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A Wecken theorem for n-valued maps

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ABSTRACT

In [6] Schirmer (1985) established that, if $\varphi : X \multimap X$ is an n-valued map defined on a compact triangulable manifold of dimension at least three, then the appropriate Nielsen number, $N(\varphi)$, is a *sharp* lower bound for the number of fixed points in the n-valued homotopy class of φ. In this note we generalize this theorem by allowing X to be any compact polyhedron without local cut points and such that no connected component is a two-manifold.

A *multifunction* $\varphi : X \multimap Y$ from a topological space X to a topological space Y is a function $\varphi : X \to \mathcal{P}(Y) - \{\emptyset\}$ where $\mathcal{P}(Y)$ is, as usual, the power set of Y. If X and Y are both compact metric spaces, and φ is a multifunction such that $\varphi(x)$ is a closed subset of Y for each $x \in X$, then φ is continuous if it is continuous as a function into the (compact) metric space of all nonempty closed subsets of Y with the Hausdorff metric. An *n-valued* map, $\varphi : X \multimap Y$ is a continuous multifunction such that $\varphi(x)$ is an unordered subset of exactly n points of Y for each $x \in X$. It is a classical theorem in this area (see [4]) that, if X is a compact, Hausdorff simply connected space, then an n-valued map $\varphi : X \multimap X$ splits into n distinct maps, i.e. $\varphi(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in X$ where each f_i is a single valued map. If X is a compact polyhedron and $\varphi : X \multimap X$ is an n-valued map, then φ is *simplicial* if its restriction to any closed simplex, $\bar{\sigma}$, splits: $\varphi|\bar{\sigma} = \{f_1, \dots, f_n\}$ where each f_i is a single-valued function that maps $\bar{\sigma}$ affinely onto a simplex of X.

In [5] Schirmer defined a Nielsen number, $N(\cdot)$, for n-valued maps, and then in [6] she established the following Wecken [2,7–9] theorem using certain general position arguments to ensure that all the homotopies involved are n-valued.

Theorem 0. *Let X be a compact manifold (with or without boundary) of dimension greater than or equal to three, and let $\varphi : X \multimap X$ be an n-valued map, then there is an n-valued homotopy between φ and an n-valued map with $N(\varphi)$ many fixed points.*

For the general position arguments to work it is crucial that X be a manifold of sufficiently high dimension as stated in Theorem 0.

In the following note we generalize Theorem 0 by allowing X to be a more general type of space. A *local cut point*, x, of a topological space X, is a point which has an open neighborhood, U, such that $U - x$ is disconnected.

Theorem 1. *Let X be a compact polyhedron without local cut points and such that no connected component is a two-manifold. Let $\varphi : X \multimap X$ be an n-valued map. Then there is an n-valued homotopy between φ and an n-valued map with $N(\varphi)$ many fixed points.*

We will establish this theorem by making fundamental use of the fact that the first step in proving Theorem 0 (obtaining an n-valued homotopy between our given n-valued map and a fix-finite one) yields a *simplicial* n-valued map w.r.t. some subdivision of the domain. This will enable us to work with PL paths and we will be able to eliminate intersections of these

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paths to produce the desired n-valued homotopy. In this way we will sidestep the general position arguments originally used by Schirmer.

Recall that the *gap* of an n-valued map $\varphi : X \multimap X$ where X is a metric space is defined as:

$$\gamma(\varphi) := \inf\{d(x_i, x_j):\ x_i \neq x_j,\ x_i, x_j \in \varphi(x),\ x \in X\}.$$

We first establish some preliminary lemmas we shall make use of in the proof of our theorem.

Throughout, X denotes a compact/finite simplicial complex without any local cut points and such that no connected component (in the ordinary point set topology sense) is a two manifold. This last condition is equivalent to the existence (possibly with respect to some subdivision) of a 1-simplex that is the common face of at least three 2-simplexes. Likewise, I will denote the unit interval.

By a PL (piecewise linear) path in X we mean a path $p : I \to X$ which, for some subdivision of L of I, maps each simplex of L affinely into a simplex of X. The image of a vertex of L is called a corner of p. A PL path is normal if (1) it doesn't pass through any vertex of X, (2) it has no multiple self-intersections and it has no self-intersections at its corners, and (3) $p(t)$ is in maximal simplexes of X for all but a finite number of values $t \in I$, and $p(t)$ goes from one maximal simplex into another when t passes across any of these exceptional values. A PL arc is a PL path with different endpoints and without self-intersections.

For the reader's convenience, we recall the definition of Schirmer's Nielsen number for n-valued maps.

Suppose $\varphi : X \multimap X$ is an n-valued map defined on a compact polyhedron. We define two fixed points of φ to be in the same fixed point class (FPC) if there is path $p : I \to X$ from one fixed point to the other, such that one has $\varphi \circ p = \{f_1, \ldots, f_n\}$ with $f_1 \sim p$ (rel. endpoints). One thus obtains a partition of $\mathrm{Fix}(\varphi)$ into finitely many FPCs [6, Theorem 5.2]. To define the index of a FPC, one first obtains an open neighborhood, U, of the FPC with $\mathrm{Fix}(\varphi) \subset \mathrm{int}(U)$, and a sufficiently close fixed finite approximation of the given n-valued map with all fixed points located in maximal simplexes, and no fixed points on $\mathrm{Fr}_X U$. One then defines the index of the FPC to be the sum of the indices of all the fixed points in U; where the index of a fixed point is defined to be its ordinary fixed point index when viewed as a fixed point of one of the maps in the splitting of the restriction of the n-valued map we have obtained to the maximal simplex it is contained in.

