \left\| u - \int_\Gamma u \right\|_2 + \left\| \int_\Gamma u \right\|_2 \leqslant C_1 \|\nabla u\|_2 + \lambda_N(\Omega) \int_\Gamma |u|$$
$$\leqslant C_2\big(\|\nabla u\|_2 + \|u\|_{2,\Gamma}\big) \tag{20}$$

where λ_N denotes the usual Lebesgue measure in $\mathbb{R}^N$ and $C_2 = C_2(N,\Omega) > 0$. This estimate completes the proof. $\square$

The space *V*. We need a further space

$$V = \big\{(u,v) \in H^2(\Omega) \times H^2(\Gamma)\colon\ u_{|\Gamma} = v\big\} \tag{21}$$

which is naturally embedded in H, and it is a Hilbert space with respect to the scalar product and norm inherited from $H^2(\Omega) \times H^2(\Gamma)$. As before we equip it with a suitable scalar product which induces a norm equivalent to that one.

Lemma 2. *If we set, for any $u, v \in V$,*

$$(u,v)_V = \int_\Omega \Delta u \Delta \overline{v} + \int_\Gamma \Delta_\Gamma u \Delta_\Gamma \overline{v} + \int_\Gamma u\overline{v}, \qquad \|u\|_V^2 = (u,u)_V, \tag{22}$$

then $\|\cdot\|_V$ is equivalent in V to the standard norm inherited by $H^2(\Omega) \times H^2(\Gamma)$.

Proof. It simply follows by elliptic regularity estimates. Indeed, for any $u \in H^2(\Omega)$ we have (see [27, p. 202])

$$\|u\|_{H^2(\Omega)} \leqslant C_3\big(\|\Delta u\|_2 + \|u_{|\Gamma}\|_{H^{3/2}(\Gamma)}\big)$$

and consequently, since $H^2(\Gamma)$ is continuously embedded in $H^{3/2}(\Gamma)$, for any $u \in V$ we get

$$\|u\|_{H^2(\Omega)} \leqslant \|\Delta u\|_2 + \|u\|_{H^2(\Gamma)} \tag{23}$$

which by (17) completes the proof. $\square$

3. Elliptic theory

This section is devoted to study the solvability of the coupled elliptic system (9) when $l > 0$, $k \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $h \in H$.

Definition. By a solution of problem (9) we mean a function $u \in V$ such that $(9)_1$ holds true in $L^2(\Omega)$, while $(9)_2$ holds true in $L^2(\Gamma)$.

Space V was just introduced in (21). Before stating the main result of this section we introduce, for any $s \geqslant 1$, the further space

$$H^s = \big\{(u,v) \in H^s(\Omega) \times H^s(\Gamma)\colon\ u_{|\Gamma} = v\big\}. \tag{24}$$

Clearly, being closed in the product space $H^s(\Omega) \times H^s(\Gamma)$, H^s is a Hilbert space equipped with the norm inherited norm, which we denote by $\|\cdot\|_{H^s}$. Moreover, it is naturally embedded in H and $H^1 = H$, $H^2 = V$ (more precisely, $\|\cdot\|_{H^1}$ and $\|\cdot\|_H$ are merely equivalent, like $\|\cdot\|_{H^2}$ and $\|\cdot\|_V$).

Our result concerning (9) is the following

Theorem 3. *There is a positive constant λ_0, depending on l, k, Ω, N, such that for $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geqslant \lambda_0$ and any $h \in H$ problem* (9) *has a unique solution $u \in V$, which also belongs to H^3. Moreover, if $h \in H^s$ for some $s \geqslant 1$, then $u \in H^{s+2}$.*

Finally, there is $C_4 = C_4(l, k, \Omega, s, \lambda) > 0$ such that

$$\|u\|_{H^{s+2}} \leqslant C_4 \|h\|_{H^s} \quad \text{for all } h \in H^s. \tag{25}$$

In order to solve elliptic problems via the variational method it is useful to introduce a sesquilinear form, which leads to weak solutions. The most natural way to perform this procedure for problem (9) would be to multiply (at least formally) the equation $-\Delta u + \lambda u = h$ by a test function $\overline{\phi} \in C^\infty(\overline{\Omega})$ and integrate over Ω to get

$$-\int_\Omega \Delta u \overline{\phi} + \lambda \int_\Omega u\overline{\phi} = \int_\Omega h\overline{\phi}.$$

Integrating by parts, when $u \in H^2(\Omega)$,

$$\int_\Omega \nabla u \nabla\overline{\phi} - \int_\Gamma u_\nu \overline{\phi} + \lambda \int_\Omega u\overline{\phi} = \int_\Omega h\overline{\phi}.$$

Then, using the boundary equation in (9) we get (when $k \neq 0$)

$$\int_\Omega \nabla u \nabla\overline{\phi} + \frac{1}{k}\int_\Gamma h\overline{\phi} - \frac{\lambda}{k}\int_\Gamma u\overline{\phi} + \frac{l}{k}\int_\Gamma \Delta_\Gamma u\overline{\phi} + \lambda\int_\Omega u\overline{\phi} = \int_\Omega h\overline{\phi}.$$

Finally, by (11) we arrive to

$$\int_\Omega \nabla u \nabla\overline{\phi} - \frac{l}{k}\int_\Gamma (d_\Gamma u | d_\Gamma \phi) - \frac{\lambda}{k}\int_\Gamma u\overline{\phi} + \lambda\int_\Omega u\overline{\phi} = -\frac{1}{k}\int_\Gamma h\overline{\phi} + \int_\Omega h\overline{\phi}. \tag{26}$$

Now, it is easy to check that the sesquilinear form in the left-hand side of (26) is indefinite in the case $k > 0$, so this procedure does not produce useful estimates. Thus, one has to look for a positive definite sesquilinear form, at least for $\operatorname{Re}\lambda$ large enough. This is exactly the content of the following two lemmas. The first one introduces the sesquilinear form which turns out to be appropriate.

