Derived Eigenvalues of Symmetric Matrices, with Applications to Distance Geometry*

A. Neumaier
Institut für Angewandte Mathematik
Universität Freiburg
D-7800 Freiburg, West Germany

Submitted by Richard A. Brualdi

ABSTRACT

The concept of derived eigenvalues of a symmetric matrix is introduced and applied to give a new characterization for the embeddability of a finite metric space into Euclidean space. The particular case of two-distance sets is discussed in more detail.

1. DERIVED EIGENVALUES

In this section we define the concept of derived eigenvalues of a symmetric matrix and discuss their fundamental properties. The results are then applied in Sections 2 and 3 to adjacency matrices of graphs and to distance matrices of two-distance sets in Euclidean space.

We shall repeatedly use the interlacing property of eigenvalues of symmetric matrices (see e.g. Parlett [11], and see Haemers [3] and Cvetković et al. [2] for applications to graphs). We denote by A^{adj} the *adjoint matrix* of A, whose matrices are the appropriately signed maximal minors of A; thus Cramer's rule takes the form

$$A^{-1} = \frac{1}{\det A} A^{\mathrm{adj}} \qquad \text{if } A \text{ is nonsingular.}$$

Furthermore, I denotes the identity matrix, J the all-one matrix, and j the

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all-one vector of arbitrary size. We begin by proving some simple matrix relations.

LEMMA 1.1. *Let* $A \in \mathbb{R}^{n\times n}$, $u, v \in \mathbb{R}^n$. *Then*

$$\det\begin{pmatrix} A & v \\ u^T & \omega \end{pmatrix} = \omega \det A - u^T A^{\mathrm{adj}} v, \tag{1}$$

$$\det\begin{pmatrix} A & v \\ u^T & \omega \end{pmatrix} = \omega \det A + \det\begin{pmatrix} A & v \\ u^T & 0 \end{pmatrix}, \tag{2}$$

$$\det(A + \omega J) = \det A - \omega \det\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix}, \tag{3}$$

$$\det\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} = -(j^T A^{-1} j)\det A \qquad \textit{if } A \textit{ is nonsingular}. \tag{4}$$

Proof. If we expand the determinant of $\begin{pmatrix} A & v \\ u^T & \omega \end{pmatrix}$ by the last column, we get (1). Since this formula is linear in ω, (2) follows. From this we get

$$\begin{aligned} \det(A + \omega J) &= \det\begin{pmatrix} A + \omega J & j \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} A & j \\ -\omega j^T & 1 \end{pmatrix} \\ &= \det A + \det\begin{pmatrix} A & j \\ -\omega j^T & 0 \end{pmatrix} \\ &= \det A - \omega \det\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix}. \end{aligned}$$

Hence (3) holds. Finally, (4) follows from (1) and Cramer's rule. ■

THEOREM 1.2. *Let A be a symmetric* $n \times n$ *matrix. Then the polynomial*

$$P_A(\xi) = \det\begin{pmatrix} \xi I - A & j \\ j^T & 0 \end{pmatrix} \tag{5}$$

has degree $n-1$, *and the coefficients of* ξ^{n-1} *and* ξ^{n-2} *are* $-n$ *and* $j^T A j - n \operatorname{tr} A$, *respectively. The zeros of* $P_A(\xi)$ *are real and interlace the eigenvalues of A. In other words, if we arrange the eigenvalues* θ_i *of A in decreasing order so that* $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$ *(counting each eigenvalue ac-*

cording to its multiplicity), then there are numbers θ_i' *with* $\theta_i \geqslant \theta_i' \geqslant \theta_{i+1}$ $(1 \leqslant i < n)$ *such that*

$$\prod_{i=1}^{n-1} (\xi - \theta_i') = -\frac{1}{n} P_A(\xi), \tag{6}$$

$$\sum_{i=1}^{n-1} \theta_i' = \operatorname{tr} A - \frac{1}{n} j^T A j. \tag{7}$$