Definition 1. Let $\varphi : X \multimap X$ be an n-valued map, and let $p : I \to X$ be an arc. Suppose that $\varphi|p(I) = \{f_1, \ldots, f_n\}$ is a (local) splitting of φ, then define:

$$S^1_{\varphi,p} := \bigcup_{1 \leqslant i,j \leqslant n,\ i \neq j} \big(f_i \circ p(I) \cap f_j \circ p(I)\big)$$

and:

$$S^2_{\varphi,p} := \bigcup_{1 \leqslant i \leqslant n} \big(f_i \circ p(I) \cap p(I)\big)$$

Definition 2. With φ and p as in Definition 1, a point $z \in S^1_{\varphi,p}$ is a multiple intersection point if:

$$z = f_i \circ p(a_1) = f_j \circ p(a_2) = f_k \circ p(a_3)$$

where i, j and k are all distinct.

Lemma 1 *(Creation of a fixed point in a simplex with prescribed characteristics). Let $\varphi : X \multimap X$ be a simplicial (possibly w.r.t. some subdivision of the domain) fixed-finite n-valued map, with all fixed points located in maximal simplexes. Suppose $p : I \to X$ is a normal PL arc (w.r.t. the subdivision of the domain) from one fixed point, x_1, to a point, $x_2 \notin \mathrm{Fix}(\varphi)$ located in a maximal simplex (w.r.t. the subdivision of the domain), σ, that has an edge that is common to at least two other simplexes, and with $\mathrm{Fix}(\varphi) \cap p(I) = \{x_1\}$. Then there is an n-valued homotopy between φ and ϕ where $\mathrm{Fix}(\phi) = \mathrm{Fix}(\varphi) \cup \{x_2\}$.*

Proof. Choose $\epsilon > 0$ small enough so that:

$$U_t = U_t(x_2; \epsilon t) = \{x \in X:\ d(x, x_2) \leqslant \epsilon t\} \subset \sigma$$

and:

$$\varphi(U_1) = \coprod_{i=1}^{n} f_i(U_1) (= V_i) \quad \text{and} \quad \varphi(U_1) \cap U_1 = \emptyset$$

and $p(I) \cap U_1$ is a line segment, where $\varphi|U_1 = \{f_1, \ldots, f_n\}$ is a (local) splitting. Now, $X - \coprod_{1 \leqslant i \leqslant n} V_i$ is path-connected, so there is a PL path $q : I \to X - \coprod_{2 \leqslant i \leqslant n} V_i$ (see Fig. 1) with $q(0) = f_1(x_2)$ s.t. $p \sim q \circ (f_1 \circ p)$ (rel. endpoints). Finally, one applies the following n-valued homotopy:

$$\Phi(x,t) = \begin{cases} \varphi(x) & \text{if } x \notin U_t \\ \{f_1((\frac{2}{\epsilon t} d(x,x_2) - 1)x + (2 - \frac{2}{\epsilon t} d(x,x_2))x_2), f_2(x), \ldots, f_n(x)\} & \text{if } 0 < \frac{\epsilon t}{2} < d(x,x_2) \leqslant \epsilon t \\ \{q(1 - t + \frac{2}{\epsilon} d(x,x_2)), f_2(x), \ldots, f_n(x)\} & \text{if } 0 \leqslant d(x,x_2) \leqslant \frac{\epsilon t}{2} \end{cases}$$

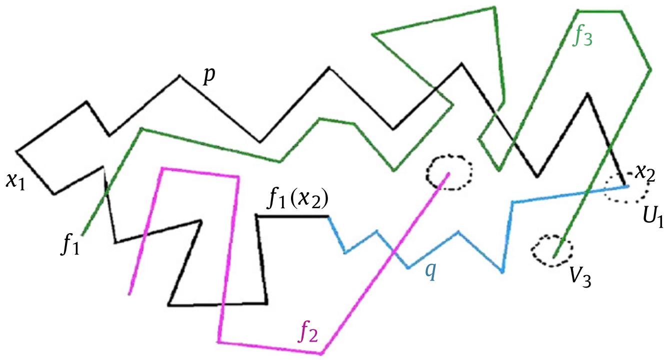

Fig. 1. Creation of fixed point.

Notice that there is a subdivision of I s.t. if $\phi|p(I) = \{f_1', \ldots, f_n'\}$ is a (local) splitting of ϕ, then each $f_i' \circ p$ is a PL path. $\square$

Lemma 2 *(Obtaining finitely many points of intersection of PL paths). Let $\varphi : X \multimap X$ be a fix-finite n-valued map with all fixed points located in maximal simplexes. Let $p : I \to X$ be a normal PL arc from x_1 to x_2, with $\mathrm{Fix}(\varphi) \cap p(I) = \{x_1, x_2\}$ s.t. if $\varphi|p(I) = \{f_1, \ldots, f_n\}$ is a (local) splitting of φ then each $f_i \circ p$ is a PL path and $f_1 \circ p \sim p$ (homotopic rel. endpoints). Then there is a partial special n-valued homotopy defined on $p(I)$ between φ and an n-valued map ϕ such that $S^1_{\phi,p} \cup S^2_{\phi,p}$ is finite.*

Proof. Label the vertices of I with respect to which p is PL as $s_0, \ldots, s_m$ and consider an intersection of the form:

$$f_i \circ p\big([s_l, s_{l+1}]\big) \cap f_j \circ p\big([s_k, s_{k+1}]\big) = f_i \circ p\big([a_1, a_2]\big)$$

or:

$$f_i \circ p\big([s_l, s_{l+1}]\big) \cap p\big([s_k, s_{k+1}]\big) = f_i \circ p\big([a_1, a_2]\big)$$

where $f_i \circ p([a_1, a_2])$ consists of more than one point, and $k \neq l$ if $i = j$. Suppose $f_i \circ p([a_1, a_2]) \subset \overline{\sigma}$, where σ is a maximal simplex, and take any point in σ that doesn't lie on the line ℓ through $f_i \circ p([a_1, a_2])$. Denote by P the plane that contains this point and the line ℓ. Next, consider all the line segments in $\bigcup_{1 \leqslant i \leqslant n} f_i \circ p(I) \cup p(I)$ (a union of PL paths) that cross ℓ at a non-zero angle and are contained in P. Let θ denote the minimum of all the angles these line segments make with ℓ if this set is non-empty (note that a line segment makes two different angles with ℓ). We consider the following cases:

