Intersection and Singleton Type Assignment Characterizing Finite Böhm-Trees

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Intersection types and type constants representing unsolvability and singleton sets of λ-terms are incorporated into the Curry version of a simple type assignment system. Two restricted forms of typability in the system turn out to be equivalent to finiteness of Böhm-trees.

1. INTRODUCTION

The notion of Böhm-trees, see [3, Chap. 10], has extensively been used in the study of type-free λ-calculus. Roughly speaking, the Böhm-tree of a λ-term M describes in a tree form the normal form of M if it exists and the limit of infinitary reduction sequence otherwise. For example, let us consider Curry's fixed point combinator $\mathbf{Y} \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$ and Turing's combinator $\mathbf{\Theta} \equiv (\lambda xf.f(xxf))(\lambda xf.f(xxf))$. They have no normal form but infinitary reduction sequences

$$\mathbf{Y} \to_\beta \lambda f.f(X) \to_\beta \lambda f.f(f(X)) \to_\beta \cdots$$
$$\mathbf{\Theta} \to_\beta \lambda f.f(\mathbf{\Theta} f) \twoheadrightarrow_\beta \lambda f.f(f(\mathbf{\Theta} f)) \twoheadrightarrow_\beta \cdots,$$

where $X \equiv (\lambda x.f(xx))(\lambda x.f(xx))$. In this case, the two reduction sequences are not confluent, but both converge to an infinite expression $\lambda f.f(f(f(\ldots)))$, and the Böhm-trees of $\mathbf{Y}$ and $\mathbf{\Theta}$ are the following tree representation.

$$\begin{array}{c} \lambda f.f \\ | \\ f \\ | \\ f \\ \vdots \end{array}$$

This tree notion is indispensable in some model theoretical considerations of λ-calculus. Indeed, in continuous models of λ-calculus, such as D_∞, P ω invented by D. Scott [13, 14] and the filter domain in [4], when two λ-terms have the same Böhm-tree their interpretations are necessarily the same. In other words, these structures may be considered as models of λ-calculus not only under β-equality but also under Böhm-tree-equality. In this regard, so far, several interesting features of the models, such as local structure, have been elucidated by means of Böhm-trees.

Among all Böhm-trees, finite ones often play an important role. For example, when we take the set of Böhm-trees with the standard order $\subseteq$ as a coherent algebraic cpo, finite trees are exactly its compact elements [3, Proposition 12.2.2]. In a sense, this order structure is inherited by the above-mentioned models. That is to say, in continuous models, the interpretation of a λ-term M is the supremum of the interpretations of λ-terms whose Böhm-trees are finite and lower than that of M with respect to the order $\subseteq$. This property, called the approximation theorem, has wide application in the study of continuous models.

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The finiteness of Böhm-trees can be characterized, in terms of reduction theory, as the weak normalizability with respect to the following restricted β-reduction.

$$C[(\lambda x.M)N] \to C[M[x := N]] \quad \text{if } C \text{ is a one-hole context and there is no unsolvable subterm of } C[(\lambda x.M)N] \text{ including the redex } (\lambda x.M)N.$$

This reduction is not effective, but would be reasonable in a theoretical sense. Indeed, it can be proved by the genericity lemma [3, Proposition 12.3.24] that if a λ-term has a normal form then one can obtain the normal form by the restricted β-reduction.

It has been an important topic in the theory of types to investigate typability of λ-terms in relation to various notions of normalizability, such as solvability, weak normalizability and strong normalizability. As classical results concerning this, it is well known that in simply typed λ-calculus and second order typed λ-calculus typable terms are all strongly normalizable [7]. Furthermore, in intersection type assignment systems [2, 6, 12], each of the three notions of normalizability above can be neatly characterized by typability under a certain limited use of the (ω)-axiom.

In this paper, we introduce a type theory that allows us to characterize the finite Böhm-trees. This kind of attempt has already been made in [10], where an intersection type assignment system with a refinement of the universal type ω is introduced to characterize the property. In our system, instead of the refinement of ω, we use intersection types, type constants representing unsolvability, and singleton types. We prove a characterization theorem for this system, which states that a restricted form of typability in this system is equivalent to finiteness of Böhm-trees.

2. PRELIMINARIES

We briefly summarize basic notations and known results concerning type-free λ-calculus, which are used in the later sections. As usual, we write Λ for the set of type-free λ-terms, and, for a λ-term M, we denote the set of free-variables of M by $\mathrm{FV}(M)$. Because of space limitation, for $n \geq 0$, we often abbreviate $\lambda x_1 \ldots x_n.M$ and $MN_1 \ldots N_n$ to $\lambda\bar{x}.M$ and $M\bar{N}$, respectively. A λ-term is said to be *solvable* if it is reduced to a λ-term of the form $\lambda\bar{x}.y\bar{M}$, and is *unsolvable* otherwise. For an unsolvable λ-term M, the result of substitution in M is also known to be unsolvable. We write $\mathsf{K}(\Lambda\mathsf{B})$ for the set of λ-terms whose Böhm-trees are finite. This set clearly contains all unsolvable terms and is closed under β-conversion, since the Böhm-tree of M is identical with that of N whenever $M =_\beta N$.

Finally, we paraphrase the set $\mathsf{K}(\Lambda\mathsf{B})$ from the viewpoint of a certain reduction strategy. The reduction strategy we consider here is the left-most counterpart, denoted $\to_l$, of the restricted β-reduction mentioned in Section 1, which is inductively defined by

1. $\lambda\bar{x}.(\lambda y.M)N\bar{P} \to_l \lambda\bar{x}.M[y := N]\bar{P}$ if $\lambda\bar{x}.(\lambda y.M)N\bar{P}$ is solvable,
2. $\lambda\bar{x}.yM_1 \ldots M_i \ldots M_m \to_l \lambda\bar{x}.yM_1 \ldots M_i' \ldots M_m$ if $M_1, \ldots, M_{i-1} \in \mathsf{NF}_\Omega$ and $M_i \to_l M_i'$.

Here the set NF_Ω, by which we intend the set of λ-terms in normal form with respect to the restricted β-reduction, is inductively defined by

1. $M \in \mathsf{NF}_\Omega$ if M is unsolvable,
2. $\lambda\bar{x}.yM_1 \ldots M_m \in \mathsf{NF}_\Omega$ if $M_1, \ldots, M_m \in \mathsf{NF}_\Omega$.