Lemma 3. *Let $h \in H$. Then $u \in V$ solves problem* (9) *if and only if*

$$a_\lambda(u, v) = (h, v)_H \quad \text{for all } v \in V, \tag{27}$$

where the sesquilinear form a_λ on V is defined by the formula

$$\begin{aligned} a_\lambda(u, v) = {} & \int_\Omega \Delta u \Delta\overline{v} + l\int_\Gamma \Delta_\Gamma u \Delta_\Gamma \overline{v} + \lambda\int_\Omega \nabla u\nabla\overline{v} + (\lambda + l)\int_\Gamma (d_\Gamma u | d_\Gamma v) \\ & - l\int_\Gamma \Delta_\Gamma u\overline{v_\nu} + k\int_\Gamma u_\nu \Delta_\Gamma\overline{v} - k\int_\Gamma u_\nu\overline{v_\nu} - k\int_\Gamma u_\nu\overline{v} + \lambda\int_\Gamma u\overline{v}. \end{aligned} \tag{28}$$

Moreover in this case $u \in H^3(\Omega)$ and $u_{|\Gamma} \in H^3(\Gamma)$.

Proof. It is divided into several steps.

(i) **Claim.** *If $u \in V$ is a solution of* (9), *then $u \in H^3(\Omega)$ and $u_{|\Gamma} \in H^3(\Gamma)$*. To recognize that our claim is true we use elliptic regularity both on Ω and Γ as follows. Since $u \in H^2(\Omega)$ we have $u_\nu \in H^{1/2}(\Gamma)$ by the Trace Theorem (see [27, Chapter I, Théorème 9.4]). So, being $h_{|\Gamma} \in H^1(\Gamma)$ and $u_{|\Gamma} \in H^{1/2}(\Gamma)$, from $(9)_2$ it follows that $-\Delta_\Gamma u + u_{|\Gamma} \in H^{1/2}(\Gamma)$, so that using the isomorphism property of $-\Delta_\Gamma + 1$ recalled in Section 2 we conclude that $u_{|\Gamma} \in H^{5/2}(\Gamma)$. Consequently, using elliptic regularity for nonhomogeneous Dirichlet problems [27, p. 202] we obtain by $(9)_1$ that $u \in H^3(\Omega)$.

From this, and using the Trace Theorem (recalled above) again, we get $u_\nu \in H^{3/2}(\Omega)$. Using $(9)_2$ again we then get $-\Delta_\Gamma u + u_{|\Gamma} \in H^1(\Gamma)$, so as before $u_{|\Gamma} \in H^3(\Gamma)$, completing the proof of our first claim.

(ii) **Claim.** *If $u \in V$ is a solution of* (9), *then formula* (27) *holds*. By the first claim we have $\Delta u \in H^1(\Omega)$. Moreover, by $(9)_1$ we get

$$(\Delta u)_{|\Gamma} = \lambda u_{|\Gamma} - h_{|\Gamma} \in H^1(\Gamma). \tag{29}$$

Consequently, we get that $\Delta u \in H$, so from $(9)_1$ we have

$$(-\Delta u, v)_H + \lambda(u, v)_H = (h, v)_H \quad \text{for all } v \in H. \tag{30}$$

By using the definition of $(\cdot,\cdot)_H$ given in (19) in the left-hand side terms of formula (30) we write it more explicitly as

$$\begin{aligned} &\int_\Omega \nabla(-\Delta u)\nabla\overline{v} + \int_\Gamma \big(d_\Gamma(-\Delta u)|d_\Gamma v\big) - \int_\Gamma \Delta u\overline{v} \\ &\quad + \lambda\int_\Omega \nabla u\nabla\overline{v} + \lambda\int_\Gamma (d_\Gamma u|d_\Gamma v) + \lambda\int_\Gamma u\overline{v} = (h, v)_H. \end{aligned} \tag{31}$$

Now, using (29) we can write $(9)_2$ in the form

$$(\Delta u)_{|\Gamma} = k u_\nu + l\Delta_\Gamma u_{|\Gamma}. \tag{32}$$

Plugging (32) into (31) we get

$$\begin{aligned} &\int_\Omega \nabla(-\Delta u)\nabla\overline{v} - k\int_\Gamma (d_\Gamma u_\nu|d_\Gamma v) - l\int_\Gamma (d_\Gamma \Delta_\Gamma u|d_\Gamma v) - k\int_\Gamma u_\nu\overline{v} \\ &\quad - l\int_\Gamma \Delta_\Gamma u\overline{v} + \lambda\int_\Omega \nabla u\nabla\overline{v} + \lambda\int_\Gamma (d_\Gamma u|d_\Gamma v) + \lambda\int_\Gamma u\overline{v} = (h, v)_H \end{aligned} \tag{33}$$

for all $v \in H$. Now we restrict to test functions $v \in V$, we integrate by parts the first integral in (33) and we use (11) in the first one to get

$$\begin{aligned} &\int_\Omega \Delta u\Delta\overline{v} - \int_\Gamma \Delta u\overline{v_\nu} - k\int_\Gamma (d_\Gamma u_\nu|d_\Gamma v) + l\int_\Gamma \Delta_\Gamma u\Delta_\Gamma\overline{v} - k\int_\Gamma u_\nu\overline{v} \\ &\quad - l\int_\Gamma \Delta_\Gamma u\overline{v} + \lambda\int_\Omega \nabla u\nabla\overline{v} + \lambda\int_\Gamma (d_\Gamma u|d_\Gamma v) + \lambda\int_\Gamma u\overline{v} = (h, v)_H. \end{aligned} \tag{34}$$

Plugging (32) once again in the second integral in the left-hand side of (34) and (11) in the third and sixth ones we finally get (27).

