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Conductor inequalities and criteria for Sobolev type two-weight imbeddings

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Dedicated to E.B. Davies on the occasion of his 60th birthday

Abstract

A typical inequality handled in this article connects the L_p-norm of the gradient of a function to a one-dimensional integral of the p-capacitance of the conductor between two level surfaces of the same function. Such *conductor inequalities* lead to necessary and sufficient conditions for multi-dimensional and one-dimensional Sobolev type inequalities involving two arbitrary measures. Compactness criteria and two-sided estimates for the essential norm of the related imbedding operator are obtained. Some counterexamples are presented to illustrate the peculiarities arising in the case of higher derivatives. Criteria for two-weight inequalities with fractional Sobolev norms of order $l < 2$ are found.

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Keywords: Conductor inequalities; Two-weight integral inequalities; Strong type capacitary inequalities; Essential norm; Multiplicative inequalities; Weighted Sobolev spaces; Fractional Sobolev spaces

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1. Introduction

Let Ω be an open set in $\mathbb{R}^n$ and let μ and ν be locally finite nonzero Borel measures on Ω. We also use the following notation: l is a positive integer, $1 \leqslant p < \infty$, $q > 0$, $\mathrm{d}x$ is an element of the Lebesgue measure m_n on $\mathbb{R}^n$, and f is an arbitrary function in $C_0^\infty(\Omega)$, i.e. an infinitely differentiable function with compact support in Ω. By M_t we mean the set $\{x \in \Omega : |f(x)| > t\}$, where $t > 0$. We shall use the equivalence relation $a \sim b$ to denote that the ratio a/b admits upper and lower bounds by positive constants depending only on n, l, p, q.

In this paper we discuss variants and applications of the inequality

$$\int_0^\infty \operatorname{cap}_p(\overline{M_{at}}, M_t)\, \mathrm{d}(t^p) \leqslant c(a, p) \int_\Omega |\operatorname{grad} f|^p\, \mathrm{d}x, \tag{1}$$

where $a = \text{const} > 1$ and cap_p is the so-called conductor p-capacitance (see (10)). A discrete version of (1) and its analogue involving second order derivatives of a nonnegative f were obtained by the author in 1972 [22].

By monotonicity of cap_p the *conductor inequality* (1) implies

$$\int_0^\infty \operatorname{cap}_p(\overline{M_t}, \Omega)\, \mathrm{d}(t^p) \leqslant C(p) \int_\Omega |\operatorname{grad} f|^p\, \mathrm{d}x, \tag{2}$$

which was also proved in [21] with the best constant

$$C(p) = p^p (p-1)^{1-p}.$$

(For $p = 2$ inequality (2) with $C(2) = 4$ was used without explicit formulation already in [19–21].)

Inequality (2) and its various extensions are sometimes called either *capacitary* or *strong type capacitary inequalities*.

They are of independent interest and have numerous applications to the theory of Sobolev spaces, linear and nonlinear partial differential equations, calculus of variations, theories of Dirichlet forms and Markov processes, etc. ([1–5,7,10–12,14,15,17,21,22,25,30,35–39]).

It is, perhaps, worth mentioning that the proof of (1) is so simple and generic that it works in a much more general frame of analysis on manifolds and metric spaces (see [13,16]).

In what follows, we deal mostly with applications of conductor inequalities to two measure Sobolev type imbeddings which seem to be unattainable with the help of capacitary strong type inequalities. In particular, we sometimes assume that $n = 1$ and we study inequalities of the type

$$\left(\int_\Omega |f|^q\, \mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |f^{(l)}|^p\, \mathrm{d}x + \int_\Omega |f|^p\, \mathrm{d}\nu\right)^{1/p}, \tag{3}$$

where $f \in C_0^\infty(\Omega)$, and their analogues involving a fractional Sobolev norm. Inequality (3) and its applications were the subject of extensive work. See, for example, books [8,18,23,28,32], papers [6,9,21,27,29,31,33,34], and references given there.