Note that this is the same as the set of λ-terms in approximate normal form, except that unsolvable parts do not collapse in the λ-terms in NF_Ω. For these notions, we have $\mathsf{K}(\Lambda\mathsf{B}) = \{M \mid \exists N \in \mathsf{NF}_\Omega \;\; M \twoheadrightarrow_l N\}$.

3. THE TYPE ASSIGNMENT SYSTEM

We introduce an extension of the Curry version of simple type assignment system, which we subsequently use to characterize finiteness of Böhm-trees. The set T of *types* is inductively defined by the following grammar

$$T ::= a \mid \Omega \mid \{M\} \mid T \to T \mid T \wedge T,$$

where a ranges over an infinite set of type-variables and M over the set Λ of type-free λ-terms. We use letters $A, B, C, \ldots$ for meta-variables standing for types. We omit parentheses in types under the assumptions that $\wedge$ connects stronger than $\to$, and that $\to$ associates to the right. For a type A, we write $\mathrm{FV}(A)$ for the set of term-variables having free occurrences in a λ-term appearing in A as the element of a singleton type. For example, $\mathrm{FV}(\{\lambda x.xy\} \to a \wedge \{z\}) = \{y, z\}$. For a λ-term M and a type A, the expression $M : A$ is called a *statement* consisting of the *subject* M and the *predicate* A. We say a finite set Γ of statements whose subjects are type-variables is a *basis* if $x \notin \mathrm{FV}(A)$ for each $x : A \in \Gamma$ and $x : A, x : B \in \Gamma$ implies $A \equiv B$. We write $\mathrm{subj}(\Gamma)$ for the set of subjects in Γ and $\mathrm{pred}(\Gamma)$ for the set of predicates in Γ. For the bases Γ and Δ, we define the basis $\Gamma \uplus \Delta$ by

$$\begin{aligned}\Gamma \uplus \Delta = {} & \{x : A \mid x : A \in \Gamma \text{ and } x \notin \mathrm{subj}(\Delta)\} \cup \\ & \{x : A \mid x \notin \mathrm{subj}(\Gamma) \text{ and } x : A \in \Delta\} \cup \\ & \{x : A \wedge B \mid x : A \in \Gamma \text{ and } x : B \in \Delta\}.\end{aligned}$$

For a basis Γ, a λ-term M and a type A, the *judgment* $\Gamma \vdash M : A$ is generated by the following natural deduction style axioms and inference rules:

$$(\mathrm{var}) \quad \Gamma \vdash x : A \quad \text{if } x : A \in \Gamma \qquad (\Omega) \quad \Gamma \vdash M : \Omega \quad \text{if } M \text{ is unsolvable}$$

$$(\to\mathrm{I}) \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad (\to\mathrm{E}) \quad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$(\wedge\mathrm{I}) \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \wedge B} \qquad (\wedge\mathrm{E}) \quad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M : A} \quad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash M : B}$$

$$(\{\}\mathrm{I}) \quad \Gamma \vdash M : \{M\} \qquad (\{\}\mathrm{E}) \quad \frac{\Gamma \vdash M[x := N] : A \quad \Gamma \vdash x : \{N\}}{\Gamma \vdash M : A}$$

where we assume $x \notin \bigcup_{C \in \mathrm{pred}(\Gamma)} \mathrm{FV}(C) \cup \mathrm{FV}(B)$ in the ($\to$I)-rule.

We impose an unusual restriction on the axiom scheme (Ω), which allows us to use logical relations in the considerations in the next section. However, the restriction makes the set of axioms nonrecursively enumerable.

Since λ-terms appear in types in this system, it would be more natural to adopt dependent product types instead of arrow types, as in [1, 8]. Actually, if we decided to use the dependent product types, we could drop the unusual side condition that $x \notin \mathrm{FV}(B)$ in the ($\to$I)-rule. However, this restriction makes the system simpler and does not induce any difficulty in the argument below.

The intersection type assignment systems presented in [2, 6] make essential use of the ($\wedge$I)-rule to ensure invariance of types under some kinds of β-expansion sequences, which allows the well-known type theoretical characterizations explained in Section 1. On the other hand, we may drop the ($\wedge$I)-rule from our system, since rules for singleton types compensate for the absence of it. Indeed, without any applications of the ($\wedge$I)-rule, the invariance of types under the expansion with respect to $\to_l$ is shown in Lemma 4.6.

Admissibility of the weakening rule, which yields $\Delta \vdash M : A$ from $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, can be easily verified by induction on the length of derivations. Thus, whenever $\Gamma \vdash M[x := N] : A$ and $x \notin \mathrm{subj}(\Gamma) \cup \mathrm{FV}(N)$, we have $\Gamma, x : \{N\} \vdash M : A$ by means of the ({}E)-rule. Although the conclusion is not derived directly from the assumption, in the later sections we often use this inference like the basic inference rules, and by the configuration

$$\dfrac{\begin{array}{c}\vdots\ \delta \\ \Gamma \vdash M[x := N] : A\end{array}}{\Gamma, x : \{N\} \vdash M : A}\ (\mathrm{weak}, \{\}\mathrm{E})$$

where δ stands for a derivation of $\Gamma \vdash M[x := N] : A$ in our type assignment system, we mean the derivation

$$\dfrac{\begin{array}{c}\vdots\ \delta' \\ \Gamma, x : \{N\} \vdash M[x := N] : A\end{array} \quad \dfrac{}{\Gamma, x : \{N\} \vdash x : \{N\}}\ (\mathrm{var})}{\Gamma, x : \{N\} \vdash M : A}\ (\{\}\mathrm{E})$$

where δ' is the derivation obtained from δ by adding the assumption $x : \{N\}$ to all bases appearing in δ. In case of a configuration with multiple use of this abbreviation, its actual form is the result of unfolding all applications of the inference (weak,{ }E) from upper ones.

4. CHARACTERIZATION THEOREM

The rest of the paper is devoted to introducing two restricted forms of typability in our type assignment system and to proving that both of them are equivalent to finiteness of Böhm-trees.