(iii) To complete the proof, we now suppose that (27) holds for some $u \in V$. We have to prove that u solves (9). Integrating by parts in the third integral in (28) and in the first one in (19) we can then write (27) as

$$\begin{aligned}
&\int_\Omega \Delta u \Delta \overline{v} + l \int_\Gamma \Delta_\Gamma u \Delta_\Gamma \overline{v} - \lambda \int_\Omega u \Delta \overline{v} + \lambda \int_\Gamma u \overline{v_\nu} + (\lambda + l) \int_\Gamma (d_\Gamma u | d_\Gamma v) \\
&\quad - l \int_\Gamma \Delta_\Gamma u \overline{v_\nu} + k \int_\Gamma u_\nu \Delta_\Gamma \overline{v} - k \int_\Gamma u_\nu \overline{v_\nu} - k \int_\Gamma u_\nu \overline{v} + \lambda \int_\Gamma u \overline{v} \\
&= - \int_\Omega h \Delta \overline{v} + \int_\Gamma h \overline{v_\nu} + \int_\Gamma (d_\Gamma h | d_\Gamma v) + \int_\Gamma h \overline{v}
\end{aligned} \tag{35}$$

for all $v \in V$. Using (11) we can write (35) as

$$\begin{aligned}
&\int_\Omega \Delta u \Delta \overline{v} + l \int_\Gamma \Delta_\Gamma u \Delta_\Gamma \overline{v} - \lambda \int_\Omega u \Delta \overline{v} + \lambda \int_\Gamma u \overline{v_\nu} - \lambda \int_\Gamma u \Delta_\Gamma \overline{v} \\
&\quad - l \int_\Gamma \Delta_\Gamma u \overline{v} - l \int_\Gamma \Delta_\Gamma u \overline{v_\nu} + k \int_\Gamma u_\nu \Delta_\Gamma \overline{v} - k \int_\Gamma u_\nu \overline{v_\nu} - k \int_\Gamma u_\nu \overline{v} + \lambda \int_\Gamma u \overline{v} \\
&= - \int_\Omega h \Delta \overline{v} + \int_\Gamma h \overline{v_\nu} - \int_\Gamma h \Delta_\Gamma \overline{v} + \int_\Gamma h \overline{v},
\end{aligned} \tag{36}$$

that is, by grouping the terms with respect to the test function,

$$\int_\Omega (\Delta u - \lambda u + h) \Delta \overline{v} + \int_\Gamma (-l \Delta_\Gamma u - k u_\nu + \lambda u - h) \overline{(-\Delta_\Gamma v + v_\nu + v)} = 0. \tag{37}$$

The form of (37) suggests now how to proceed. Indeed if we restrict to test functions $v \in C_c^\infty(\Omega)$, at least to get $(9)_1$, we get that $\Delta(-\Delta u + \lambda u - h) = 0$ in distributional sense, which is not $(9)_1$. Then it is more useful to start by proving $(9)_2$. With this aim, we restrict (37) to test functions $\mathbb{D}v$, where $v \in H^2(\Gamma)$, to get

$$\int_\Gamma (-l \Delta_\Gamma u - k u_\nu + \lambda u - h) \overline{(-\Delta_\Gamma v + \mathcal{A} v + v)} = 0, \tag{38}$$

where $\mathcal{A}$ denotes the Dirichlet-to-Neumann operator already introduced.

We now claim that by (38) it follows that

$$\int_\Gamma (-l \Delta_\Gamma u - k u_\nu + \lambda u - h) \overline{\phi} = 0 \quad \text{for all } \phi \in L^2(\Gamma), \tag{39}$$

from which clearly one has that $(9)_2$ holds in $L^2(\Gamma)$. To prove our claim it is enough to recognize that, given an arbitrary $\phi \in L^2(\Gamma)$, the problem

$$-\Delta_\Gamma v + \mathcal{A}v + v = \phi \tag{40}$$

has a solution $w \in H^2(\Gamma)$, which turns out to be unique. Hence, our claim is nothing but a refinement, in this particular case, of a previous result of the authors [34, Lemma 1] which says that given $\tilde{l} > 0$ and $\tilde{k} \in \mathbb{R}$ there is $\overline{\Lambda} \geqslant 0$ such that for $\Lambda \geqslant \overline{\Lambda}$ the problem

$$-\tilde{l}\Delta_\Gamma v - \tilde{k}\mathcal{A}v + \Lambda v = \phi \tag{41}$$

has unique solution $v \in H^2(\Gamma)$ for any $\phi \in L^2(\Gamma)$. In particular, our claim is proved if we prove that, when $\tilde{k} < 0$, then we can take $\overline{\Lambda} = 1$. To prove this fact we argue as in the quoted paper, writing (41) in the more explicit form

$$-\int_\Gamma \tilde{k}\mathcal{A}v\overline{\psi} + \tilde{l}\int_\Gamma (d_\Gamma v|d_\Gamma \psi) + \Lambda \int_\Gamma v\overline{\psi} = \int_\Gamma \phi\overline{\psi} \quad \text{for all } \psi \in H^1(\Gamma), \tag{42}$$

and then we apply Lax–Milgram theorem (see [11, p. 376]) to the sesquilinear form

$$a(v,\psi) = -\int_\Gamma \tilde{k}\mathcal{A}v\overline{\psi} + \tilde{l}\int_\Gamma (d_\Gamma v|d_\Gamma \psi) + \Lambda \int_\Gamma v\overline{\psi}, \quad v,\psi \in H^1(\Gamma),$$

which is trivially Hermitian (by (15)) and continuous. To recognize that it is also coercive for $\Lambda \geqslant 1$ we simplify the argument of [34]. Indeed, since $\tilde{k} < 0$ we have by (15)

$$a(v,v) = -\tilde{k}\|\nabla(\mathbb{D}v)\|_2^2 + \tilde{l}\|d_\Gamma v\|_{2,\Gamma}^2 + \Lambda\|v\|_{2,\Gamma}^2 \geqslant \min\{\tilde{l},\Lambda\}\|v\|_{H^1(\Gamma)}^2,$$

so that the form is coercive whenever $\Lambda > 0$. Then, from Lax–Milgram theorem we get the existence of a solution $v \in H^1(\Gamma)$ of (41). By the isomorphism property of $-\Delta_\Gamma + 1$ recalled in Section 2 it then follows that $v \in H^2(\Gamma)$, completing the proof of our claim.