Proof. As a symmetric matrix A has a spectral decomposition of the form $A = QDQ^{-1}$ with a diagonal matrix $D = \operatorname{Diag}(\theta_1, \dots, \theta_n)$ whose entries are the eigenvalues of A and an orthogonal matrix Q whose columns are eigenvectors of A. By (1) we have

$$P_A(\xi) = \det\begin{pmatrix} \xi I - A & j \\ j^T & 0 \end{pmatrix} = -j^T(\xi I - A)^{\mathrm{adj}} j$$
$$= -j^T Q(\xi I - D)^{\mathrm{adj}} Q^{-1} j,$$

and with $a := Q^{-1} j = Q^T j$ we get

$$P_A(\xi) = -\sum_{i=1}^{n} a_i^2 \prod_{j \neq i} (\xi - \theta_j). \tag{8}$$

Thus $P_A(\xi)$ is a polynomial of degree $\leqslant n-1$ with leading coefficient $-\sum a_i^2 = -a^T a = -j^T Q Q^T j = -j^T j = -n$, and the coefficient of ξ^{n-2} is

$$\sum a_i^2 \Big(\sum_{j \neq i} \theta_j \Big) = \sum a_i^2 (\theta_i - \operatorname{tr} A) = a^T D a - a^T a \operatorname{tr} A$$
$$= j^T Q D Q^{-1} j - n \operatorname{tr} A = j^T A j - n \operatorname{tr} A.$$

In particular, this implies (6) and (7) with possibly complex numbers θ_i', the zeros of $P_A(\xi)$. We now show that the θ_i' are real and interlace the θ_i. If we divide (8) by the polynomial $\det(\xi I - A) = \prod_{j=1}^{n} (\xi - \theta_i)$, we get the rational function

$$p_A(\xi) := \frac{P_A(\xi)}{\det(\xi I - A)} = -\sum_{i=1}^{n} \frac{a_i^2}{\xi - \theta_i}. \tag{9}$$

This formula shows that if an eigenvalue θ of A of multiplicity f is a pole of $p_A(\xi)$, then θ is a zero of $P_A(\xi)$ of multiplicity $f-1$ if $f>1$ and not a zero if $f=1$. And if θ is not a pole of $p_A(\xi)$, then it is a zero of $P_A(\xi)$ of multiplicity f. Moreover, between any consecutive poles $\theta<\theta'$ of $p_A(\xi)$ the value of $p_A(\xi)$ increases continuously and monotonically from $-\infty$ to $+\infty$, so that there is precisely one zero of $P_A(\xi)$ between θ and θ'. In this way we find $n-1$ real zeros of $P_A(\xi)$. Hence all zeros of $P_A(\xi)$ are real, and they interlace the eigenvalues of A. ∎

We call the zeros θ_i' $(1\leqslant i<n)$ of $P_A(\xi)$ the *derived eigenvalues* of the symmetric $n\times n$ matrix A. The largest and smallest derived eigenvalue of A are denoted by $\theta'_{\max}(A)=\theta_1'$ and $\theta'_{\min}(A)=\theta'_{n-1}$, respectively. We shall see that the derived eigenvalues behave very much like ordinary eigenvalues.

PROPOSITION 1.3. *Let θ be an f-fold eigenvalue of A. Then θ is an f-fold derived eigenvalue of A if the eigenspace of θ is orthogonal to j, and an $(f-1)$-fold derived eigenvalue of A otherwise.*

Proof. Continuing with the notation of the previous proof, we observe that θ is not a pole of $p_A(\xi)$ precisely when $a_i=0$ for every i with $\theta_i=\theta$, i.e. iff $x^Tj=0$ for every x in the space spanned by the ith columns of Q where $\theta_i=\theta$. But this space is just the eigenspace of θ. ∎

COROLLARY 1.4. *If A has constant row sums k, then the derived eigenvalues of A are precisely those eigenvalues of A which are distinct from k (with the same multiplicities) and, in addition, if k is an eigenvalue of A of multiplicity $f>1$, the number k with multiplicity $f-1$.*

Proof. In this case $Aj=kj$, so that j is an eigenvalue for the eigenvalue $\theta_1=k$, and all other eigenspaces are orthogonal to j. ∎