We first consider one of the restrictions, which is defined by the types in which singleton types do not appear. We write $T_{-\{\}}$ for the set of such types and study the inhabitants of the types in $T_{-\{\}}$ under bases whose predicates are all in $T_{-\{\}}$. (Note that we do not impose any restrictions on types appearing on the way to derive conclusions.) Then, by means of a logical relation over λ-terms, it is shown in Lemma 4.4 that the set of such inhabitants is a subset of $\mathsf{K}(\Lambda\mathsf{B})$.

To see our proof, we begin by introducing some model theoretical notation. We use the letter ξ to denote a mapping which assigns λ-terms to term-variables. For a λ-term N, $\xi(x : N)$ stands for the mapping such that $\xi(x : N)(y)$ is N if $y \equiv x$, and $\xi(y)$ otherwise. For a mapping ξ, a λ-term M, and a type A, we write $M\theta_\xi$ for the result of simultaneously substituting $\xi(x)$ for each free occurrence of x in M, and inductively define the type $A\theta_\xi$, as follows:

1. $a\theta_\xi \equiv a$,
2. $\Omega\theta_\xi \equiv \Omega$,
3. $\{M\}\theta_\xi \equiv \{M\theta_\xi\}$,
4. $(A \to B)\theta_\xi \equiv A\theta_\xi \to B\theta_\xi$,
5. $(A \wedge B)\theta_\xi \equiv A\theta_\xi \wedge B\theta_\xi$.

We define the *logical relation* R as the mapping that inductively assigns subsets of Λ to types in T, as follows:

1. $R(a) = \mathsf{K}(\Lambda\mathsf{B})$,
2. $R(\Omega) = \mathsf{K}(\Lambda\mathsf{B})$,
3. $R(\{M\}) = \{N \mid N =_\beta M\}$,
4. $R(A \wedge B) = R(A) \cap R(B)$,
5. $R(A \to B) = \{M \mid \forall N \in R(A)\ MN \in R(B)\}$.

Note that, for each type A, $R(A)$ can be shown to be closed under β-conversion by simple induction on the structure of A.

LEMMA 4.1. *For each* $A \in T_{-\{\}}$,

(1) *If* $M_1, \ldots, M_m \in \mathsf{K}(\Lambda\mathsf{B})$ *then* $xM_1 \ldots M_m \in R(A)$.

(2) *If* $M \in R(A)$ *then* $M \in \mathsf{K}(\Lambda\mathsf{B})$.

Proof. By simultaneous induction on the structure of A.

Case 1. Suppose $A \equiv a$. Then (1) follows from $xM_1 \ldots M_m \in \mathsf{K}(\Lambda\mathsf{B})$, and (2) is clear from the definition of R.

Case 2. Suppose $A \equiv \Omega$. Then (1) and (2) follow as in Case 1.

Case 3. Suppose $A \equiv B \wedge C$. Then the definition of $T_{-\{\}}$ entails $B, C \in T_{-\{\}}$. Thus, for (1), the induction hypothesis implies $xM_1 \ldots M_m \in R(B) \cap R(C) = R(B \wedge C)$. For (2), $M \in R(B \wedge C) \subseteq R(B)$, which together with the induction hypothesis implies $M \in \mathsf{K}(\Lambda\mathsf{B})$.

Case 4. Suppose $A \equiv B \to C$. Then the definition of $T_{-\{\}}$ entails $B, C \in T_{-\{\}}$. To see (1), let us assume $N \in R(B)$. Then $N \in \mathsf{K}(\Lambda\mathsf{B})$ follows from the induction hypothesis for (2), and $xM_1 \ldots M_m N \in R(C)$ by the induction hypothesis for (1). Hence we obtain $xM_1 \ldots M_m \in R(B \to C)$. As for (2), for a term-variable z, we obtain $z \in R(B)$ by the induction hypothesis for (1), and furthermore $Mz \in R(C)$. Thus we have $Mz \in \mathsf{K}(\Lambda\mathsf{B})$ by the induction hypothesis for (2), concluding $M \in \mathsf{K}(\Lambda\mathsf{B})$. ■

LEMMA 4.2. *For each $A \in T$, if $M\theta_{\xi(x:N)}\bar{P} \in R(A)$ then $(\lambda x.M)\theta_\xi N\bar{P} \in R(A)$.*

Proof. As mentioned above, for each $A \in T$, $R(A)$ is closed under β-conversion. Furthermore, we obtain $M\theta_{\xi(x:N)}\bar{P} =_\beta (\lambda x.M)\theta_\xi N\bar{P}$ as follows.

$$\begin{aligned} M\theta_{\xi(x:N)}\bar{P} &\equiv M[x := z]\theta_{\xi(z:z)}[z := N]\bar{P} \\ &=_\beta \big(\lambda z.\big(M[x := z]\theta_{\xi(z:z)}\big)\big)N\bar{P} \\ &\equiv (\lambda z.M[x := z])\theta_\xi N\bar{P} \\ &\equiv (\lambda x.M)\theta_\xi N\bar{P}, \end{aligned}$$

where z is a fresh variable. These facts ensure the statement of the lemma. ■

We use the standard notation for validity, that is we write $\xi \models \Gamma$ if and only if $\xi(x) \in R(A\theta_\xi)$ for any $x : A \in \Gamma$, and $\Gamma \models M : A$ if and only if $M\theta_\xi \in R(A\theta_\xi)$ for any ξ satisfying $\xi \models \Gamma$. Then the next lemma states soundness of the system. In [11], this kind of assertion is called the basic lemma of the logical relations.

LEMMA 4.3 (Basic lemma). *If $\Gamma \vdash M : A$ then $\Gamma \models M : A$.*

Proof. By induction on the length of derivation.