Case 1: Suppose that either $1 \leqslant l < m - 1$ and $1 \leqslant i \leqslant n$, or $l \in \{0, m-1\}$ and $i \neq 1$ in $f_i \circ p([s_l, s_{l+1}])$. Pick $y \in (P - \ell) \cap \sigma$ such that (see Fig. 2):

(i) if θ_1 is the angle made between the line segment $[f_i \circ p(a_1), y]$ and $f_i \circ p([a_1, a_2])$, θ_2 is the angle between the line segment $[f_i \circ p(a_2), y]$ and $f_i \circ p([a_1, a_2])$, then $\max\{\theta_1, \theta_2\} < \theta$
(ii) if L denotes the set of all line segments in $\bigcup_{1 \leqslant i \leqslant n} f_i \circ p(I) \cup p(I)$ that lie in P but don't intersect $f_i \circ p([a_1, a_2])$ then $\mathrm{dist}(y, f_i \circ p([a_1, a_2])) < \mathrm{dist}(L, f_i \circ p([a_1, a_2]))$
(iii) $0 < \epsilon = \mathrm{dist}(y, f_i \circ p([a_1, a_2])) < \min\{\gamma(\varphi), \min\{d(f_i(x), x)\colon\ x \in p([s_l, s_{l+1}])\}\}$ (note that $\min\{d(f_i(x), x)\colon\ x \in p([s_l, s_{l+1}])\} > 0$ since f_i is fixed point free on $p([s_l, s_{l+1}])$.

Case 2: Suppose that $l \in \{0, m-1\}$ and $i = 1$. In this case we are considering an intersection of the form:

$$f_1 \circ p\big([0, s_1]\big) \cap p\big([0, s_1]\big) = f_1 \circ p\big([0, a_1]\big)$$

or:

$$f_1 \circ p\big([s_{m-1}, 1]\big) \cap p\big([s_{m-1}, 1]\big) = f_1 \circ p\big([a_1, 1]\big)$$

Choose y exactly as in Case 1, except that this time:

$$0 < \epsilon = \mathrm{dist}\big(y, f_1 \circ p\big([0, a_1]\big)\big) < \gamma(\varphi)$$

Now in Case 1, define: f_i' to be equal to f_i on $p([0, a_1]) \cup p([a_2, 1])$ and to map $p([a_1, \frac{a_1+a_2}{2}])$ affinely to $[f_i \circ p(a_1), y]$, $p([\frac{a_1+a_2}{2}, a_2])$ affinely to $[y, f_i \circ p(a_2)]$; define f_i' similarly in Case 2. In all cases, there is clearly an ϵ-homotopy between f_i and f_i' that is constant on $p([0, a_1]) \cup p([a_2, 1])$. Now, consider $\Phi : p(I) \times I \multimap X$ given by: $\Phi(x, t) = \{f_1(x), \ldots, f_{i_t}(x), \ldots, f_n(x)\}$. This multimap is n-valued since $\epsilon < \gamma(\varphi)$; it is a *special* n-valued homotopy in Case 1 because $\epsilon < \min\{d(f_i(x), x)\colon\ x \in p([s_l, s_{l+1}])\}$, and for obvious reasons in Case 2.

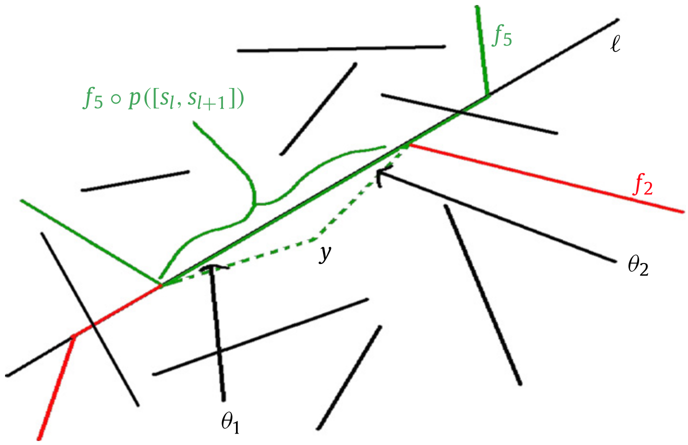

Fig. 2. Finitely many intersections of PL paths.

Repeat the above procedure finitely many times until one obtains ϕ as in the statement of the lemma (this follows from the fact that there are only finitely many intersections of the forms described above). □

Lemma 3 *(Elimination of multiple intersections). Let $\varphi : X \multimap X$ be a fixed finite n-valued map with all fixed points located in maximal simplexes, and let $p : I \to X$ be a normal PL arc from x_1 to x_2 with $\mathrm{Fix}(\varphi) \cap p(I) = \{x_1, x_2\}$. Suppose that, if $\varphi|p(I) = \{f_1, \dots, f_n\}$ is a (local) splitting of φ then each $f_i \circ p$ is a PL path, and $f_1 \circ p \sim p$ (rel. endpoints). Suppose too, that $S^1_{\varphi,p} \cup S^2_{\varphi,p}$ is finite. Then there is a partial special n-valued homotopy defined on $p(I)$ between φ and ϕ where $S^1_{\phi,p} \cup S^2_{\phi,p}$ is finite and there are no multiple intersections in $S^1_{\phi,p}$.*

Proof. Suppose $z \in S^1_{\varphi,p}$ is a multiple intersection. We consider the following cases.