PROPOSITION 1.5. *Let $\xi\in\mathbb{R}$. Then*

$$\xi>\theta'_{\max}(A)\quad\Rightarrow\quad(-1)^{n-1}\det\begin{pmatrix}A-\xi I & j\\ j^T & 0\end{pmatrix}<0, \tag{10}$$

$$\xi<\theta'_{\min}(A)\quad\Rightarrow\quad\det\begin{pmatrix}A-\xi I & j\\ j^T & 0\end{pmatrix}<0. \tag{11}$$

Proof. Use (6), and note that

$$\det\begin{pmatrix} A-\xi I & j \\ j^T & 0 \end{pmatrix} = (-1)^{n-1}\det\begin{pmatrix} \xi I - A & j \\ j^T & 0 \end{pmatrix}. \tag{12}$$

■

PROPOSITION 1.6. *A number $\xi \in \mathbb{R}$ is a derived eigenvalue of the symmetric matrix A iff $\det(A - \xi I + \omega J)$ is independent of ω.*

Proof. Use (3) and (12). ■

COROLLARY 1.7. *If θ' is a derived eigenvalue of the symmetric matrix A, then, for $p, q, r \in \mathbb{R}$ and $p \neq 0$, the number $p\theta' + q$ is a derived eigenvalue of $pA + qI + rJ$.* ■

THEOREM 1.8. *Let A be a symmetric $n \times n$ matrix with $n > 2$, and let A^- be the principal submatrix of order $n-1$ obtained from A by deleting for some i the ith row and column of A. Then the derived eigenvalues of A^- interlace those of A.*

Proof. For $t \in \mathbb{R}$, denote by $\theta_i(t)$ and $\theta_i^-(t)$ the ith largest eigenvalues of the matrices

$$\begin{pmatrix} A & tj \\ tj^T & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^- & tj \\ tj^T & 1 \end{pmatrix},$$

respectively. By Cauchy's interlacing theorem, the ith largest eigenvalue θ_i of A lies between $\theta_i(t)$ and $\theta_{i+1}(t)$; hence $\theta_{i+1} \leqslant \theta_{i+1}(t) \leqslant \theta_i$ for $1 \leqslant i < n$. Since

$$\det\begin{pmatrix} A - \theta_{i+1}(t)I & tj \\ tj^T & 1 - \theta_{i+1}(t) \end{pmatrix} = t^2 \det\begin{pmatrix} A - \theta_{i+1}(t)I & j \\ j^T & t^{-2}[1-\theta_{i+1}(t)] \end{pmatrix},$$

every accumulation point of $\theta_{i+1}(t)$ for $t \to \infty$ must be a derived eigenvalue, and since there is only one in the interval $[\theta_{i+1}, \theta_i]$, we must have $\lim_{t\to\infty}\theta_{i+1}(t) = \theta_i'$. Again by Cauchy's interlacing theorem, the $\theta_i^-(t)$ interlace the $\theta_i(t)$, and for $t \to \infty$ we find that the derived eigenvalues of A^- interlace those of A. ■

There is also a relation between derived eigenvalues of A and the signature of $\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix}$. Denote by $\pi(A)$ the number of positive eigenvalues of a symmetric matrix A, each eigenvalue counted according to its multiplicity. We need the following auxiliary result:

LEMMA 1.9. *Let A be a symmetric matrix with $\pi(A) = i$. Then*

$$\pi\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} = \begin{cases} i+1 & \text{if } (-1)^{n-i}\det\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} > 0, \\ i & \text{otherwise}. \end{cases}$$

Proof. Clearly, $\pi(A - \xi I)$ is the number of eigenvalues $> \xi$ of A. Since the eigenvalues of A interlace those of $\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix}$, we see that $\pi(A - \xi I) = l$ implies that

$$\pi\begin{pmatrix} A - \xi I & j \\ j^T & -\xi \end{pmatrix} = \begin{cases} l+1 & \text{if } (-1)^{n-i}\det\begin{pmatrix} A - \xi I & j \\ j^T & -\xi \end{pmatrix} > 0, \\ = l & \end{cases}$$

otherwise. Specializing to $\xi = 0$, $l = i$ proves the claim. ■

PROPOSITION 1.10. *Let A be a symmetric matrix. Then*

$$\pi\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} = 0 \quad \Leftrightarrow \quad \theta'_{\max}(A) = \theta_{\max}(A) = 0, \tag{13}$$

$$\pi\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} \leqslant 1 \quad \Leftrightarrow \quad \theta'_{\max}(A) \leqslant 0. \tag{14}$$