Let $n = 1$, $x \in \mathbb{R}$, $d > 0$, and let $\sigma_d(x)$ denote the open interval $(x - d, x + d)$. The equivalence of the inequality

$$\left(\int_\Omega |f|^q\, \mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |f'|^p\, \mathrm{d}x + \int_\Omega |f|^p\, \mathrm{d}\nu\right)^{1/p} \tag{4}$$

with an arbitrary $f \in C_0^\infty(\Omega)$ and $q \geqslant p$, and the statement

$$\mu(\sigma_d(x))^{p/q} \leqslant \mathrm{const}(\tau^{1-p} + \nu(\sigma_{d+\tau}(x))), \tag{5}$$

where x, d and τ are such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$, is valid without complementary assumptions about μ and ν. Criterion (5) is a particular case of a general multi-dimensional condition equivalent to the inequality

$$\left(\int_\Omega |f|^q \, \mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |\phi(x, \operatorname{grad} f)|^p \, \mathrm{d}x + \int_\Omega |f|^p \, \mathrm{d}\nu\right)^{1/p} \tag{6}$$

obtained in [21] (see also [23, Theorem 2.3.7]). The condition just referenced is formulated in terms of the conductor capacitance generated by the integral

$$\int_\Omega |\phi(x, \operatorname{grad} f)|^p \, \mathrm{d}x,$$

where the function: $\Omega \times \mathbb{R}^n \ni (x, y) \to \phi(x, y)$ is positively homogeneous in y of degree 1 and subject to the Caratheodory condition. In the one-dimensional case, when this capacitance is calculated explicitly (see either [21, Lemma 4] or [23, Lemma 2.2.2/2]), the general criterion just mentioned takes a much simpler form, which is given in (5).

We conclude Introduction with a brief outline of the contents of the paper. A proof of (1) is given in Section 2. In Sections 3 and 4 we discuss inequality (4) and give a criterion for its multiplicative analogue. A necessary and sufficient condition for the compactness and two-sided estimates of the essential norm of the imbedding operator associated with (4) are obtained in Section 5.

In Section 6 we characterise the inequality

$$\left(\int_\Omega |f|^q \, \mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |f''(x)|^p \, \mathrm{d}x + \int_\Omega |f|^p \, \mathrm{d}\nu\right)^{1/p} \tag{7}$$

with $1 < p \leqslant q < \infty$, restricted to nonnegative functions $f \in C_0^\infty(\Omega)$, by requiring the condition

$$\mu(\sigma_d(x))^{p/q} \leqslant \mathrm{const}(\tau^{1-2p} + \nu(\sigma_{d+\tau}(x)) \tag{8}$$

to be valid for all intervals $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. A simple example shows that (8) does not guarantee (7) for all $f \in C_0^\infty(\Omega)$. We also give counterexamples showing that the necessary condition for (3)

$$\mu(\sigma_d(x))^{p/q} \leqslant \mathrm{const}(\tau^{1-lp} + \nu(\sigma_{d+\tau}(x)) \tag{9}$$

is not sufficient if $l \geqslant 3$.

Section 7 is dedicated to multi-dimensional conductor (p, l)-capacitance inequalities for fractional Sobolev L_p-norms of order l in (0, 1) and (1, 2). The article is concluded with necessary and sufficient conditions for two-measure multi-dimensional inequalities of type (6) involving fractional norms.

2. Inequality (1)

Let g and G denote arbitrary bounded open sets in $\mathbb{R}^n$ subject to $\bar{g} \subset G, \bar{G} \subset \Omega$. We introduce the p-capacitance of the conductor $G \backslash \overline{g}$ (in other terms, the relative p-capacity of the set $\bar{g}$ with respect to G) as

$$\operatorname{cap}_p(\bar{g}, G) = \inf\left\{\int_\Omega |\operatorname{grad}\varphi(x)|^p\,\mathrm{d}x : \varphi\in C_0^\infty(G),\ 0\leqslant\varphi\leqslant 1 \text{ on } G \text{ and } \varphi = 1 \text{ on a neighborhood of } \overline{g}\right\}. \tag{10}$$

This infimum does not change if the class of admissible functions φ is enlarged to

$$\{\varphi\in C^\infty(\Omega) : \varphi\geqslant 1 \text{ on } g,\ \varphi\leqslant 0 \text{ on } \Omega\backslash G\} \tag{11}$$

(see [23, Section 2.2]).

Now, we derive a generalization of the conductor inequality (1).