Case 1. If the last step of the derivation is by (var) yielding $\Gamma \vdash x : A$ then it is immediate from the assumption that $\xi \models \Gamma$.
Case 2. If the last step of the derivation is by (Ω) yielding $\Gamma \vdash M : \Omega$ then $M\theta_\xi \in R(\Omega) = R(\Omega\theta_\xi)$. This is because $M\theta_\xi$ is unsolvable.
Case 3. Suppose the last step of the derivation is by ($\to$I), which yields $\Gamma \vdash \lambda x.M : A \to B$ from $\Gamma, x : A \vdash M : B$. Suppose also $\xi \models \Gamma$ and $N \in R(A\theta_\xi)$. Then we obtain $\xi(x : N) \models \Gamma, x : A$. This is because

$$\begin{aligned} \xi(x : N)(x) &\equiv N \\ &\in R(A\theta_\xi) \\ &= R\big(A\theta_{\xi(x:N)}\big), \end{aligned}$$

and for each $y : C \in \Gamma$

$$\begin{aligned} \xi(x : N)(y) &\equiv \xi(y) \\ &\in R(C\theta_\xi) \\ &= R\big(C\theta_{\xi(x:N)}\big). \end{aligned}$$

Note that $x \notin \mathrm{FV}(A) \cup \mathrm{FV}(C)$ in the formulas above because of the side conditions for basis and for the ($\to$I)-rule. It follows from the induction hypothesis and the side-condition of the ($\to$I)-rule that $M\theta_{\xi(x:N)} \in R(B\theta_{\xi(x:N)}) = R(B\theta_\xi)$. Now applying Lemma 4.2 to the preceding formula, we have $(\lambda x.M)\theta_\xi N \in R(B\theta_\xi)$, and accordingly $(\lambda x.M)\theta_\xi \in R(A\theta_\xi \to B\theta_\xi) = R((A \to B)\theta_\xi)$.
Case 4. Suppose the last step of the derivation is by ($\to$E), which yields $\Gamma \vdash MN : B$ from $\Gamma \vdash M : A \to B$ and $\Gamma \vdash N : A$. By the induction hypothesis we have $M\theta_\xi \in R((A \to B)\theta_\xi) = R(A\theta_\xi \to B\theta_\xi)$ and $N\theta_\xi \in R(A\theta_\xi)$. These immediately imply $(MN)\theta_\xi \equiv M\theta_\xi N\theta_\xi \in R(B\theta_\xi)$.
Case 5. Suppose the last step of the derivation is by ($\wedge$I), which yields $\Gamma \vdash M : A \wedge B$ from $\Gamma \vdash M : A$ and $\Gamma \vdash M : B$. By the induction hypothesis we have $M\theta_\xi \in R(A\theta_\xi)$ and $M\theta_\xi \in R(B\theta_\xi)$. Thus $M\theta_\xi \in R(A\theta_\xi) \cap R(B\theta_\xi) = R(A\theta_\xi \wedge B\theta_\xi) = R((A \wedge B)\theta_\xi)$.
Case 6. Suppose the last step of the derivation is by ($\wedge$E), which yields $\Gamma \vdash M : A$ from $\Gamma \vdash M : A \wedge B$. Then the condition is clear from the induction hypothesis and the inclusion $R(A \wedge B) \subseteq R(A)$.
Case 7. If the last step of the derivation is by ($\{\}$I) yielding $\Gamma \vdash M : \{M\}$, then $M\theta_\xi \in \{N | N =_\beta M\theta_\xi\} = R(\{M\theta_\xi\}) = R(\{M\}\theta_\xi)$.

Case 8. Suppose the last step of the derivation is by ({}E), which yields $\Gamma \vdash M : A$ from $\Gamma \vdash x : \{N\}$ and $\Gamma \vdash M[x := N] : A$. Suppose also that $\xi \models \Gamma$. Then we obtain $\xi(x) \in R(\{N\}\theta_\xi) = R(\{N\theta_\xi\})$ and $M[x := N]\theta_\xi \in R(A\theta_\xi)$ by the induction hypothesis. The first formula implies $\xi(x) =_\beta N\theta_\xi$, and therefore we have

$$\begin{aligned} M[x := N]\theta_\xi &\equiv M[x := z]\theta_{\xi(z:z)}[z := N\theta_\xi] \\ &=_\beta M[x := z]\theta_{\xi(z:z)}[z := \xi(x)] \\ &\equiv M\theta_\xi, \end{aligned}$$

where z is a fresh variable. Now we obtain $M\theta_\xi \in R(A\theta_\xi)$, since $R(A\theta_\xi)$ is closed under β-conversion. ■

The converse of Lemma 4.3 does not hold for the following two reasons. First, types are not closed under β-conversion in our type assignment system, even though the interpretation of each type by itself is. Second, the property is violated by singleton types. To see it, let us consider the basis

$$\Gamma = \{x : \{y\} \to \{y\} \wedge \{yy\},\ y : \{\lambda x.xx\}\}.$$

For every ξ, if $\xi \models \Gamma$ then we have $\xi(x)\xi(y) =_\beta \xi(y)$, $\xi(x)\xi(y) =_\beta \xi(y)\xi(y)$ and $\xi(y) =_\beta \lambda x.xx$, from which $\lambda x.xx =_\beta (\lambda x.xx)(\lambda x.xx)$ follows. This is a contradiction. Hence we have $\xi \not\models \Gamma$ for every ξ and conclude $\Gamma \models z : a$. However, on the other hand, we cannot derive $\Gamma \vdash z : a$.

Under the preparation above, we can now show half of our main theorem.

LEMMA 4.4. *If there exist a basis* Γ *and* $A \in T_{-\{\}}$ *such that* $\mathrm{pred}(\Gamma) \subseteq T_{-\{\}}$ *and* $\Gamma \vdash M : A$ *then* $M \in \mathsf{K}(\Lambda\mathsf{B})$.

Proof. By virtue of Lemma 4.1 (1), taking identity mapping ι, we have $\iota(y) \equiv y \in R(B)$ for each $y : B \in \Gamma$; that is $\iota \models \Gamma$. Thus we obtain $M \equiv M\theta_\iota \in R(A\theta_\iota) = R(A)$ by the basic lemma shown above. Therefore, by Lemma 4.1 (2), we conclude that the Böhm-tree of M is finite. ■

Next we consider the other form of restricted typability, which is based on two subsets of $T_{-\{\}}$. The sets T_l and T_r are simultaneously defined by the following grammar:

$$\begin{aligned} T_l &::= a \mid T_r \to T_l \mid T_l \wedge T_l, \\ T_r &::= a \mid \Omega \mid T_l \to T_r. \end{aligned}$$

Then we focus our attention on the inhabitants of the types in T_r under bases whose predicates are all in T_l and show that each element of $\mathsf{K}(\Lambda\mathsf{B})$ turns out to be such an inhabitant.