Case 1: Suppose $z \in \sigma$ where σ is a maximal simplex, and $z = f_m \circ p([a_1, a_2])$ where $0 < a_1, a_2 < 1$ and $1 \leqslant m \leqslant n$. Then choose $a_3, a_4 \in (0, 1)$ with $a_3 < a_1 \leqslant a_2 < a_4$ s.t. the following conditions are satisfied:

(i) $f_m \circ p([a_3, a_4]) \subset \sigma$
(ii) $f_m \circ p([a_3, a_4])$ is either a line segment or a union of two line segments meeting at "z"
(iii) $0 < \delta_1 < \mathrm{dist}(f_m \circ p([a_3, a_4]), S^1_{\varphi,p} - \{z\})$.

Now pick $y \in \sigma - (\ell_1 \cup \ell_2)$ (where ℓ_i, $i = 1, 2$ are the lines through $f_m \circ p([a_3, a_1])$ and $f_m \circ p([a_1, a_4])$ respectively — note that it may be the case that $\ell_1 = \ell_2$) s.t. the following conditions are satisfied:

$$\text{(i)}\quad 0 < \delta = d(y, z) < \min\{\gamma(\varphi), \delta_1, \delta_2, \delta_3, \delta_4\}$$

and:

$$\text{(ii)}\quad \max\{\theta_1, \theta_2\} < \min\{\theta'_1, \theta'_2\}$$

where:

$$0 < \delta_2 = \min\{d(p(y), f_m \circ p(y)) \colon y \in [a_3, a_4]\}$$
$$0 < \delta_3 = \mathrm{dist}(L_1, f_m \circ p([a_3, a_4]))$$
$$0 < \delta_4 = \mathrm{dist}(L_2, f_m \circ p\ ([a_3, a_4]))$$

and L_i, $i = 1, 2$ are defined analogously to L in Lemma 2 with P replaced by P_i, the planes containing y and ℓ_i, $i = 1, 2$ respectively, θ'_i, $i = 1, 2$ are defined analogously to θ (in Lemma 2) but considering the planes P_i separately.

Next, define f'_m to be equal to f_m on $p(I - (a_3, a_4))$ and to map $[p(a_3), p(a_1)]$ affinely to $[f_m \circ p(a_3), x]$ and $[p(a_1), p(a_4)]$ affinely to $[x, f_m \circ p(a_4)]$. Obviously, there is a δ-homotopy between f_m and f'_m which yields a special n-valued homotopy between φ and ϕ where ϕ has one less multiple intersection.

Case 2: If $z \in \sigma$ and σ is a non-maximal simplex with $\dim(\sigma) \geqslant 1$, then proceed similarly to Case 1, with $y \in \sigma \cap f_m \circ p([a_3, a_4])$ if (say) $f_m \circ p([a_3, a_4]) \subset \sigma$.

Case 3: If z is a vertex (of X) then let γ be a normal PL arc (w.r.t. some subdivision of $[a_3, a_4]$ as shown in Fig. 3) that is sufficiently close to $f_m \circ p$ (the estimate is similar to the one in Case 1) and define:

$$f'_m(x) = \begin{cases} f_m(x) & \text{if } x \in p(I - (a_3, a_4)) \\ \gamma \circ p^{-1}(x) & \text{if } x \in p([a_3, a_4]) \end{cases}$$

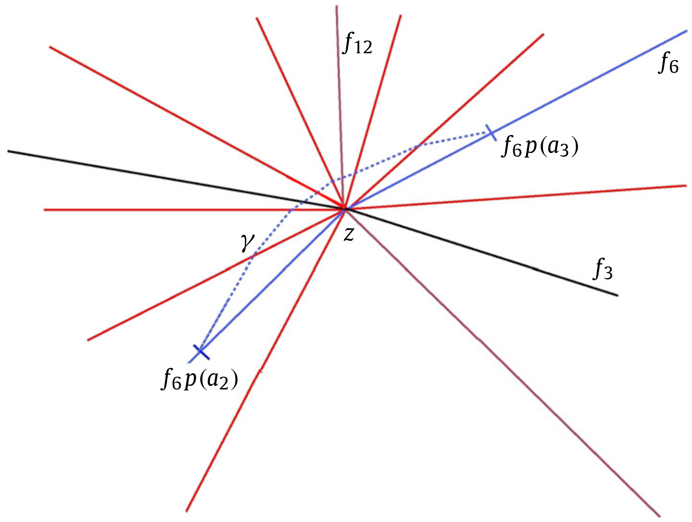

Fig. 3. Elimination of multiple intersections.

Clearly f'_m is homotopic to f_m (by a homotopy with track equal to the above mentioned estimate), and setting: $\phi = \{f_1, \dots, f'_m, \dots, f_n\}$ we obtain an n-valued map with one less multiple intersection.

Finally, repeat the above procedure until all (finitely many) multiple intersections have been eliminated.