Proposition 1. *For all $f\in C_0^\infty(\Omega)$ and for an arbitrary $a>1$ inequality* (1) *holds with*

$$c(a,p) = \frac{p\log a}{(a-1)^p}.$$

Proof. We show first that the function $t\to\operatorname{cap}_p(\overline{M_{at}}, M_t)$ is measurable. Let us introduce the open set $\mathscr{S} := \{t>0 : |\operatorname{grad} f|>0 \text{ on } \partial M_t\}$ whose complement has zero one-dimensional Lebesgue measure by the Morse theorem. Let $t_0\in\mathscr{S}$. Given an arbitrary $\varepsilon>0$, there exists a function $\varphi\in C_0^\infty(M_{t_0})$, $\varphi=1$ on a neighbourhood $\overline{M_{at_0}}$, and such that

$$\|\operatorname{grad}\varphi\|_{L_p}^p \leqslant \operatorname{cap}_p(\overline{M_{at_0}}, M_{t_0}) + \varepsilon.$$

Since $t_0\in\mathscr{S}$ we deduce from (10) that for all sufficiently small $\delta>0$

$$\|\operatorname{grad}\varphi\|_{L_p}^p \geqslant \operatorname{cap}_p(\overline{M_{a(t_0-\delta)}}, M_{t_0+\delta}).$$

Therefore,

$$\operatorname{cap}_p(\overline{M_{a(t_0\pm\delta)}}, M_{t_0\pm\delta}) \leqslant \operatorname{cap}_p(\overline{M_{at_0}}, M_{t_0}) + \varepsilon,$$

which means that the function $t\to\operatorname{cap}_p(\overline{M_{at}}, M_t)$ is upper semicontinuous on $\mathscr{S}$. The measurability of this function follows.

Let γ denote a locally integrable function on $(0,\infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. Then there exists the improper integral

$$\int_0^\infty (\gamma(t)-\gamma(at))\,\frac{\mathrm{d}t}{t} := \lim_{\varepsilon\to 0_+, N\to+\infty}\int_\varepsilon^N (\gamma(t)-\gamma(at))\,\frac{\mathrm{d}t}{t},$$

and the following identity

$$\int_0^\infty (\gamma(t)-\gamma(at))\,\frac{\mathrm{d}t}{t} = (\gamma(0)-\gamma(\infty))\log a \tag{12}$$

holds. Setting here

$$\gamma(t) := \int_{M_t} |\operatorname{grad} f|^p\,\mathrm{d}x,$$

we obtain

$$\int_\Omega |\mathrm{grad}\, f|^p\, \mathrm{d}x \geqslant \frac{1}{\log\, a} \int_0^\infty \int_{M_t \setminus M_{at}} |\mathrm{grad}\, f|^p\, \mathrm{d}x\, \frac{\mathrm{d}t}{t}.$$

By (10) the right-hand side exceeds

$$\frac{(a-1)^p}{p\, \log\, a} \int_0^\infty \mathrm{cap}_p(\overline{M_{at}}, M_t)\, \mathrm{d}(t^p)$$

and (1) follows. □

3. Applications of (1)

The following lemma, essentially resulting from (1), is a particular case of the general result from [21] and mentioned in Introduction.

Lemma 1. *Let* $1 \leqslant p \leqslant q$. *The inequality*

$$\left(\int_\Omega |f|^q\, \mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |\mathrm{grad}\, f|^p\, \mathrm{d}x + \int_\Omega |f|^p\, \mathrm{d}v\right)^{1/p} \tag{13}$$

holds for all $f \in C_0^\infty(\Omega)$ *if and only if there exists a constant* $K > 0$ *such that for all open bounded setsg and* G, *subject to* $\bar{g} \subset G$, $\bar{G} \subset \Omega$, *the inequality*

$$\mu(g)^{1/q} \leqslant K(\, \mathrm{cap}_p(\bar{g}, G) + v(G))^{1/p} \tag{14}$$

is valid.

We prove this lemma here for readers' convenience.

Proof. The necessity is proved simply by putting any function φ from class (11) into (13). Let us prove the sufficiency of (14). We use the obvious identity

$$|f|^q = \left(\int_0^\infty \chi_{M_t}\, \mathrm{d}(t^p)\right)^{q/p},$$

where χ_{M_t} stands for the characteristic function of the set M_t. Hence

$$\|f\|_{L_q(\mu)} = \left\|\int_0^\infty \chi_{M_t}\, \mathrm{d}(t^p)\right\|^{1/p}_{L_{q/p}(\mu)}, \tag{15}$$

where the notation

$$\|f\|_{L_q(\mu)} = \left(\int_\Omega |f|^q\, \mathrm{d}\mu\right)^{1/q}$$

is used. Since $q \geqslant p$, it follows by Minkowski's inequality that the right-hand side in (15) does not exceed

$$\left(\int_0^\infty \|\chi_{M_t}\|_{L_{q/p}(\mu)}\, \mathrm{d}(t^p)\right)^{1/p}.$$