Our proof below is analogous to that in [2, 6]. Considerations in Section 2 show that it is sufficient to prove that each λ-term in NF_Ω is typable in the restricted sense introduced above and that types are invariant under the expansion with respect to $\to_l$, which is verified in Lemmas 4.5 and 4.6, respectively. The proof below is based on considering derivations in a special form where the use of singleton types is restricted. A derivation is said to be *proper* if all instances of the ({}E)-rule and all statements whose predicates are singleton types always appear in subderivations having one of the following two forms:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash M[x_1 := N_1]\dots[x_n := N_n] : A\end{array}}{\Gamma, x_n : \{N_n\} \vdash M[x_1 := N_1]\dots[x_{n-1} := N_{n-1}] : A}\ \text{(weak,\{\}E)}}{\Gamma \vdash \lambda x_n.M[x_1 := N_1]\dots[x_{n-1} := N_{n-1}] : \{N_n\} \to A}\ (\to\text{I})$$

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash \lambda x_1\dots x_n.M : \{N_1\} \to \cdots \to \{N_n\} \to A\end{array} \qquad \dfrac{}{\Gamma \vdash N_1 : \{N_1\}}\ (\{\}\text{I})}{\Gamma \vdash (\lambda x_1\dots x_n.M)N_1 : \{N_2\} \to \cdots \to \{N_n\} \to A}\ (\to\text{E})$$

where $n \geq 1$, all x_i are mutually distinct, and $x_i \notin \bigcup_{i=1}^n \mathrm{FV}(N_i) \cup \mathrm{FV}(A)$ for each $i \in \{1, \dots, n\}$,

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma \vdash (\lambda x.M)N_1 : \{N_2\} \to \cdots \{N_n\} \to A \\ \vdots \\ \Gamma \vdash (\lambda x.M)N_1 \ldots N_{n-1} : \{N_n\} \to A\end{array} \quad \dfrac{}{\Gamma \vdash N_1 : \{N_1\}}\,(\{\}\mathrm{I})}{\Gamma \vdash (\lambda x.M)N_1 \ldots N_n : A}\,(\to\mathrm{E})}{}
$$

where the first step is

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash M[x := N_1] : \{N_2\} \to \cdots \{N_n\} \to A\end{array}}{\Gamma, x : \{N_1\} \vdash M : \{N_2\} \to \cdots \{N_n\} \to A}\,(\mathrm{weak},\{\}\mathrm{E})}{\Gamma \vdash \lambda x.M : \{N_1\} \to \cdots \{N_n\} \to A}\,(\to\mathrm{I}) \quad \dfrac{}{\Gamma \vdash N_1 : \{N_1\}}\,(\{\}\mathrm{I})}{\Gamma \vdash (\lambda x.M)N_1 : \{N_2\} \to \cdots \{N_n\} \to A}\,(\to\mathrm{E})
$$

where $n \geq 1$ and $x \notin \bigcup_{i=1}^{n} \mathrm{FV}(N_i) \cup \mathrm{FV}(A)$.

LEMMA 4.5. *If* $M \in \mathsf{NF}_\Omega$ *then there exist* Γ, $A \in T_r$ *and* δ *such that* $\mathrm{pred}(\Gamma) \subseteq T_l$ *and* δ *is a proper derivation of* $\Gamma \vdash M : A$.

Proof. By induction on the structure of M.

Case 1. If M is unsolvable then we have $\vdash M : \Omega$ by the (Ω)-axiom.

Case 2. Suppose $M \equiv \lambda x_1 \ldots x_n.yM_1 \ldots M_m$ such that $M_1, \ldots, M_m \in \mathsf{NF}_\Omega$. Then for each $i \in \{1, \ldots, m\}$, applying the induction hypothesis to M_i, we can find Γ_i, $A_i \in T_r$ and δ_i such that $\mathrm{pred}(\Gamma_i) \subseteq T_l$ and δ_i is a proper derivation of $\Gamma_i \vdash M_i : A_i$. Here, for a certain type-variable a, let us denote the basis $\biguplus_{i=1}^{m} \Gamma_i \uplus \{y : A_1 \to \cdots \to A_m \to a\}$ by Σ. Then it is clear from the definition of T_l that $A_1 \to \cdots \to A_m \to a \in T_l$, which together with the definition of $\uplus$ guarantees that $\mathrm{pred}(\Sigma) \subseteq T_l$. Furthermore we obtain a proper derivation δ_i' of $\Sigma \vdash M_i : A_i$ based on δ_i, since for each $z : C \in \Gamma_i$ we have $\Sigma \vdash z : C$ by means of only (var) and $(\wedge\mathrm{E})$. Likewise we can construct a proper derivation of $\Sigma \vdash y : A_1 \to \cdots \to A_m \to a$. These immediately ensure the existence of a proper derivation of $\Sigma \vdash yM_1 \ldots M_m{:}a$. Now repeated application of $(\to\mathrm{I})$ to the preceding derivation yields a proper derivation of $\Delta \vdash \lambda x_1 \ldots x_n.yM_1 \ldots M_m : B_1 \to \ldots \to B_n \to a$ where $\Delta = \{z : C \mid z : C \in \Sigma \text{ and } z \notin \{x_1, \ldots, x_n\}\}$ and B_i is a certain type-variable if $x_i \notin \mathrm{subj}(\Sigma)$ and $x_i : B_i \in \Sigma$ otherwise, for each i. Here $B_1 \to \cdots \to B_n \to a \in T_r$ is trivial. ■

From the preceding proof, we can observe that the use of sets T_l and T_r is the minimum requirement to give types to every λ-terms in NF_Ω. This is the reason that T_l and T_r are similar to the set of types appearing in principal basis schemes of intersection type assignment systems and to the set of principal type schemes, respectively. (For the definition of these schemes, see [5].) Note also that the $(\wedge\mathrm{E})$-rule is used in an essential way, which is in contrast to the lack of essential use of the $(\wedge\mathrm{I})$-rule, as mentioned in Section 3.