Proof. (i) By Lemma 1.9,

$$\pi\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} = 0 \quad \text{iff} \quad \pi(A) = 0 \text{ and } (-1)^n \det\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} \leqslant 0.$$

Since $\pi(A) = 0$ forces $\theta'_{\max}(A) \leqslant \theta_{\max}(A) \leqslant 0$, Proposition 1.5 gives the equivalence (13).

(ii) By Lemma 1.9,

$$\pi\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} \leqslant 1,$$

holds iff either $\pi(A)=0$, or

$$\pi(A)=1 \quad \text{and} \quad (-1)^{n-1}\det\begin{pmatrix} A & j \\ j^T & 0 \end{pmatrix} \leqslant 0.$$

Since $\theta_2 \leqslant \theta'_{\max}(A) \leqslant \theta_{\max}(A)$, the equivalence (14) again follows from Proposition 1.5. ■

The preceding result has an important application to distance geometry. The *distance matrix* of a sequence of n vectors $x^{(1)},\dots,x^{(n)} \in \mathbb{R}^m$ is the symmetric $n \times n$ matrix C whose entries are the *squared distances*

$$c_{ik} = \left(x^{(i)} - x^{(k)}, x^{(i)} - x^{(k)}\right) = \sum_{j=1}^{m} \left(x_j^{(i)} - x_j^{(k)}\right)^2.$$

Menger [6], Schoenberg [12], and Seidel [13] derived necessary and sufficient conditions for a matrix C to be the distance matrix of a sequence of vectors in Euclidean space; see also Neumaier [7, 8] for the spherical and the non-Euclidean case, respectively. Here we give a new such condition.

THEOREM 1.11. *A symmetric $n \times n$ matrix C is the distance matrix of a sequence of n vectors in $\mathbb{R}^m$ iff $c_{ii}=0$ $(i=1,\dots,n)$, $\theta'_{\max}(C) \leqslant 0$, and the multiplicity of 0 as a derived eigenvalue of C is $\geqslant n-1-m$.*

Proof. By Proposition 1.10,

$$\theta'_{\max}(C) \leqslant 0 \qquad \text{iff} \quad \pi\begin{pmatrix} C & j \\ j^T & 0 \end{pmatrix} \leqslant 1.$$

Moreover, by definition of derived eigenvalues,

$$\operatorname{rk}\begin{pmatrix} C & j \\ j^T & 0 \end{pmatrix} \leqslant m+2$$

precisely when 0 is a derived eigenvalue of C of multiplicity $\geqslant n-1-m$. Thus the theorem is reduced to Menger's characterization in [6] by the conditions

$$c_{ii}=0 \quad (i=1,\dots,n), \qquad \pi\begin{pmatrix} C & j \\ j^T & 0 \end{pmatrix} \leqslant 1, \qquad \operatorname{rk}\begin{pmatrix} C & j \\ j^T & 0 \end{pmatrix} \leqslant m+2. \quad \blacksquare$$

2. THE DERIVED SPECTRUM OF GRAPHS

In this section we specialize the concept of derived eigenvalues to adjacency matrices of graphs. Let Γ be a graph (finite, undirected, without loops or multiple edges) with adjacency relation $\sim$. The *adjacency matrix* of Γ is the matrix A indexed by the pairs of vertices of Γ whose (x,y) entry A_{xy} equals 1 when $x \sim y$ and 0 otherwise. The (derived) eigenvalues of A are called the *(derived) eigenvalues* of Γ, and the list of (derived) eigenvalues is called the *(derived) spectrum* of Γ. When giving a specific (derived) spectrum, multiplicities are written as exponents.