Hence

$$\|f\|^p_{L_q(\mu)} \leqslant \int_0^\infty \mu(M_t)^{p/q}\, \mathrm{d}(t^p). \tag{16}$$

Let $a \in (1, \infty)$. By (16) and (14)

$$\|f\|^p_{L_q(\mu)} \leqslant a^p \int_0^\infty \mu(M_{at})^{p/q}\, \mathrm{d}(t^p) \leqslant a^p K^p \int_0^\infty (\mathrm{cap}_p(\overline{M_{at}}, M_t) + v(M_t))\, \mathrm{d}(t^p),$$

which, together with Proposition 1, implies

$$\|f\|^p_{L_q(\mu)} \leqslant K^p \left(\frac{p\, a^p \,\log a}{(a-1)^p} \int_\Omega |\mathrm{grad}\, f|^p\, \mathrm{d}x + a^p \int_\Omega |f|^p\, \mathrm{d}v \right).$$

The sufficiency of (14) follows. □

From Lemma 1, we shall deduce a sufficient condition for (13) which does not involve the p-capacitance.

Corollary 1. *Let $n < p \leqslant q$. If for all bounded open sets g* and *G in $\mathbb{R}^n$ such that $\overline{g} \subset G$, $\overline{G} \subset \Omega$*

$$\mu(g)^{1/q} \leqslant K((\mathrm{dist}(\partial g, \partial G)^{n-p} + v(G))^{1/p}, \tag{17}$$

then (13) *holds for all $f \in C_0^\infty(\Omega)$.*

Proof. Let φ be an arbitrary admissible function in (10). By Sobolev's integral representation (see, for example, [23, 1.1.10]), we have for all $y \in g$ and $z \in \Omega \backslash G$

$$1 \leqslant (\varphi(y) - \varphi(z))^p \leqslant c\, |y - z|^{p-n} \int_\Omega |\mathrm{grad}\, \varphi(x)|^p\, \mathrm{d}x,$$

which implies

$$(\mathrm{dist}(\partial g, \partial G))^{n-p} \leqslant c\ \mathrm{cap}_p(\overline{g}, G).$$

It remains to refer to Lemma 1. □

Let us see how criterion (5) follows from Lemma 1.

Theorem 1. *Let $n = 1$ and $1 \leqslant p \leqslant q < \infty$. Inequality* (4) *holds for all $f \in C_0^\infty(\Omega)$ if and only if condition* (5) *is satisfied. The sharp constant C in* (4) *is equivalent to*

$$\sup_{x,d,\tau} \frac{\mu(\sigma_d(x))^{1/q}}{(\tau^{1-p} + v(\sigma_{d+\tau}(x)))^{1/p}},$$

where x, d, τ are the same as in (5).

Proof. Let $g_0=(a,b)$, $G_0=(A,B)$ and $A<a<b<B$. It is an easy exercise to show that

$$\operatorname{cap}_p(\bar{g}_0, G_0)=(a-A)^{1-p}+(B-b)^{1-p}. \tag{18}$$

(For the proof of a more general formula for a weighted p-capacitance see either Lemma 4 in [21] or Lemma 2.2.2/2 in [23].) Hence, by setting $g=\sigma_d(x)$ and $G=\sigma_{d+\tau}(x)$ into (14), we obtain

$$\mu(\sigma_d(x))^{1/q}\leqslant K(2\tau^{1-p}+\nu(\sigma_{d+\tau}(x))^{1/p},$$

which implies the necessity of (5). In order to prove the sufficiency we need to obtain (14) for all admissible sets g and G. Let G be the union of nonoverlapping intervals G_i and let $g_i=G_i\cap g$. Denote by h_i the smallest interval containing g_i and by τ_i the minimal distance from h_i to $\mathbb{R}\backslash G_i$. By definition of the p-capacitance (10) in the one-dimensional case, we have

$$\operatorname{cap}_p(\bar{g}_i, G_i)=\operatorname{cap}_p(\bar{h}_i, G_i)$$

and

$$\operatorname{cap}_p(\bar{g}, G)=\sum_i \operatorname{cap}_p(\bar{g}_i, G_i).$$