If $z = f_m \circ p([0, a_1])$ (or $z = f_m \circ p([a_1, 1])$) then choose $\epsilon > a_1$ s.t. $f_m \circ p(\epsilon)$ is sufficiently close to z and define:

$$H\big(p(x), t\big) = \begin{cases} f_m((1-t)p(x) + tp(\epsilon)) & \text{if } 0 \leqslant x \leqslant \epsilon \\ f_m \circ p(x) & \text{if } \epsilon \leqslant x \leqslant 1 \end{cases}$$

(This is the homotopy between f_m and f'_m in this case.) $\square$

Lemma 4 *(Eliminating intersections with $p(I)$). Let $\varphi : X \multimap X$ be a fixed-finite n-valued map with all fixed points located in maximal simplexes. Let $p : I \to X$ be a normal PL arc from x_1 to x_2 with $\mathrm{Fix}(\varphi) \cap p(I) = \{x_1, x_2\}$ and s.t. if $\varphi|p(I) = \{f_1, \dots, f_n\}$ is a (local) splitting, then each $f_i \circ p$ is a PL path, and $f_1 \circ p \sim p$ (rel. endpoints). Suppose that $S^1_{\varphi,p} \cup S^2_{\varphi,p}$ is finite, and there are no multiple intersections in $S^1_{\varphi,p}$. Then there is a special n-valued homotopy defined on $p(I)$ between φ and ϕ where if $\phi = \{g_1, \dots, g_n\}$ is a splitting of ϕ, then $g_1 \circ p(I) \cap p(I) = \{x_1, x_2\}$ and for all other values of i, $g_i \circ p(I) \cap p(I) = \emptyset$, while $S^1_{\phi,p}$ is finite and there are no multiple intersections in this set.*

Proof. First, suppose $z \in \sigma$ and $z \in f_i \circ p(I) \cap f_j \circ p(I) \cap p(I)$ where $\dim(\sigma) \geqslant 3$. Let $0 < \epsilon_1 = \mathrm{dist}(z, S^1_{\varphi,p} - \{z\})$ and $0 < \epsilon_2 = \frac{\gamma(\varphi)}{2}$. By an $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$ PL perturbation of f_i and f_j in a neighborhood of z we can eliminate this intersection with $p(I)$.

Next, let $p([s_i, s_j])$ be a portion of $p(I)$ s.t. each line segment is contained in a simplex of dimension two, and that is maximal (order by inclusion) with respect to this property.

Case 1: Suppose $[s_i, s_j] \subset (0, 1)$. Consider the line segment $p([s_l, s_{l+1}])$ (where $i \leqslant l \leqslant j-1$). Take a line segment l_l in $\mathrm{carr}(p(s_l))$ centered on $p(s_l)$ and another line segment, l_{l+1} in $\mathrm{carr}(p(s_{l+1}))$ centered on $p(s_{l+1})$ such that the line segments between respective endpoints of l_l and l_{l+1} are both parallel to $p([s_l, s_{l+1}])$. Denote the parallelogram one obtains in this way by P_l. Let T_1 denote an equilateral triangle containing the edge l_i that is contained in the closure of a maximal simplex of dimension greater than two that abuts the simplex containing P_i. Choose T_1 such that $(T_1 - \ell_i) \cap p(I) = \emptyset$. Likewise choose T_2. Let $S_1 = T_1 \cup (\bigcup_{\ell \leqslant i \leqslant j-1} P_l) \cup T_2$ and let $g : S_1 \to S_2 (\subseteq \mathbb{R}^2)$ be a homeomorphism that maps each T_i $(i = 1, 2)$ and each P_l $(i \leqslant l \leqslant j-1)$ affinely as shown in Figs. 4, 5. Next, label the images under g of the intersections of the different $f_j \circ p(I)$ with S_1 as L_i again as shown in Figs. 4, 5. If $L_i = f_j \circ p([x_1, x_2])$, refer to x_1 as the "left endpoint of L_i" and x_2 as the "right endpoint of L_i". Given $z \in L_i \cap g \circ p(I)$ where $z = g \circ f_j \circ p(x_1) = g \circ p(x_2)$ and $x_1 < x_2$ we say L_i can be moved to the right, else we say L_i can be moved to the left. Now, a posteriori, notice that we could have chosen the ℓ_i so small that there are no intersections among the different L_i in $g(S_1 - p(I))$, s.t. after performing any necessary PL perturbations all intersections of L_i with S_1 are bas in Figs. 4, 5, and such that we have the following situation. Consider any L_i that can be moved to the left, then its left endpoint is greater than the right endpoints of all other L_j to the left of this that can be moved to the right. Obviously, with a little care we can ensure that if two different L_i intersect at a point in $p(I)$ and can both be moved to the right, then we can move them off $p([s_i, s_j])$ in such a way that the corresponding f'_k, f'_l we obtain in this way don't map a single point of $p(I)$ to the same point of X. One can now, by performing PL "moves" as shown in Figs. 4, 5, obtain a special n-valued homotopy between $\varphi|p(I)$

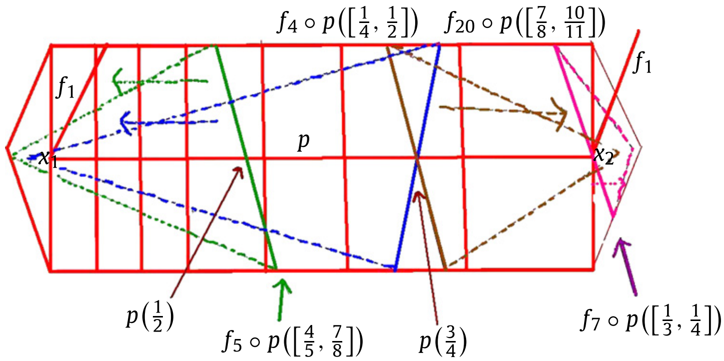

Fig. 4. Eliminating intersections with $p(I)$.

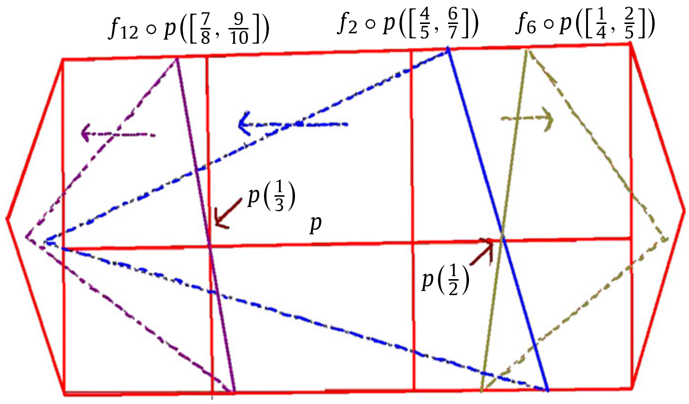

Fig. 5. Eliminating intersections with $p(I)$.

and an n-valued map defined on $p(I)$ such that the images of the usual single-valued maps in its splitting don't intersect $p([s_i, s_j])$.