Now, to prove $(9)_1$, we use $(9)_2$ in (37) to get

$$\int_\Omega (\Delta u - \lambda u + h)\Delta\overline{v} = 0 \quad \text{for all } v \in V, \tag{43}$$

which clearly implies $(9)_1$ since for any $\psi \in L^2(\Omega)$ there are $v \in V$ such that $\Delta v = \psi$, for example by taking the unique solution $v \in H^2(\Omega) \cap H_0^1(\Omega)$ with homogeneous Dirichlet boundary conditions. The proof is then complete. □

The following key estimate shows that the sesquilinear form (28) is appropriate.

Lemma 4. *There are positive constants λ_0 and C_5, depending on l, k, Ω, N, such that for all $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geqslant \lambda_0$ we have*

$$\operatorname{Re} a_\lambda(u,u) \geqslant C_5\|u\|_V^2 \quad \textit{for all } u \in V.$$

Proof. By (28) we have

$$a_\lambda(u,u) = \|\Delta u\|_2^2 + l\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \lambda\|\nabla u\|_2^2 + (\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2 + \lambda\|u\|_{2,\Gamma}^2$$
$$- l\int_\Gamma \Delta u\overline{u}_\nu + k\int_\Gamma u_\nu\Delta_\Gamma\overline{u} - k\|u_\nu\|_{2,\Gamma}^2 - k\int_\Gamma u_\nu\overline{u}$$

and then

$$
\begin{aligned}
\operatorname{Re} a_\lambda(u,u) &= \|\Delta u\|_2^2 + l\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \operatorname{Re}\lambda\|\nabla u\|_2^2 + (\operatorname{Re}\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2 \\
&\quad + \operatorname{Re}\lambda\|u\|_{2,\Gamma}^2 + (k-l)\int_\Gamma \operatorname{Re}[\Delta_\Gamma u \bar{u}_\nu] - k\|u_\nu\|_{2,\Gamma}^2 - k\int_\Gamma \operatorname{Re}[u_\nu \bar{u}] \\
&\geqslant \|\Delta u\|_2^2 + l\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \operatorname{Re}\lambda\|\nabla u\|_2^2 + (\operatorname{Re}\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2 \\
&\quad + \operatorname{Re}\lambda\|u\|_{2,\Gamma}^2 - \big(|k|+l\big)\int_\Gamma |\Delta_\Gamma u||u_\nu| - |k|\|u_\nu\|_{2,\Gamma}^2 - |k|\int_\Gamma |u_\nu||u|. \qquad (44)
\end{aligned}
$$

By Young inequality we estimate

$$
|k|\int_\Gamma |u_\nu||u| \leqslant \frac{|k|}{2}\|u_\nu\|_{2,\Gamma}^2 + \frac{|k|}{2}\|u\|_{2,\Gamma}^2, \qquad (45)
$$

and, given any $\varepsilon > 0$ to be fixed later, by weighted Young inequality

$$
\big(|k|+l\big)\int_\Gamma |\Delta_\Gamma u||u_\nu| \leqslant \frac{(|k|+l)\varepsilon}{2}\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \frac{|k|+l}{2\varepsilon}\|u_\nu\|_{2,\Gamma}^2. \qquad (46)
$$

Plugging (45) and (46) into (44), we get

$$
\begin{aligned}
\operatorname{Re} a_\lambda(u,u) &\geqslant \|\Delta u\|_2^2 + \left[l - \frac{(|k|+l)\varepsilon}{2}\right]\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \operatorname{Re}\lambda\|\nabla u\|_2^2 \\
&\quad + \left(\operatorname{Re}\lambda - \frac{|k|}{2}\right)\|u\|_{2,\Gamma}^2 + (\operatorname{Re}\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2 \\
&\quad - \left(\frac{|k|+l}{2\varepsilon} + \frac{3}{2}|k|\right)\|u_\nu\|_{2,\Gamma}^2. \qquad (47)
\end{aligned}
$$

Then, by choosing $\varepsilon = l/(|k|+l)$, we get

$$
\begin{aligned}
\operatorname{Re} a_\lambda(u,u) &\geqslant \|\Delta u\|_2^2 + \frac{l}{2}\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \operatorname{Re}\lambda\|\nabla u\|_2^2 + \left(\operatorname{Re}\lambda - \frac{|k|}{2}\right)\|u\|_{2,\Gamma}^2 \\
&\quad + (\operatorname{Re}\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2 - C_6\|u_\nu\|_{2,\Gamma}^2, \qquad (48)
\end{aligned}
$$

where $C_6 = C_6(k,l) = (\frac{(|k|+l)^2}{2l} + \frac{3}{2}|k|)$. To estimate the last term in the right-hand side of (48), we note that by the embedding $H^{7/4}(\Omega) \hookrightarrow H^{3/2}(\Omega)$ and by the Trace Theorem there is $C_7 = C_7(\Omega) > 0$ such that

$$
\|u_\nu\|_{2,\Gamma}^2 \leqslant C_7\|u\|_{H^{7/4}(\Omega)}^2 \quad \text{for all } u \in H^2(\Omega).
$$