Hence, and by (18) applied to the intervals h_i and G_i,

$$\operatorname{cap}_p(\bar{g}, G)\geqslant \sum_i \tau_i^{1-p}. \tag{19}$$

Using (5), we obtain

$$\mu(g_i)^{1/q}\leqslant \mu(h_i)^{1/q}\leqslant A(\tau_i^{1-p}+\nu(G_i))^{1/q},$$

where A is a positive constant independent of g and G. Since $q\geqslant p$, we have

$$\mu(g)^{p/q}\leqslant \sum_i \mu(g_i)^{p/q},$$

which, together with (19), implies

$$\mu(g)^{p/q}\leqslant A^p \sum_i(\tau_i^{1-p}+\nu(G_i))\leqslant A^p(\operatorname{cap}_p(\bar{g}, G)+\nu(G)).$$

The result follows from Lemma 1. □

In the next remark some other straightforward extensions of Theorem 1 are collected.

Remark 1. We obtain from (16) that the left-hand side in (4) can be replaced with

$$\left(\int_0^\infty \mu(M_t)^{p/q}\,\mathrm{d}(t^p)\right)^{1/p}$$

without affecting Theorem 1. In other words, the space $L_q(\mu)$ can be changed for the Lorentz space $L_{q,p}(\mu)$.

Another possible modification of Theorem 1 concerns the Orlicz space $L_M(\mu)$, where M is an arbitrary convex function on $(0,\infty)$, $M(+0)=0$. Let N denote the complementary convex function to M.

One can easily show (compare with Theorem 4 in [21]) that the condition

$$\mu(\sigma_d(x))N^{-1}\left(\frac{1}{\mu(\sigma_d(x))}\right) \leqslant \text{const}(\tau^{1-p} + \nu(\sigma_{d+\tau}(x)))^{1/p}$$

is necessary and sufficient for the inequality

$$\int_0^\infty \mu(M_\tau)\, N^{-1}\left(\frac{1}{\mu(M_\tau)}\right) \mathrm{d}(\tau^p) \leqslant c\left(\int_\Omega |f'|^p\, \mathrm{d}x + \int_\Omega |f|^p\, \mathrm{d}\nu\right)$$

as well as for the inequality

$$\|\, |u|^p \,\|_{L_M(\mu)} \leqslant c\left(\int_\Omega |f'|^p\, \mathrm{d}x + \int_\Omega |f|^p\, \mathrm{d}\nu\right).$$

It is well known that the weight w in the integral

$$\int_\Omega |f'(x)|^p w(x)\, \mathrm{d}x$$

can be removed by the change of the variable x:

$$\xi = \int \frac{\mathrm{d}x}{w(x)^{1/(p-1)}}.$$

Therefore, Theorem 1 leads to a criterion for three-weight inequality

$$\left(\int_\Omega |f|^q\, \mathrm{d}\mu\right)^{1/q} \leqslant \left(\int_\Omega |f'|^p\, \mathrm{d}\lambda + \int_\Omega |f|^p\, \mathrm{d}\nu\right)^{1/p},$$

where λ is a nonnegative measure. Note that the singular part of λ does not influence the validity of the last inequality (compare with [27] and [23, Section 1.3.1]).

Remark 2. Let $n = 1$. With $p \in (1, \infty)$ and the measure ν, we associate a function $\mathscr{R}$ of an interval $\sigma_d(x)$ by the equality

$$\mathscr{R}(\sigma_d(x)) = \sup\{\tau : \tau^{1-p} > \nu(\sigma_{d+\tau}(x))\}, \tag{20}$$

with $\overline{\sigma_{d+\tau}(x)} \subset \Omega$ as everywhere. Clearly,

$$\mathscr{R}(\sigma_d(x))^{1-p} \leqslant \inf\{\tau : \tau^{1-p} + \nu(\sigma_{d+\tau}(x))\} \leqslant 2\mathscr{R}(\sigma_d(x))^{1-p}, \tag{21}$$

which shows that criterion (5) can be written as

$$\sup_{\overline{\sigma_d(x)} \subset \Omega} \mathscr{R}(\sigma_d(x))^{(p-1)/p} \mu(\sigma_d(x))^{1/q} < \infty.$$

Remark 3. According to Theorem 2 in [21] (see also Theorem 2.1.3 in [23]), inequality (4) with $p=1$, $q \geqslant 1$, is equivalent to the inequality

$$\mu(g)^{1/q} \leqslant C(2 + \nu(g)),$$

where g is an arbitrary interval and C is the same constant as in (4).