It is well known (see e.g. Cvetković et al. [2]) that the eigenvalues of a graph reflect many of its properties. Here we show that the derived eigenvalues of a graph are useful invariants, too.

THEOREM 2.1. *Let Γ be a graph with n vertices, eigenvalues $\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n$, and derived eigenvalues $\theta'_1 \geqslant \cdots \geqslant \theta'_{n-1}$. Then:*

(i) *The derived eigenvalues of Γ interlace the eigenvalues of Γ:*

$$\theta_i \geqslant \theta'_i \geqslant \theta_{i+1} \qquad (1 \leqslant i < n).$$

(ii) *The average valency of Γ equals $-\sum_{i=1}^{n-1}\theta'_i$.*
(iii) *If Γ is regular of valency k, then*

$$\theta'_i = \theta_{i+1} \qquad (1 \leqslant i < n).$$

(iv) *The derived eigenvalues of the complement $\bar{\Gamma}$ of Γ are the numbers*

$$\bar{\theta}'_i = -1-\theta'_{n-i} \qquad (1 \leqslant i < n).$$

(v) *The derived eigenvalues of the subgraphs $\Gamma\backslash\{x\}$ (obtained by deleting from Γ the vertex x and all edges containing x) interlace the derived eigenvalues of Γ.*

Proof. (i) and (ii) follow from Theorem 1.2 by noting that $\mathrm{tr}(A) = 0$ and $j^T Aj$ is the total number of ordered pairs of adjacent vertices of Γ. (iii) follows from Corollary 1.4; (iv) from Corollary 1.7, since $\overline{\Gamma}$ has adjacency matrix $J - I - A$; and (v) from Theorem 1.8. ■

We now compute the derived spectrum of some particular graphs. A useful tool is the following result on coclique extensions. A *coclique extension* of a graph Γ with *weights* w_x ($x \in \Gamma$) is a graph Γ_0 which can be partitioned into cocliques C_x ($x \in \Gamma$) such that two vertices $a, b \in \Gamma_0$ are adjacent iff $a \in C_x$, $b \in C_y$ with $x \sim y$.

PROPOSITION 2.2. *Let Γ_0 be a coclique extension of Γ with weights w_x ($x \in \Gamma$). In terms of the adjacency matrix A of Γ and the diagonal matrix W with diagonal entries $W_{xx} = w_x$ ($x \in \Gamma$), the nonzero derived eigenvalues of Γ_0 are precisely the nonzero roots of the polynomial*

$$P_{A,w}(\xi) = \det\begin{pmatrix} \xi I - AW & j \\ j^T W & 0 \end{pmatrix}.$$

Proof. We first note that for I and j of dimension $s \times s$ and s, respectively,

$$\det\begin{pmatrix} \xi I & ja^T \\ bj^T & B \end{pmatrix} = \det \xi I \det\left[B - bj^T(\xi I)^{-1} ja^T\right]$$

$$= \xi^s \det(B - bs\xi^{-1}a^T)$$

$$= \xi^{s-1} \det\begin{pmatrix} \xi & a^T \\ bs & B \end{pmatrix}.$$

We apply this identity to each diagonal block of the matrix

$$\begin{pmatrix} \xi I - A_0 & j \\ j^T & B \end{pmatrix},$$

where A_0 is the adjacency matrix of Γ_0, partitioned into blocks corresponding to the cocliques C_x ($x \in \Gamma$) of size w_x. The result is

$$\det\begin{pmatrix} \xi I - A_0 & j \\ j^T & 0 \end{pmatrix} = \xi^{\Sigma(w_x - 1)} \det\begin{pmatrix} \xi I - AW & j \\ j^T W & 0 \end{pmatrix}.$$

Now the assertion follows from the definition of derived eigenvalues. ■

PROPOSITION 2.3.