Consequently, by interpolation inequality (see [27]),

$$
\|u_\nu\|_{2,\Gamma}^2 \leqslant C_7\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{3/2} \quad \text{for all } u \in H^2(\Omega).
$$

Using weighted Young inequality we then get, for any $\delta > 0$ (to be fixed below),

$$\|u_\nu\|_{2,\Gamma}^2 \leqslant \frac{C_7}{4\delta}\|u\|_{H^1(\Omega)}^2 + \frac{3C_7\delta}{4}\|u\|_{H^2(\Omega)}^2 \quad \text{for all } u \in H^2(\Omega).$$

By applying (20) and (23) in the last formula, we get

$$\|u_\nu\|_{2,\Gamma}^2 \leqslant \frac{C_8}{\delta}\big(\|\nabla u\|_2^2 + \|u\|_{2,\Gamma}^2\big) + C_8\delta\big(\|\Delta u\|_2^2 + \|\Delta_\Gamma u\|_{2,\Gamma}^2 + \|u\|_{2,\Gamma}^2\big) \tag{49}$$

for all $u \in V$, where $C_8 = C_8(\Omega) > 0$. Plugging (49) into (48) we derive

$$\begin{aligned}\operatorname{Re} a_\lambda(u,u) \geqslant{}& (1 - C_6C_8\delta)\|\Delta u\|_2^2 + \frac{l}{4}(2 - 4C_6C_8\delta/l)\|\Delta_\Gamma u\|_{2,\Gamma}^2\\ &+ (\operatorname{Re}\lambda - C_6C_8/\delta)\|\nabla u\|_2^2 + (\operatorname{Re}\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2\\ &+ \left[\operatorname{Re}\lambda - \frac{|k|}{2} - C_6C_8(\delta + 1/\delta)\right]\|u\|_{2,\Gamma}^2.\end{aligned} \tag{50}$$

Choosing $\delta = \delta_0 = \min\{2, l\}/(4C_6C_8)$ we rewrite (50) as

$$\begin{aligned}\operatorname{Re} a_\lambda(u,u) \geqslant{}& \frac{1}{2}\|\Delta u\|_2^2 + \frac{l}{4}\|\Delta_\Gamma u\|_{2,\Gamma}^2 + (\operatorname{Re}\lambda - C_6C_8/\delta_0)\|\nabla u\|_2^2\\ &+ (\operatorname{Re}\lambda + l)\|d_\Gamma u\|_{2,\Gamma}^2 + \left[\operatorname{Re}\lambda - \frac{|k|}{2} - C_6C_8(\delta_0 + 1/\delta_0)\right]\|u\|_{2,\Gamma}^2.\end{aligned} \tag{51}$$

Now, by setting

$$\lambda_0 = \max\big\{C_6C_8/\delta_0, |k| + 2C_6C_8(\delta_0 + 1/\delta_0)\big\}, \tag{52}$$

we clearly have, when $\operatorname{Re}\lambda \geqslant \lambda_0$, that

$$\operatorname{Re}\lambda - C_6C_8/\delta_0 \geqslant 0 \quad \text{and} \quad \operatorname{Re}\lambda - \frac{|k|}{2} - C_6C_8(\delta_0 + 1/\delta_0) \geqslant \lambda_0/2,$$

so by (51) we finally obtain

$$\operatorname{Re} a_\lambda(u,u) \geqslant \frac{1}{2}\|\Delta u\|_2^2 + \frac{l}{4}\|\Delta_\Gamma u\|_{2,\Gamma}^2 + \frac{\lambda_0}{2}\|u\|_{2,\Gamma}^2.$$

By setting $C_5 = \min\{\frac{1}{2}, \frac{l}{4}, \frac{\lambda_0}{2}\}$ and using (22) the proof is complete. $\square$

Remark 1. It is clear from the proof that $\lambda_0 \geqslant 4/l$ and $C_5 \leqslant l/4$, so that $\lambda_0 \to +\infty$ and $C_5 \to 0$ as $l \to 0^+$. This instability property will be confirmed in Remark 2.

We can now give the

Proof of Theorem 3. By Lemma 3, problem (9) can be equivalently written as (27). The sesquilinear form a_λ in V is trivially continuous and, by Lemma 4, it is also coercive when $\operatorname{Re}\lambda \geqslant \lambda_0$. We then apply Lax–Milgram theorem (see [11, p. 376]) to get the existence of a unique solution u of (9) in V. By Lemma 3 we also have $u \in H^3$.

We now suppose that $h \in H^s$, $s > 1$. To recognize that $u \in H^{s+2}$ we apply the same bootstrap procedure applied in Lemma 3. More precisely, we shall prove that for any $n \in \mathbb{N}_0$ we have

$$u \in H^{\min\{s+2,n+7/2\}}(\Omega) \quad \text{and} \quad u_{|\Gamma} \in H^{\min\{s+2,n+4\}}(\Gamma), \tag{53}$$

from which our claim follows for n large enough. We prove (53) by induction on n. To prove that (53) holds when $n = 0$ we recognize that, by $(9)_1$,

$$\Delta u = \lambda u - h \in H^{\min\{s,3\}}(\Omega) \quad \text{and} \quad u_{|\Gamma} \in H^3(\Gamma)$$

so by elliptic regularity (see [27, p. 202]) we have

$$u \in H^{\min\{s+2,5,7/2\}}(\Omega) = H^{\min\{s+2,7/2\}}(\Omega),$$

which is the required regularity on Ω when $n = 0$. By the Trace Theorem (see [27, Chapter I, Théorème 9.4]) we then have $u_\nu \in H^{\min\{s+1/2,2\}}(\Gamma)$. Hence, since $ku_\nu \in H^{\min\{s+1/2,2\}}(\Gamma)$, $\lambda u \in H^3(\Gamma)$ and $h \in H^s(\Gamma)$, by $(9)_2$ we have