Similarly to (4), we can characterise the inequality

$$\left(\int_\Omega |f|^q \,\mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |f'|^p \,\mathrm{d}x\right)^{\delta/p}\left(\int_\Omega |f|^r \,\mathrm{d}v\right)^{(1-\delta)/r} \tag{22}$$

by using the following assertion proved in [22].

Lemma 2 (*see [21, Theorem 5] or [23, Theorem 2.3.9]*). *Let $n \geqslant 1$, $p \geqslant 1$ and $\delta \in [0, 1]$. If the inequality*

$$\left(\int_\Omega |f|^q \,\mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |\mathrm{grad}\, f|^p \,\mathrm{d}x\right)^{\delta/p}\left(\int_\Omega |f|^r \,\mathrm{d}v\right)^{(1-\delta)/r} \tag{23}$$

is valid for all $f \in C_0^\infty(\Omega)$ and some positive r and q, then there exists a constant α such that for all open bounded subsets g and G of Ω such that $\bar{g} \subset G$, $\overline{G} \subset \Omega$, the inequality

$$\mu(g)^{p/q} \leqslant \alpha \,\mathrm{cap}_p(\bar{g}, G)^\delta v(G)^{(1-\delta)p/r} \tag{24}$$

holds.

If (24) holds for all g and G as above, then (23) is valid for all functions $f \in C_0^\infty(\Omega)$ with $1/q \leqslant (1-\delta)/r + \delta/p$.

Arguing as in the proof of Theorem 1, we arrive at the following criterion for (22).

Theorem 2. *Let $n = 1$, $p \geqslant 1$ and $\delta \in [0, 1]$. If inequality (22) holds for all $f \in C_0^\infty(\Omega)$ and some positive r and q, then there exists a constant $\beta > 0$ such that*

$$\mu(\sigma_d(x))^{1/q} \leqslant \frac{\beta}{\tau^{\delta(p-1)/p}} v(\sigma_{d+\tau}(x))^{(1-\delta)/r} \tag{25}$$

for all $x \in \Omega$, $d > 0$ and $\tau > 0$ such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. Conversely, if (25) is true for some positive r and q such that $1/q \leqslant (1-\delta)/r + \delta/p$, then (22) holds.

Note that for $p = 1$ condition (25) is simplified

$$\mu(\sigma_d(x))^{r/q(1-\delta)} \leqslant \mathrm{const}\; v(\sigma_d(x)).$$

For the particular case $\mu = v$, inequality (22) admits the following simpler characterisation which results from [21, Theorem 5].

Theorem 3. (i) *Let $n = 1$, and let the inequality*

$$\mu(\sigma_d(x))^\alpha \leqslant \mathrm{const}\; \tau^{(1-p)/p} \tag{26}$$

with $p \geqslant 1$, and $\alpha > 0$, hold for all $x \in \Omega$, $d > 0$ and $\tau > 0$ such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. Furthermore, let q be a positive number satisfying one of the conditions: (i) $q \leqslant \alpha^{-1}$ if $\alpha p \leqslant 1$ *or* (ii) $q < \alpha^{-1}$ *if $\alpha p > 1$. Then the inequality*

$$\left(\int_\Omega |f|^q \,\mathrm{d}\mu\right)^{1/q} \leqslant C\left(\int_\Omega |f'|^p \,\mathrm{d}x\right)^{\delta/p}\left(\int_\Omega |f|^r \,\mathrm{d}\mu\right)^{(1-\delta)/r} \tag{27}$$

with $r \in (0, q)$ and $\delta = (q-r)/(1-\alpha r)q$, is valid for any function $f \in C_0^\infty(\Omega)$.

(ii) *Conversely, let* $p \geqslant 1$, $\alpha > 0$ *and* $r \in (0, \alpha^{-1}]$. *Furthermore, let the inequality* (27) *with* $\delta = (q-r)/(1-\alpha r)q$ *be fulfilled for any function* $f \in C_0^\infty(\Omega)$. *Then* (26) *holds for all* x *and* d *such that* $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

Remark 4. Comparing Theorems 1 and 3 we see that the multiplicative inequality (27) is equivalent to

$$\left(\int_\Omega |f|^{1/\alpha}\, \mathrm{d}\mu\right)^\alpha \leqslant C\left(\int_\Omega |f'|^p\, \mathrm{d}x\right)^{1/p}$$

if $\alpha p \leqslant 1$.