(i) *The void graph nK_1 with n vertices has derived spectrum $\Theta' = 0^{n-1}$.*

(ii) *The complete graph K_n with n vertices has derived spectrum $\Theta' = (-1)^{n-1}$.*

(iii) *The complete bipartite graph $K_{m,n}$ with classes of size m and n has derived spectrum*

$$\Theta' = 0^{m+n-2}\left(\frac{-2mn}{m+n}\right)^1.$$

(iv) *The disjoint union $K_m + K_n$ of an m-clique and an n-clique has derived spectrum*

$$\Theta' = \left(\frac{2mn}{m+n} - 1\right)^1 (-1)^{m+n-2}.$$

(v) *The derived spectrum of a complete multipartite graph with t_i classes of size m_i ($i = 1, \dots, s$), where $m_1 < m_2 < \cdots < m_s$, consists of*

$$0^{\Sigma t_i(m_i - 1)}(-m_1)^{t_1 - 1} \cdots (-m_s)^{t_s - 1}$$

together with the zeros of the rational function $\Sigma_{i=1}^s t_i m_i/(\xi + m_i)$.

(vi) *The derived spectrum of the disjoint union of t_i m_i-cliques ($i = 1, \dots, s$), where $m_1 < m_2 < \cdots < m_s$, consists of*

$$(m_s - 1)^{t_s - 1} \cdots (m_1 - 1)^{t_1 - 1}(-1)^{\Sigma t_i(m_i - 1)}$$

together with the zeros of the rational function $\Sigma_{i=1}^s t_i m_i/(m_i - 1 - \xi)$.

Proof. (i) and (iii) are special cases of (v), and (ii), (iv), (vi) are obtained from (i), (iii), and (v) by applying Theorem 2.1(iv). Thus it suffices to prove (v). But a complete multipartite graph is a coclique extension of a complete graph Γ (with adjacency matrix $A = J - I$). Since

$$\det\begin{pmatrix} \xi I - AW & j \\ j^T W & 0 \end{pmatrix} = \det\begin{pmatrix} \xi I - AW + jW & j \\ j^T W & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} \xi I + W & j \\ j^T W & 0 \end{pmatrix}$$

$$= -j^T W(\xi I + W)^{-1} j \det(\xi I + W)$$

$$= -\left(\sum_{i=1}^{s} \frac{t_i m_i}{\xi + m_i} \right) \prod_{i=1}^{s} (\xi + m_i)^{t_i},$$

the result follows from Proposition 2.2. ■

Note that by (ii) and (iv), the derived spectrum of a disjoint union of graphs does not have a simple relationship with that of its components.

The previous proposition covers most graphs with at most four vertices, and by computing the missing cases directly we get the results in Table 1.

The following proposition can be considered as a first (simple) step in the classification of graphs whose smallest derived eigenvalue is bounded from below. See Cvetković et al. [2], Hoffman [4], and Neumaier and Bussemaker [10] for discussions of the corresponding problem for ordinary eigenvalues.

PROPOSITION 2.4. *Let Γ be a graph. Then:*

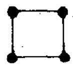
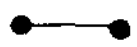
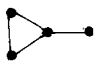
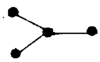
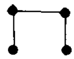
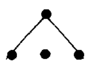
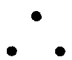
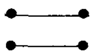
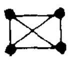
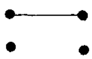
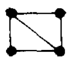
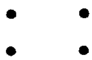
$$\theta'_{\min}(\Gamma) \begin{cases} = 0 & \Leftrightarrow \quad \Gamma \textit{ is empty}, \\ = -1 & \Leftrightarrow \quad \Gamma \textit{ is nonempty, a disjoint union of cliques}, \\ = -\frac{4}{3} & \Leftrightarrow \quad \Gamma \textit{ is a path with three vertices}, \\ \leqslant \frac{1}{2}(-1-\sqrt{3}) & \textit{otherwise}. \end{cases}$$

Proof. By Theorem 2.1(v) and Table 1, if Γ contains an edge, then $\theta'_{\min}(\Gamma) \leqslant -1$, and if Γ contains an induced path with three vertices, then $\theta'_{\min}(\Gamma) \leqslant -\frac{4}{3}$. But adding another vertex (and some edges) to such a path already yields $\theta'_{\min}(\Gamma) \leqslant \frac{1}{2}(-1-\sqrt{3})$. This implies the assertion. ■

I should like to challenge the reader to classify the graphs whose smallest derived eigenvalue is > -2. These graphs cannot contain induced circuits of even length; thus their structure must be very restricted and can probably be determined completely.