$$-l\Delta_\Gamma u_{|\Gamma} \in H^{\min\{s,s+1/2,2,3\}}(\Gamma) = H^{\min\{s,2\}}(\Gamma)$$

and consequently $-\Delta_\Gamma u_{|\Gamma} + u_{|\Gamma} \in H^{\min\{s,2\}}(\Gamma)$. By the isomorphism property of $-\Delta_\Gamma + 1$ recalled in Section 2 we then get $u_{|\Gamma} \in H^{\min\{s+2,4\}}(\Gamma)$ which completes the proof when $n = 0$. To complete the induction process we now suppose that (53) holds. Arguing as in the case $n = 0$ by $(9)_1$ we get

$$\Delta u = \lambda u - h \in H^{\min\{s,n+7/2\}}(\Omega) \quad \text{and} \quad u_{|\Gamma} \in H^{\min\{s+2,4+n\}}(\Gamma)$$

so by elliptic regularity $u \in H^{\min\{s+2,n+9/2\}}(\Omega)$. By the Trace Theorem we then have $u_\nu \in H^{\min\{s+1/2,n+3\}}(\Gamma)$, so by using $(9)_2$, $-\Delta_\Gamma u + u_{|\Gamma} \in H^{\min\{s,n+3\}}(\Gamma)$. As before $u_{|\Gamma} \in H^{\min\{s+2,n+5\}}(\Gamma)$, completing the induction process.

Finally, to prove (25) we set up the operator $A_\lambda : D(A_\lambda) \to H^s$, where

$$D(A_\lambda) = \{(u, v) \in H^{s+2}\colon\ (\Delta u)_{|\Gamma} = ku_\nu + l\Delta_\Gamma v\}$$

and

$$A_\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u + \lambda u \\ -ku_\nu - l\Delta_\Gamma v + \lambda v \end{pmatrix}.$$

One easily sees that $D(A_\lambda)$ is closed in H^{s+2}, so it is a Hilbert space with respect to the scalar product inherited by it. Moreover A_λ is bounded, and $u \in H^{s+2}$ solves (9) if and only if $u \in D(A_\lambda)$ and $A_\lambda u = h$. By previous analysis A_λ is bijective, so (25) follows by the Closed Graph Theorem. □

4. Analysis of problem (1)

We will use here the results of the previous section to analyze problem (1), thus proving Theorems 1 and 2. We start by setting up the unbounded operator $A : D(A) \subset H \to H$ by

$$D(A) = \{(u, v) \in H^3\colon\ (\Delta u)_{|\Gamma} = ku_\nu + l\Delta_\Gamma v\} \tag{54}$$

and

$$A\binom{u}{v} = \begin{pmatrix} \Delta u \\ ku_\nu + l\Delta_\Gamma v \end{pmatrix}. \tag{55}$$

Our main results are a consequence of the following one.

Theorem 4. *Operator A generates an analytic semigroup $\{S(t),\ t \geqslant 0\}$ in H, and*

$$\|S(t)\|_{\mathcal{L}(H)} \leqslant e^{\lambda_0 t} \quad \textit{for all } t \geqslant 0, \tag{56}$$

where λ_0 is the positive number given in Theorem 3, so $\{S(t),\ t \geqslant 0\}$ is quasi-contractive.

Proof. We introduce the unbounded operator B in H by $D(B) = D(A)$ and $B = A - \lambda_0 I$. Then, given any $u \in D(B)$, we have that u solves (9) when $\lambda = \lambda_0$ and $h = -Bu$. Hence, by (27),

$$(Bu, u)_H = -(h, u)_H = -a_{\lambda_0}(u, u) \quad \text{for all } u \in D(B). \tag{57}$$

Then, by Lemma 4 we get that $\operatorname{Re}(Bu, u)_H \leqslant 0$, for all $u \in D(B)$, i.e. B is a dissipative operator in H. Moreover, by Theorem 3, $R(I - B) = H$. We then apply [29, Theorem 4.6, p. 16] to get that $D(B)$ is dense in H. Moreover, given any $u \in D(B)$, by Lemma 4 and (57) we have

$$\operatorname{Re}(-Bu, u)_H = \operatorname{Re} a_{\lambda_0}(u, u) \geqslant C_5 \|u\|_V^2, \tag{58}$$

while by (57) and the continuity of a_{λ_0}

$$\left|\operatorname{Im}(-Bu, u)_H\right| \leqslant \left|a_{\lambda_0}(u, u)\right| \leqslant C_9 \|u\|_V^2 \tag{59}$$

for some $C_9 = C_9(k, l, N, \Omega) > 0$. Combining (58) and (59) we get that $-B$ is a densely defined m-sectorial operator in H. We then apply semigroup theory (see for example [22, Theorem 5.9, p. 37]) which shows that B generates an analytic contraction semigroup $\{T(t),\ t \geqslant 0\}$ in H, and consequently A generates an analytic semigroup $\{S(t),\ t \geqslant 0\}$, given by $S(t) = e^{\lambda_0 t} T(t)$, $t \geqslant 0$, so clearly (56) follows. $\square$

Now we can give the proofs of our main results.