LEMMA 4.6. *Let* C *be a one-hole context,* δ *a proper derivation of* $\Gamma \vdash C[M[x := N]] : A$ *and* $C[(\lambda x.M)N] \to_l C[M[x := N]]$. *Then there exists a proper derivation of* $\Gamma \vdash C[(\lambda x.M)N] : A$.

Proof. By induction on the length of the derivation.

Case 1. Suppose C is the trivial context, namely $C \equiv [\,]$. Then the following derivation ensures the condition

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta\\ \Gamma \vdash M[x := z][z := N] : A\end{array}}{\Gamma, z : \{N\} \vdash M[x := z] : A}\,(\mathrm{weak},\{\}\mathrm{E})}{\Gamma \vdash \lambda z.M[x := z] : \{N\} \to A}\,(\to\mathrm{I}) \quad \dfrac{}{\Gamma \vdash N : \{N\}}\,(\{\}\mathrm{I})}{\Gamma \vdash (\lambda x.M)N : A}\,(\to\mathrm{E})
$$

where $z \notin \mathrm{FV}(N) \cup \mathrm{FV}(A)$.

Case 2. Suppose C is nontrivial. Then we distinguish cases according to the last rule applied in δ.

Subcase 2.1. The case where the last step of δ is by (var) is impossible since we now assume C is nontrivial.

Subcase 2.2. Suppose the last step of δ is by (Ω). Then $C[M[x := N]]$ is necessarily unsolvable, and we can deduce $\Gamma \vdash C[(\lambda x.M)N] : \Omega$ by (Ω).

Subcase 2.3. Suppose the last step of δ is by ($\to$I), yielding $\Gamma \vdash \lambda y.C'[M[x := N]] : A \to B$ from $\Gamma, y : A \vdash C'[M[x := N]] : B$. (Note that we do not have to consider the case where the subject of the conclusion is just $M[x := N]$.) Then it follows from the definition of $\to_l$ that $\lambda y.C'[(\lambda x.M)N] \to_l \lambda y.C'[M[x := N]]$ entails $C'[(\lambda x.M)N] \to_l C'[M[x := N]]$. Thus we can find a proper derivation of $\Gamma, y : A \vdash C'[(\lambda x.M)N] : B$ by induction and a derivation of $\Gamma \vdash \lambda y.C'[(\lambda x.M)N] : A \to B$ by the use of rule ($\to$I).

Subcase 2.4. Suppose the last step of δ is by ($\to$E). Then we distinguish cases again according to the predicate of its minor premise. The case of a singleton type is studied in the first two cases, together covering the two cases in the definition of proper derivation; the other possible form of the predicate is considered in the third case.

Subsubcase 2.4.1. Suppose δ is of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash P[y_1 := Q_1]\ldots[y_n := Q_n] : B}}{\Gamma, y_n : \{Q_n\} \vdash P[y_1 := Q_1]\ldots[y_{n-1} := Q_{n-1}] : B}\,(\text{weak},\{\}\text{E})}{\Gamma \vdash \lambda y_n.P[y_1 := Q_1]\ldots[y_{n-1} := Q_{n-1}] : \{Q_n\} \to B}\,(\to\text{I})}{\begin{array}{c}\vdots\\ \Gamma \vdash \lambda y_1 \ldots y_n.P : \{Q_1\} \to \cdots \to \{Q_n\} \to B \qquad \dfrac{}{\Gamma \vdash Q_1 : \{Q_1\}}\,(\{\}\text{I}) \\ \hline \Gamma \vdash (\lambda y_1 \ldots y_n.P)Q_1 : \{Q_2\} \to \cdots \to \{Q_n\} \to B \end{array}}\,(\to\text{E})$$

where $n \geq 1$, all y_i are mutually distinct, and $y_i \notin \bigcup_{i=1}^{n} \mathrm{FV}(Q_i) \cup \mathrm{FV}(B)$ for each $i \in \{1, \ldots, n\}$. Suppose also that $C[(\lambda x.M)N] \to_l C[M[x := N]] \equiv (\lambda y_1 \ldots y_n.P)Q_1$. Then the only possible case to consider is that $M[x := N] \equiv \lambda y_1 \ldots y_n.P$, or equivalently $C[\,] \equiv [\,]Q_1$. This is because the assumption that $M[x := N]$ occurs in $\lambda y_2 \ldots y_n.P$ or Q_1 contradicts the definition of $\to_l$ and furthermore $M[x := N] \not\equiv (\lambda y_1 \ldots y_n.P)Q_1$ by the assumption for C. Taking these facts under consideration, we now obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\;\delta'}{\Gamma \vdash M[x := z][z := N] : \{Q_1\} \to A}}{\Gamma, z : \{N\} \vdash M[x := z] : \{Q_1\} \to A}\,(\text{weak},\{\}\text{E})}{\Gamma \vdash \lambda x.M : \{N\} \to \{Q_1\} \to A}\,(\to\text{I}) \qquad \dfrac{}{\Gamma \vdash N : \{N\}}\,(\{\}\text{I})}{\Gamma \vdash (\lambda x.M)N : \{Q_1\} \to A}\,(\to\text{E}) \qquad \dfrac{}{\Gamma \vdash Q_1 : \{Q_1\}}\,(\{\}\text{I})}{\Gamma \vdash C[(\lambda x.M)N] : A}\,(\to\text{E})$$

where $A \equiv \{Q_2\} \to \cdots \to \{Q_n\} \to B$, z is a fresh variable and δ' is the subderivation of δ above deducing $\Gamma \vdash \lambda y_1 \ldots y_n.P : \{Q_1\} \to \cdots \to \{Q_n\} \to B$. This is clearly proper.