TABLE 1

THE DERIVED SPECTRUM OF GRAPHS WITH AT MOST FOUR VERTICES

	$\varnothing$		$0^2(-2)^1$
	$(-1)^1$		$\left(\frac{-1+\sqrt{3}}{2}\right)^1(-1)^1\left(\frac{-1-\sqrt{3}}{2}\right)^1$
	0^1		$0^2(-\frac{3}{2})^1$
	$(-1)^2$		$\left(\frac{-1+\sqrt{5}}{2}\right)^1\left(-\frac{1}{2}\right)^1\left(\frac{-1-\sqrt{5}}{2}\right)^1$
	$0^1(-\frac{4}{3})^1$		$(\frac{1}{2})^1(-1)^2$
	$(\frac{1}{3})^1(-1)^1$		$\left(\frac{-1+\sqrt{3}}{2}\right)^1 0^1\left(\frac{-1-\sqrt{3}}{2}\right)^1$
	0^2		$1^1(-1)^2$
	$(-1)^3$		$(\frac{1}{2})^1 0^1(-1)^1$
	$0^1(-1)^1(-\frac{3}{2})^1$		0^3

3. GRAPHS AND TWO-DISTANCE SETS

We now discuss the relations between derived eigenvalues of graphs and two-distance sets in Euclidean space.

A set S of vectors in $\mathbb{R}^n$ is called a *two-distance set* (cf. Larman et al. [5], Neumaier [8, 9], Blokhuis [1]) if the squared distance between pairs of distinct vectors in S takes precisely two distinct values α, β ($\alpha > \beta > 0$). The graphs $\Gamma_+(S)$ and $\Gamma_-(S)$ which have as vertices the vectors in S and as edges the pairs at squared distance α and β, respectively, are called the *large-* and *small-distance graphs* of S, respectively. Clearly, the two graphs are complementary, and they are neither empty nor complete.

The number $m = \alpha/(\alpha - \beta)$ is invariant under scaling of S and is called the *type* of S; note that $m > 1$. Multiplication of all vectors of S by

$\sqrt{2/(\alpha-\beta)}$ turns S into a two-distance set with squared distances $\alpha' = 2m$ and $\beta' = 2m-2$; we call such a two-distance set *normalized*. Clearly, the distance graphs are unaffected by normalization.

EXAMPLE 3.1. Let L be a *partial linear space*, i.e. a collection of subsets of a finite set X called *lines* such that every pair of distinct lines have at most one common point. If all lines contain the same number m of points, we may construct a two-distance set S as follows. Let $X = \{a_1, \dots, a_n\}$, and for each line $l \in L$, let $x(l)$ denote the vector whose ith entry is 1 if $a_i \in l$ and 0 otherwise. Then $S = \{x(l) \mid l \in L\}$ is a normalized two-distance set of type m. (In fact, S is a *spherical* two-distance set, since it is contained in the sphere of radius $\sqrt{m}$). The small-distance graph of S is called the *line graph* of L.

THEOREM 3.2. *Let Γ be a graph, neither empty nor complete. Then:*

(i) *Γ is isomorphic to the small-distance graph of a two-distance set of type m iff $\theta'_{min}(\Gamma) \geqslant -m$.*
(ii) *Γ is isomorphic to the large-distance graph of a two-distance set of type m iff $\theta'_{\max}(\Gamma) \leqslant m-1$.*

Proof. Apply Theorem 1.11 to $C = 2m(J-I) - 2A$ and $C = 2(m-1)(J-I) + 2A$, where A is the adjacency matrix of Γ, and simplify using Corollary 1.7. ∎

COROLLARY 3.3. *Let Γ be a graph with v vertices and smallest derived eigenvalue $-m$ of multiplicity f. Then Γ is isomorphic to the small-distance graph of a two-distance set of type m in $\mathbb{R}^{v-1-f}$.*

Proof. Use the dimension statement in Theorem 1.11. ∎

Of course, a similar statement holds for the largest derived eigenvalue.

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