Case 2: This case is handled similarly. $\square$

Lemma 5 *(Eliminating intersections of PL paths in local splittings). Let $\varphi : X \multimap X$ be a fixed-finite n-valued map with all fixed points located in maximal simplexes. Let $p : I \to X$ be a normal PL arc from x_1 to x_2 with $\mathrm{Fix}(\varphi) \cap p(I) = \{x_1, x_2\}$ and s.t. x_1 is contained in a maximal simplex that has an edge that is common to at least two other simplexes. Suppose that if $\varphi|p(I) = \{f_1, \dots, f_n\}$ is a (local) splitting of φ, then each $f_i \circ p$ is a PL path, and $f_1 \circ p \sim p$. Suppose too that $S^1_{\varphi,p}$ is finite, there are no multiple intersections in $S^1_{\varphi,p}$, and $f_1 \circ p(I) \cap p(I) = \{x_1, x_2\}$ whereas $f_i \circ p(I) \cap p(I) = \emptyset$ for all other values of i. Then there is a special n-valued homotopy defined on $p(I)$ between φ and ϕ where if $\phi = \{g_1, \dots, g_n\}$ is a splitting of ϕ, then $g_1 = f_1$, and for $1 < i \leqslant n$ one has $g_i \circ p(I)$ reduced to a single point.*

Proof. Consider n copies of the unit interval, $I_1, \dots, I_n$. On I_i $(1 \leqslant i \leqslant n)$, label $[a_1, a_2]$ as d_{ij} if there is a point of intersection:

$$z = f_i \circ p([a_1, a_2]) = f_j \circ p([a_3, a_4])$$

where $a_2 < a_3$ if $i = j$ and $[a_1, a_2]$ is maximal w.r.t. this property (order by inclusion). There is naturally an ordering on d_{ij} $(1 \leqslant j \leqslant n)$; Label the kth element as d_{il_k}. Introduce the following equivalence relation on d_{ij_k}:

$$d_{ij_m} \sim d_{ji_l} \quad \text{if } f_i \circ p([a_1, a_2]) = f_j \circ p([a_3, a_4])$$

and: d_{ij_m} corresponds to $[a_1, a_2]$ whilst d_{ji_l} corresponds to $[a_3, a_4]$.

Case 1. Now, suppose that $d_{ij_1} \sim d_{ji_1}$ with $i, j \neq 1$, $i \neq j$ where d_{ij_1} corresponds to $[a_1, a_2]$ and d_{ji_1} corresponds to $[a_3, a_4]$. If $a_2 > a_4$, then letting $\epsilon > 0$ be small enough that $a_4 + \epsilon < a_5$ where d_{ji_2} corresponds to $[a_5, a_6]$, we can "slide" $f_j \circ p([0, a_4 + \epsilon])$ through itself and past this point of intersection without creating any new points of intersection. The homotopy that accomplishes this is given by:

$$f_{j_t}(p(x)) = \begin{cases} f_j \circ p(t(a_4 + \epsilon) + (1 - t)x) & \text{if } x \in [0, a_4 + \epsilon] \\ f_j \circ p(x) & \text{if } x \in [a_4 + \epsilon, 1] \end{cases}$$

If $a_2 < a_4$ one proceeds similarly.

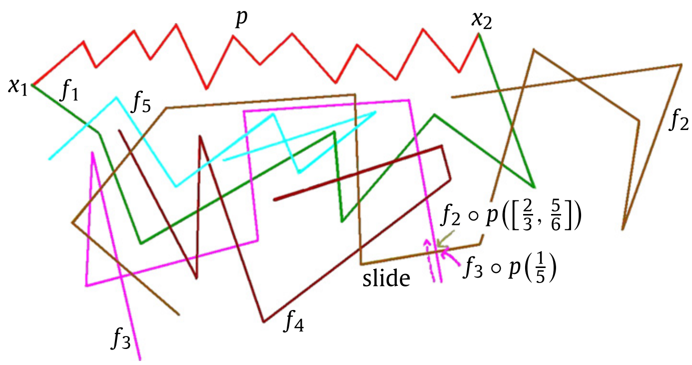

Fig. 6. Eliminating intersections of PL paths in local splittings.

Case 2. Now, suppose $d_{ii_1} \sim d_{ii_l}$ $(i, l > 1)$ (so f_i has a self-intersection). Then, again, for a suitably chosen $\epsilon > 0$, we can "slide" $f_i \circ p([0, a_2 + \epsilon])$ (where d_{ii_1} corresponds to $[a_1, a_2]$) through itself and past this point of intersection without creating any new intersections exactly as above.

Case 3. If we don't have any of the intersections of the types considered in Case 1 or Case 2 we claim that we must have:

$$d_{ij_1} \sim d_{ji_l} \quad (l > 1,\ i \neq j)$$

where d_{ij_l} corresponds to $[a_3, a_4]$, d_{ij_1} corresponds to $[a_1, a_2]$ and $a_3 > a_2$, in which case we can "slide", for a suitably chosen $\epsilon > 0$, $f_i \circ p([0, a_2 + \epsilon])$ past the point of intersection as above/before. (See Fig. 6.)