Subsubcase 2.4.2. Suppose δ is of the form

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash P[y := Q_1] : \{Q_2\} \to \cdots \{Q_n\} \to A}}{\Gamma, y : \{Q_1\} \vdash P : \{Q_2\} \to \cdots \{Q_n\} \to A}\,(\text{weak},\{\}\text{E})}{\Gamma \vdash \lambda y.P : \{Q_1\} \to \cdots \{Q_n\} \to A}\,(\to\text{I}) \qquad \dfrac{}{\Gamma \vdash Q_1 : \{Q_1\}}\,(\{\}\text{I})}{\Gamma \vdash (\lambda y.P)Q_1 : \{Q_2\} \to \cdots \{Q_n\} \to A}\,(\to\text{E}) \\ \vdots \\ \Gamma \vdash (\lambda y.P)Q_1 \cdots Q_{n-1} : \{Q_n\} \to A \end{array} \qquad \dfrac{}{\Gamma \vdash Q_n : \{Q_n\}}\,(\{\}\text{I})}{\Gamma \vdash (\lambda y.P)Q_1 \cdots Q_n : A}\,(\to\text{E})$$

where $n \geq 1$ and $y \notin \bigcup_{i=1}^{n} \mathrm{FV}(Q_i) \cup \mathrm{FV}(B)$. Suppose also that $C[(\lambda x.M)N] \to_l C[M[x := N]] \equiv (\lambda y.P)Q_1 \ldots Q_n$. Then, as in the preceding case, the assumption that $M[x := N]$ occurs in P or Q_i for some $i \in \{1, \ldots, n\}$ contradicts the definition of $\to_l$ and furthermore $M[x := N] \not\equiv (\lambda y.P)Q_1 \ldots Q_n$ by the assumption for C. Thus the only possible case to consider is that $M[x := N] \equiv (\lambda y.P)Q_1 \ldots Q_i$, or $C[\,] \equiv [\,]Q_{i+1} \ldots Q_n$, for some $i \in \{0, 1, \ldots, n-1\}$. Now we obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \delta'}{\Gamma \vdash M[x:=z][z:=N] : \{Q_{i+1}\} \to \cdots \{Q_n\} \to A}}{\Gamma, z:\{N\} \vdash M[x:=z] : \{Q_{i+1}\} \to \cdots \{Q_n\} \to A}\ (\text{weak}, \{\}\text{E})}{\Gamma \vdash \lambda x.M : \{N\} \to \{Q_{i+1}\} \to \cdots \{Q_n\} \to A}\ (\to\text{I}) \quad \dfrac{}{\Gamma \vdash N : \{N\}}\ (\{\}\text{I})}{\Gamma \vdash (\lambda x.M)N : \{Q_{i+1}\} \to \cdots \{Q_n\} \to A}\ (\to\text{E})$$

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash (\lambda x.M)N Q_{i+1} \ldots Q_{n-1} : \{Q_n\} \to A} \quad \dfrac{}{\Gamma \vdash Q_n : \{Q_n\}}\ (\{\}\text{I})}{\Gamma \vdash C[(\lambda x.M)N] : A}\ (\to\text{E})$$

where z is a fresh variable and δ' is the subderivation of δ above which gives $\Gamma \vdash (\lambda y.P)Q_1 \ldots Q_i : \{Q_{i+1}\} \to \cdots \to \{Q_n\} \to A$. This is clearly proper.

Subsubcase 2.4.3. Suppose the last step of δ is by ($\to$E) yielding $\Gamma \vdash C'[M[x:=N]]P : B$ from $\Gamma \vdash C'[M[x:=N]] : A \to B$ and $\Gamma \vdash P : A$, in which A is not a singleton type. Then it follows from the definition of $\to_l$ that $C'[(\lambda x.M)N]P \to_l C'[M[x:=N]]P$ entails $C'[(\lambda x.M)N] \to_l C'[M[x:=N]]$. Thus we can find a proper derivation of $\Gamma \vdash C'[(\lambda x.M)N] : A \to B$ by the induction hypothesis and moreover that of $\Gamma \vdash C'[(\lambda x.M)N]P : B$ by ($\to$E). The case where the last ($\to$E) yields $\Gamma \vdash PC'[M[x:=N]] : B$ from $\Gamma \vdash P : A \to B$ and $\Gamma \vdash C'[M[x:=N]] : A$ can be verified analogously.

Subcase 2.5. Suppose the last step of δ is by ($\wedge$E), yielding $\Gamma \vdash C[M[x:=N]] : A$ from $\Gamma \vdash C[M[x:=N]] : A \wedge B$. Then we obtain a proper derivation of $\Gamma \vdash C[(\lambda x.M)N] : A \wedge B$ by the induction hypothesis and a derivation of $\Gamma \vdash C[(\lambda x.M)N] : A$ by the use of rule ($\wedge$E).

Subcase 2.6. If the last step of δ is by ({}I) or ({}E) then it contradicts properness of δ. So we need not verify these cases. ■

Now we prove the implication which states that the finiteness of Böhm-trees implies the restricted typability. In conjunction with Lemma 4.4, this completes the proof of our main theorem, since the restricted form of typability introduced earlier in this paper allows us to validate the opposite implication.

LEMMA 4.7. *If* $M \in \mathsf{K}(\Lambda\mathsf{B})$ *then there exists a basis* Γ *and* $A \in T_r$ *such that* $\text{pred}(\Gamma) \subseteq T_l$ *and* $\Gamma \vdash M : A$.

Proof. Suppose $M \in \mathsf{K}(\Lambda\mathsf{B})$. Then, as mentioned in Section 2, there is a λ-term N such that $M \twoheadrightarrow_l N \in \mathsf{NF}_\Omega$. For this N, Lemma 4.5 ensures existence of Γ, $A \in T_r$ and δ such that $\text{pred}(\Gamma) \subseteq T_l$ and δ is a proper derivation of $\Gamma \vdash N : A$. Therefore we obtain the statement of the lemma by repeated application of Lemma 4.6. ■

THEOREM 4.4 (Main). *For each* $M \in \Lambda$, *the following are equivalent*:

(1) $M \in \mathsf{K}(\Lambda\mathsf{B})$,

(2) *there exist a basis* Γ *and* $A \in T_{-\{\}}$ *such that* $\text{pred}(\Gamma) \subseteq T_{-\{\}}$ *and* $\Gamma \vdash M : A$,

(3) *there exist a basis* Γ *and* $A \in T_r$ *such that* $\text{pred}(\Gamma) \subseteq T_l$ *and* $\Gamma \vdash M : A$.