Proof of Theorem 1. By Theorem 4 the operator A generates the analytic, and hence differentiable, quasi-contractive semigroup $\{S(t),\ t \geqslant 0\}$ in H. Then, by semigroup theory (see [29, §4.1]) given any $u_0 \in H$ there is a unique solution

$$u \in C\big([0, \infty); H\big) \cap C^1\big((0, \infty); H\big) \tag{60}$$

of the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t > 0, \\ u(0) = u_0. \end{cases} \tag{61}$$

Clearly, (60) is nothing but (53), and (61) is the abstract form of problem (1). Moreover, (4) is nothing but (56) due to Lemma 3. Next, by using the differentiability property of the semigroup $\{S(t),\ t \geqslant 0\}$ and [29, §2.4] we get that $u \in C^\infty((0, \infty); H)$ and consequently $Bu = Au - \lambda_0 u = u' - \lambda_0 u \in C^\infty((0, \infty); H)$. By (25) (when $s = 1$) then we get that $u \in C^\infty((0, \infty); H^3)$. A standard bootstrap procedure then gives that $u \in C^\infty((0, \infty); H^{2n+1})$ for all $n \in \mathbb{N}$. By Morrey's theorem (see for example [5, Corollaire IX.13]) we then get that (5) holds. $\square$

Proof of Theorem 2. We introduce, by recurrence on $n \in \mathbb{N}$, the space

$$D(B^n) = \{u \in D(B^{n-1}):\ Bu \in D(B^{n-1})\} \tag{62}$$

endowed with the graph norm

$$\|u\|^2_{D(B^n)} = \sum_{i=0}^{n} \|D^i u\|^2_H.$$

By Theorem 3 it is immediate to recognize that

$$D(B^n) = \{u \in H^{2n+1}:\ (\Delta^i u)_{|\Gamma} = k(\Delta^{i-1} u)_\nu + l\Delta_\Gamma(\Delta^{i-1} u)_{|\Gamma},\ i = 1, \dots, n\}, \tag{63}$$

and that the graph norm is equivalent to the norm of H^{2n+1} introduced in Section 3. Since B is a dissipative operator in H and $R(I - B) = H$ we are able to apply the procedure outlined in the proof of [5, Théorème VII.5] (see also [6, Chapter 1]) in the real case, which works as well in the complex one. Consequently, since $u_0 \in D(B^n)$, we get

$$u \in C([0,\infty); D(B^n)) \cap C^1([0,\infty); D(B^{n-1})) \cap \cdots \cap C^n([0,\infty); H)$$

which, by previous remark, is nothing but (7). Finally, if $u_0 \in C^\infty(\Omega)$ and (6) holds for all $i \in \mathbb{N}$ we apply previous analysis, for any $n \in \mathbb{N}$, together with Morrey's theorem to get (8). □

5. Limit behavior as $l \to 0^+$

This section is devoted to study the limit behavior of the solution of problem (1) when $l \to 0^+$. The motivation of this study is to understand what happens to the solution of (1) when the Laplace–Beltrami term, which makes it well-posed, becomes more and more negligible. The limit problem, at least formally, is given by

$$\begin{cases} u_t - \Delta u = 0 & \text{in } Q = (0,\infty) \times \Omega, \\ u_t = k u_\nu & \text{on } [0,\infty) \times \Gamma, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{64}$$

which has been studied in [33] (see also [3]). We want to show here how the ill-posed problem (64) is approximated by well-posed problems like (1). We recall the following definition and result from that paper. In what follows we restrict to the real-valued case.

Definition 1. (See [33, Definition 1].) Given $u_0 \in H^1(\Omega)$ we say that

$$u \in C([0,T); H^1(\Omega)), \quad T > 0, \tag{65}$$

is a weak solution of (64) if

$$u(0) = u_0 \tag{66}$$

and

$$-\int_0^T\int_\Omega u\varphi_t + \int_0^T\int_\Omega \nabla u\nabla\varphi + \frac{1}{k}\int_0^T\int_\Gamma u\varphi_t = 0 \tag{67}$$

for all $\varphi \in C_c^\infty((0,T)\times\overline{\Omega})$.

Theorem 5. *(See [33, first part of Theorem 1].) If $N \geqslant 2$ there is $u_0 \in C^\infty(\overline{\Omega})$ satisfying the compatibility conditions*

$$\Delta^n u_0 = k\big(\Delta^{n-1}u_0\big)_\nu \quad \text{on } \Gamma \text{ for all } n \in \mathbb{N},$$

such that problem (64) *has no weak solutions.*

The first step in our analysis is the following

Lemma 5. *Theorem* 5 *holds also if* (65) *is weakened to*

$$u \in C_w\big([0,T); H^1(\Omega)\big), \tag{68}$$

that is it concerns also weakly continuous solutions.

Proof. Looking at the proof in the quoted paper one immediately sees that the continuity of u was used only at two places: at first in order that (67) makes sense, and at second to recognize that the functions $t \mapsto \langle u(t), \Phi_n'\rangle$ are continuous in $[0,T)$, where $\langle\cdot,\cdot\rangle$ denotes an equivalent scalar product in $H^1(\Omega)$ and Φ_n', $n \in \mathbb{N}$, are eigenfunctions of a suitable eigenvalue problem, which belong to $C^\infty(\overline{\Omega})$. Both facts continue to hold when (65) is weakened to (68). □

We can now state the main result of this section.

Theorem 6. *Let $u_0 \in H^1(\Omega)$ be an initial datum such that problem* (64) *has no weak solutions $u \in C_w([0,T); H^1(\Omega))$ for any $T > 0$, and denote by u^l the solution of* (1) *corresponding to u_0 and l given by Theorem* 1.[1] *Then, for any $T > 0$, we have*

$$\max_{t\in[0,T]} \big\|u^l(t)\big\|_{H^1(\Omega)} \to \infty \quad \text{as } l \to 0^+. \tag{69}$$

Proof. We suppose by contradiction that (69) fails for some fixed $T > 0$. Then there is a sequence $l_n \to 0^+$, such that

$$\big\|u^n\big\|_{C([0,T];H^1(\Omega))} \leqslant C_{10} \quad \text{for all } n \in \mathbb{N}, \tag{70}$$

where we denoted $u^n = u^{l_n}$ for simplicity, and $C_{10} = C_{10}(T, u_0, \Omega) > 0$. Since, by Theorem 1, we have $u^n \in C^\infty((0,\infty)\times\overline{\Omega})$ we are allowed, for any $t \in (0,T)$, to multiply the heat equation by a test function $v \in C_c^\infty(\Omega)$ and to integrate by parts to get

$$\left|\int_\Omega u_t^n(t)v\right| \leqslant \big\|u^n(t)\big\|_{H^1(\Omega)}\|\nabla v\|_2. \tag{71}$$

[1] Which is real-valued since u_0 is real-valued.