We establish this by a short argument by contradiction. Suppose the claim is false. Consider the m copies of I_i that have labeled subsets. Upon relabeling the I_i (if necessary) one obtains a string of inequalities:

$$d_{12_1} > d_{21_{k_1}} > d_{23_1} > d_{32_{k_2}} > \cdots > d_{j(j+1)_1} > d_{(j+1)j_{k_j}} > \cdots > d_{lj_1} > d_{jl_{k_l}}$$

where: $d_{i(i+1)_1} \sim d_{(i+1)i_{k_i}} k_i > 1$ and $d_{ij_1} > d_{ji_t}$ if d_{ij_1} corresponds to $[a_1, a_2]$, d_{ji_t} corresponds to $[a_3, a_4]$, and $a_1 > a_4$. But, one has:

$$d_{j(j+1)_1} > d_{jl_{k_l}} \quad \text{and} \quad d_{jl_{k_l}} > d_{j(j+1)_1}$$

a contradiction.

Finally, if $i = 1$ in any of the above cases, a little extra care is required in removing the intersection. That is, one can't "slide" $f_1 \circ p([0, a_2 + \epsilon])$ through itself since this homotopy would not be special (x_1 wouldn't remain a fixed point as time varies). So, one eliminates the intersection by homotoping f_j. $\square$

Lemma 6 *(Obtaining a non-surjective homotopy). Let X be a finite/compact simplicial complex without local cut points and such that no connected component is a two manifold. Let p, q be two normal PL arcs from x_1 to x_2 that are homotopic rel. endpoints, with $p \circ q^{-1}$ homeomorphic to S^1, and let $z_1, \ldots, z_{n-1}$ be $n-1$ distinct points in $X - (p(I) \cup q(I))$. Then there is a homotopy (rel. endpoints) that avoids $z_1, \ldots, z_{n-1}$, between q and a normal PL arc, r, and there is another homotopy (rel. endpoints) between r and p that avoids a two simplex, σ.*

Proof. First, suppose that X contains a maximal simplex, σ_1, of dimension greater than two. Subdivide X so that $p \circ q^{-1}$ is a one-dimensional subcomplex. We may assume, WLOG, that (w.r.t. some subdivision of I^2) the path homotopy, H, from p to q is simplicial on $I \times \{0, 1\}$. By the Relative Simplicial Approximation Theorem [10], upon further subdivision of I^2, one can obtain a simplicial approximation to H that is equal to H on $I \times \{0, 1\}$. This homotopy then, obviously misses σ_1. In this case we may take $r = q$.

Next suppose that X is a two complex. Subdivide X so that $p \circ q^{-1}$ is a one subcomplex, and (upon application of the Relative Simplicial Approximation Theorem) let H denote a simplicial (w.r.t. some subdivision of I^2) homotopy (rel. endpoints) between p and q as above. Let $P = H^{-1}(p(I))$, notice that WLOG we may assume that there is no path in P from $\{0\} \times I$ to $\{1\} \times I$, and consider one of its connected components, P_1. Label the connected components of $I^2 - P$ that are contained in discs bounded by simple closed loops in P_1, as $C_1, \ldots, C_m$, and consider one such connected component, C_1. Let $H_1 : I^2 \to I^2$ be a map with $H_1(C_1) = \ell_1$, $H_1(\ell_2) = \ell_1$, $H_1|B \cup A \cup \ell_3 \cup P_1^c = \mathrm{id}$ as in Fig. 7. Repeat this procedure with all the connected components C_i. Next proceed as above with P_2, then P_3 until all such connected components have been considered. Notice that one obtains in this fashion a (path) homotopy $\overline{H} = H \circ H_k \circ \cdots \circ H_1$ between p and q such that $I^2 - \overline{H}^{-1}(p(I))$ is path connected. Subdivide if necessary and let $\sigma \subset X - (p(I) \cup q(I) \cup \{z_1, \ldots, z_{n-1}\})$ be a two simplex such that $X - \sigma$ has the same defining properties as X. Define a PL arc, γ, in $I^2 - (\overline{H}^{-1}(p(I)) \cup (0, 1) \times \{0\})$ from $(0, 0)$ to $(1, 0)$ such that $\overline{H}(\gamma)$ is a normal PL arc with $\overline{H}^{-1}(\sigma)$ contained in the disc bounded by $I \times \{0\} \cup \gamma$, and with $z_1, \ldots, z_{n-1}$

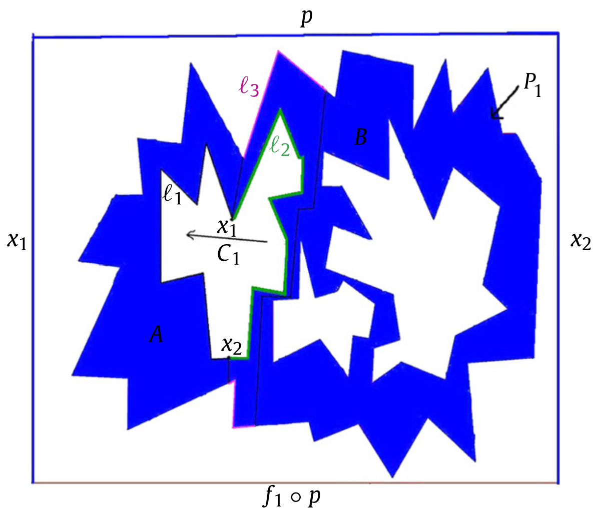

Fig. 7. Obtaining a non-surjective homotopy step I.