5. CONCLUDING REMARKS

A variety of intersection type assignment systems are well known for their nice theoretical aspects, most of which are based on the property that in those systems types are invariant under subject β-conversion. Let us discuss here one of those systems, which is comparable with the system presented in [6]; its types and inference rules are obtained from the system defined in Section 3 by eliminating singleton types and by replacing the type constant Ω and the axiom (Ω) with ω and the following, respectively.

$$(\omega) \quad \Gamma \vdash M : \omega$$

This axiom and the rules for intersection types are applied to ensure the invariance of types under subject β-expansion. Here we recall an example exhibited in [9], in which we find an essential use of intersection types. Consider the reduction $(\lambda x.xx)\mathbf{I} \to_\beta \mathbf{I}\,\mathbf{I}$, where $\mathbf{I}$ stands for $\lambda x.x$. Then we can assign type $a \to a$ to the contractum, as follows.

$$\dfrac{\dfrac{\dfrac{}{x : a \to a \vdash x : a \to a}\,(\text{var})}{\vdash \mathbf{I} : (a \to a) \to a \to a}\,(\to\text{I}) \qquad \dfrac{\dfrac{}{x : a \vdash x : a}\,(\text{var})}{\vdash \mathbf{I} : a \to a}\,(\to\text{I})}{\vdash \mathbf{I}\,\mathbf{I} : a \to a}\,(\to\text{E})$$

In this derivation, we assign two different types $(a \to a) \to a \to a$ and $a \to a$ to the two occurrences of $\mathbf{I}$. Thus, in order to assign $a \to a$ to the redex $(\lambda x.xx)\mathbf{I}$, it is natural to assign the same type to each x occurring in its function body. However proper application of the $(\to\text{I})$-rule demands that types assigned to both occurrences of x coincide. This problem can be solved by making intersection of the two types, which together with the $(\wedge\text{E})$-rule enables proper application of $(\to\text{I})$. Further we can assign the intersection type to $\mathbf{I}$ by virtue of the $(\wedge\text{I})$-rule. Accordingly we obtain the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : B \vdash x : B}\,(\text{var})}{x : B \vdash x : A \to A}\,(\wedge\text{E}) \qquad \dfrac{\dfrac{}{x : B \vdash x : B}\,(\text{var})}{x : B \vdash x : A}\,(\wedge\text{E})}{x : B \vdash xx : A}\,(\to\text{E})}{\vdash \lambda x.xx : B \to A}\,(\to\text{I}) \qquad \dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\,(\text{var})}{\vdash \mathbf{I} : A \to A}\,(\to\text{I}) \qquad \dfrac{\dfrac{}{x : a \vdash x : a}\,(\text{var})}{\vdash \mathbf{I} : A}\,(\to\text{I})}{\vdash \mathbf{I} : B}\,(\wedge\text{I})}{\vdash (\lambda x.xx)\mathbf{I} : A}\,(\to\text{E})$$

where $A \equiv a \to a$ and $B \equiv (A \to A) \wedge A$.

From the discussion above, the ordinary device of intersection types seems sufficient to guarantee Lemma 4.6. However, this is not the case and we cannot eliminate singleton types from our system. This is mainly because the translation using only intersection types changes structures of derivations globally and because in our system the type constant Ω is assigned only to unsolvable terms.

To see it, let us consider the reduction $(\lambda y.x(y\mathbf{Z}))\mathbf{Z} \to_\beta x(\mathbf{Z}\mathbf{Z})$, where $\mathbf{Z}$ stands for $\lambda x.xx$, in our system with complete disregard for singleton types. Then we have the following derivation for the contractum.

$$\dfrac{\dfrac{}{x : \Omega \to a \vdash x : \Omega \to a}\,(\text{var}) \qquad \dfrac{}{x : \Omega \to a \vdash \mathbf{Z}\mathbf{Z} : \Omega}\,(\Omega)}{x : \Omega \to a \vdash x(\mathbf{Z}\mathbf{Z}) : a}\,(\to\text{E})$$

If there is a type A such that $x : \Omega \to a \vdash \mathbf{Z} : A$ and $x : \Omega \to a, y : A \vdash y\mathbf{Z} : \Omega$ then we can simply obtain the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \Omega \to a}\,(\text{var}) \qquad \begin{array}{c}\vdots\\ \Gamma \vdash y\mathbf{Z} : \Omega\end{array}}{\Gamma \vdash x(y\mathbf{Z}) : a}\,(\to\text{E})}{x : \Omega \to a \vdash \lambda y.x(y\mathbf{Z}) : A \to a}\,(\to\text{I}) \qquad \begin{array}{c}\vdots\\ x : \Omega \to a \vdash \mathbf{Z} : A\end{array}}{x : \Omega \to a \vdash (\lambda y.x(y\mathbf{Z}))\mathbf{Z} : a}\,(\to\text{E})$$

where $\Gamma = \{x : \Omega \to a,\ y : A\}$. However, the existence of such types immediately implies that the judgment $\vdash \mathbf{Z}\mathbf{Z} : \Omega$ is derivable by using only the axiom (var) and the rules $(\to\text{I})$, $(\to\text{E})$, $(\wedge\text{I})$, and $(\wedge\text{E})$. (Note that $y\mathbf{Z}$ is solvable and we cannot apply the (Ω)-axiom in any derivations of $x : \Omega \to a, y : A \vdash y\mathbf{Z} : \Omega$.) This contradicts the well-known fact (see [2, 12]) that the typability under those type assignment rules coincides with the strong normalizability.

In contrast, in the system with singleton types, the type $\{\mathbf{Z}\}$ enables the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta \vdash x : \Omega \to a}\,(\text{var}) \qquad \dfrac{\dfrac{}{x : \Omega \to a \vdash \mathbf{Z}\mathbf{Z} : \Omega}\,(\Omega)}{\Delta \vdash y\mathbf{Z} : \Omega}\,(\text{weak}, \{\}\text{E})}{\Delta \vdash x(y\mathbf{Z}) : a}\,(\to\text{E})}{x : \Omega \to a \vdash \lambda y.x(y\mathbf{Z}) : \{\mathbf{Z}\} \to a}\,(\to\text{I}) \qquad \dfrac{}{x : \Omega \to a \vdash \mathbf{Z} : \{\mathbf{Z}\}}\,(\{\}\text{I})}{x : \Omega \to a \vdash (\lambda y.x(y\mathbf{Z}))\mathbf{Z} : a}\,(\to\text{E})$$

where $\Delta = \{x : \Omega \to a,\ y : \{\mathbf{Z}\}\}$.

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