By a standard density argument (71) holds true for all $v \in H_0^1(\Omega)$ and then, by (70), we get the second estimate we need, that is

$$\|u_t^n\|_{L^\infty((0,T);H^{-1}(\Omega))} \leqslant C_{10} \quad \text{for all } n \in \mathbb{N}. \tag{72}$$

By (70) we get that, up to a subsequence,

$$u^n \to u \quad \text{weakly}^* \text{ in } L^\infty\big((0,T);H^1(\Omega)\big). \tag{73}$$

Moreover, by combining (70) with (72) and using the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ we also get, by the Aubin–Lions compactness lemma, that

$$u^n \to v \quad \text{strongly in } C\big([0,T];H^{-1}(\Omega)\big). \tag{74}$$

Moreover, by (73) we get that $u^n \to u$ weakly* in $L^\infty((0,T);H^{-1}(\Omega))$, while by (74) $u^n \to v$ in the same sense, so $u = v$. Then we can combine (73)–(74) to

$$u^n \to u \quad \text{weakly}^* \text{ in } L^\infty\big((0,T);H^1(\Omega)\big) \text{ and strongly in } C\big([0,T];H^{-1}(\Omega)\big). \tag{75}$$

We now claim that u is a weak solution of (64) in the class (68). Once this claim is proved the proof is complete, since we are in contradiction with Theorem 5 as extended by Lemma 5. By applying [30, Theorem 2.1], we get

$$u \in C_w\big([0,T];H^1(\Omega)\big). \tag{76}$$

Moreover, by (75) it immediately follows that (66) holds. Multiplying the heat equation in (1) by a test function $\psi \in C_c^\infty(0,T) \times \overline{\Omega}$, integrating by parts in Ω, using the boundary condition in (1), using (11) and finally integrating by parts in time twice we get that u^n satisfies the distribution identity

$$-\int_0^T\int_\Omega u^n\varphi_t + \int_0^T\int_\Omega \nabla u^n \nabla\varphi + \frac{1}{k}\int_0^T\int_\Gamma u^n\varphi_t - \frac{l_n}{k}\int_0^T\int_\Gamma u^n \Delta_\Gamma\varphi = 0 \tag{77}$$

for all $\varphi \in C_c^\infty((0,T) \times \overline{\Omega})$. By (75) we can pass to the limit as $n \to \infty$ in (77), and we get that u satisfies (67), completing the proof of our claim. □

Remark 2. Theorem 6 shows that the instability property of $\lambda_0 = \lambda_0(l)$ pointed out in Remark 1 does not depend on our estimates obtained in Lemma 4. Indeed, suppose by contradiction that there is $\tilde\lambda_0 = \tilde\lambda_0(l,k,\Omega,N) > 0$ such that for all $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geqslant \tilde\lambda_0$ we have

$$\operatorname{Re} a_\lambda(u,u) \geqslant C_5\|u\|_V^2 \quad \text{for all } u \in V, \quad \text{and} \quad \varliminf_{l\to 0^+} \tilde\lambda_0 < \infty$$

when k, Ω and N are fixed. Then there is a sequence $l_n \to 0^+$ such that $\tilde\lambda_0(l_n) \leqslant \bar\lambda < \infty$ for all $n \in \mathbb{N}$. By repeating our proofs with $\tilde\lambda_0$ instead of λ_0 we get

$$\|u^{l_n}(t)\|_{H^1(\Omega)}^2 \leqslant e^{2\bar\lambda T}\|u_0\|_H^2$$

for $t \in [0,T]$, $T > 0$ fixed, which contradicts (69).

6. Open problems and final remarks

Although Theorems 1–2 give existence and uniqueness of solutions to problem (1) in a Hilbert framework, building a satisfactory theory for C^∞ initial data, many interesting problems are still open, both of theoretical and applied nature.

1. We are not able to produce a satisfactory regularity theory in even order spaces H^{2n}, $n \geqslant 1$, which is particularly bad for $n = 1$. A new estimate in V would be necessary.
2. The extension of the analysis to more general problems, like the ones considered by [9,10] has still to be done. In particular, Lemma 3 has to be properly extended.
3. Our arguments, which are based on Lax–Milgram theorem, cannot be extended to the case of Banach spaces. Now, it would be natural to consider the case of $u_0 \in W^{1,p}(\Omega)$, $u_{0|\Gamma} \in W^{1,p}(\Gamma)$.
4. We do not give explicit representation formulas of the solution u, even for regular data. In particular we are not able to apply the Fourier method, which would be based on the study of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -k u_\nu - l\Delta_\Gamma u = \lambda u & \text{on } \Gamma. \end{cases} \tag{78}$$

The elliptic theory developed in Section 3 allows to prove in a simple way that

$$\Sigma := \{\lambda \in \mathbb{C}\colon \text{(78) has a nontrivial solution } u \in V\}$$

is at most countable, but it is far from giving an exhaustive spectral theory, since the operator A or equivalently A_λ is not symmetric in H. Actually formula (26) suggests some symmetry of the operator A_λ, but in a framework of Krein spaces. This study is left to specialists in Krein spaces theory.

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