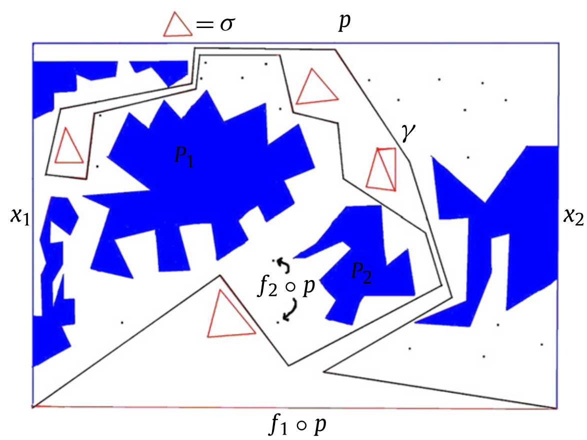

Fig. 8. Obtaining a non-surjective homotopy step II.

contained in the complement of this disc in I^2. See Fig. 8 for a graphic description of γ. Finally, setting $r = \gamma$ we clearly have the two path homotopies in the statement of the lemma. □

Theorem 1. *Let X be a compact polyhedron without local cut points and such that no connected component is a two-manifold. Let $\varphi : X \multimap X$ be an n-valued map. Then there is an n-valued homotopy between φ and an n-valued map, ϕ, with $N(\varphi)$ many fixed points.*

Proof. The proof proceeds in several steps.

Step 1. Apply the Hopf construction for n-valued maps, to obtain a simplicial (possibly w.r.t. some subdivision of the domain) fixed-finite n-valued map, φ' with all fixed points located in maximal simplexes.

Step 2. Let σ_1 denote a maximal simplex that has an edge that is common to at least two other simplexes and, if φ' has l FPCs, pick l distinct points z_i, $1 \leqslant i \leqslant l$ in σ_1 and choose representatives from each Nielsen class, x_i, $1 \leqslant i \leqslant l$, together with normal PL arcs $p_i : I \to X$ $(1 \leqslant i \leqslant l)$ as in [3] from z_i to x_i such that $p_i(I) \cap p_j(I) = \emptyset$ when $i \neq j$, and such that $\mathrm{Fix}(\varphi) \cap p_i(I) = \{x_i\}$, for $1 \leqslant i \leqslant l$. For each $\{z_i, x_i\}$ and p_i, $1 \leqslant i \leqslant l$, apply Lemmas 1–5 in that order. Let $\varphi_i' = \{f_{1_i}, \ldots, f_{n_i}\}$ denote the splitting of the n-valued map φ_i' we obtain upon application of Lemmas 1–5. Now, apply Lemma 6 with $p = p_i$, $q = f_{1_i} \circ p_i$, $z_{k_i} = f_{k+1_i} \circ p(I)$ to obtain $r = \overline{f}_{1_i} \circ p_i$. Now, let σ_{2_i} denote the maximal simplex that is missed by H_i (the special homotopy of special paths $(p_{\epsilon_i}(s) = p_i(s - \epsilon \sin(s\pi))$ and $\overline{f}_{1_i} \circ p_i)$ given by [3]), and let q_{j_i}, $2 \leqslant j \leqslant n$ denote arcs with $q_{k_i}(I) \cap q_{l_i}(I) = \emptyset$, $k \neq l$ and $q_{j_i}(I) \cap (\overline{f}_{1_i} \circ p(I) \cup p(I)) = \emptyset$ from z_{k_i}, $1 \leqslant k \leqslant n-1$, to distinct points in σ_{2_i}. Now, apply a special n-valued homotopy that "moves" z_{k_i}, $2 \leqslant k \leqslant n$ into σ_{2_i} along q_{k_i}.

Step 3: Let U_i denote a small simply connected open neighborhood of $p_i(I)$ with $U_i \cap \sigma_{2_i} = \emptyset$. Apply [3, Theorem 5.2] to obtain a special homotopy between $\overline{f}_{1_i}$ and a map f_{1_i}' that is a proximity map w.r.t. a sufficiently fine subdivision of X that ensures that $f_{1_i}'(p(I)) \subset U_i$. The corresponding n-valued homotopy is obtained by letting the other maps the splitting remain constant. Let V_i denote a small open neighborhood of $p_i([0,1))$ with $z_i \in \overline{V}_i$, and extend the special n-valued homotopy that is constant on $X - V_i$ and is the composition of all the special n-valued homotopies given by the application of Lemmas 1–6 on $p(I)$, to a special n-valued homotopy defined on all of X. Now choose a small simply connected neighborhood of $p([0,1))$, $W_i \subset X - \sigma_{2_i}$ with $z_i \in \overline{W}_i$ such that, if $\varphi_i''|W_i = \{f'_{1_i}, \dots, f'_{n_i}\}$ is a (local) splitting of the n-valued map we have obtained so far, then $f'_{j_i}(W_i) \subset \sigma_{2_i}$ for $2 \leqslant j \leqslant n$, and $f'_{1_i}(W_i) \subset U_i$. Apply lemmas in [1] to move x_i along $p(I)$ to z_i by a homotopy with support contained in W_i. Letting f'_{j_i} for $2 \leqslant j \leqslant n$ we obtain an n-valued homotopy (defined on all of X) between φ_i'' and an n-valued map φ_i''' with $\mathrm{Fix}(\varphi_i''') = \mathrm{Fix}(\varphi_i'') - \{x_i\} \cup \{z_i\}$.

The above procedure is repeated l times to obtain a representative in σ_1 for each FPC.

Step 4: Now, obtain a system of normal PL Nielsen arcs from the fixed points in the different FPCs to the z_i that are contained in $X - \bigcup_{1 \leqslant i \leqslant l} V_i$ and intersect each other only at their final end-points. This is important as we would like the maps in our (local) splittings of the restriction of the n-valued map obtained at each stage to these arcs to be simplicial — w.r.t. some subdivision of the domain-and in specially extending our n-valued homotopy in Step 3, we obtain an n valued map that is *not* simplicial on V_i. Use these arcs as above to obtain an n-valued homotopy to an n-valued map that has $\{z_1, \dots, z_l\}$ as fixed point set. As a final step eliminate all the fixed points that have index zero as in [6], to obtain the n-valued map in the statement of the theorem. □

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