The next assertion concerning an arbitrary charge λ (not a nonnegative measure as elsewhere) follows directly from [23, Theorem 2.3.8].

Theorem 4. *Let* $n = 1$, *and let* λ^+ *and* λ^- *denote the positive and negative parts of the charge* λ, *respectively.*

(i) *Let* $\varepsilon \in (0, 1)$ *and* $p > 1$. *If the inequality*

$$\lambda^+(\sigma_d(x)) \leqslant C_\varepsilon \tau^{1-p} + (1-\varepsilon)\lambda^-(\sigma_{d+\tau}(x))$$

holds for all $x \in \Omega$, $d > 0$, $\tau > 0$, *such that* $\overline{\sigma_{d+\tau}(x)} \subset \Omega$, *then for all* $f \in C_0^\infty(\Omega)$

$$\int_\Omega |f|^p \mathrm{d}\lambda \leqslant C \int_\Omega |f'|^p\, \mathrm{d}x. \tag{28}$$

(ii) *If* (28) *is true, then*

$$\lambda^+(\sigma_d(x)) \leqslant C\tau^{1-p} + \lambda^-(\sigma_{d+\tau}(x)) \tag{29}$$

for all $x \in \Omega$, $d > 0$, $\tau > 0$, *such that* $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

Example 1. We show that (29) is not sufficient for (28). Let λ^+ and λ^- be the Dirac measures concentrated at the points 0 and 1, respectively. We introduce the sequence of piecewise linear functions $\{\varphi_m\}_{m=1}^\infty$ on $\mathbb{R}$ by

$$\varphi_m(x) = 0 \quad \text{for } |x| > m^{p/(p-1)},$$

$$\varphi_m(0) = 1, \quad \varphi_m(1) = 1 - m^{-1}.$$

Then

$$\int_{\mathbb{R}} |\varphi_m|^p\, \mathrm{d}\lambda = \frac{p}{m}(1 + o(1)) \quad \text{and} \quad \int_{\mathbb{R}} |\varphi_m'|^p\, \mathrm{d}x \sim m^{-p} \quad \text{as } m \to \infty$$

and therefore (28) fails. However, condition (29) holds with $C = 1$. In order to check this, we need to consider only the case $\lambda^+(\sigma_d(x)) = 1$ and $\lambda^-(\sigma_{d+\tau}(x)) = 0$, when clearly $\tau \leqslant 1$ and $\tau^{1-p} \geqslant 1$.

4. A p-capacity depending on v and its applications to inequalities (4) and (13)

Let $n \geqslant 1$ and let K denote a compact subset of Ω. We introduce a relative p-capacity of K with respect to Ω, depending on the measure v, by

$$\operatorname{cap}_p(K, \Omega, v) = \inf \left(\|\operatorname{grad} \varphi\|^p_{L_p} + \int_\Omega |\varphi|^p \, \mathrm{d}v \right), \tag{30}$$

where infimum is extended over all functions $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geqslant 1$ on K. Arguing as in [23, Section 2.2], one can show that the infimum in (30) will be the same if the set of admissible functions is replaced with $\{\varphi \in C_0^\infty(\Omega) : \varphi = 1 \text{ on } K \ 0 \leqslant \varphi \leqslant 1 \text{ on } \Omega\}$.

Making small changes in the proof of Proposition 1, one arrives at the inequality

$$\int_0^\infty \operatorname{cap}_p(\overline{M_{at}}, M_t, v) \, \mathrm{d}(t^p) \leqslant c(p) \left(\|\operatorname{grad} f\|^p_{L_p} + \int_\Omega |f|^p \, \mathrm{d}\mu \right),$$

where $a = const > 1$ and $f \in C_0^\infty(\Omega)$. By this inequality one can easily obtain the following condition, necessary and sufficient for (13) with $q \geqslant p$:

$$\mu(g)^{p/q} \leqslant \text{const } \operatorname{cap}_p(g, \Omega, v) \tag{31}$$

for all bounded open sets g with $\overline{g} \subset \Omega$.

The next lemma shows directly that (31) is equivalent to (14).

Lemma 3. *The equivalence relation holds*,

$$\operatorname{cap}_p(K, \Omega, v) \sim \inf_G (\operatorname{cap}_p(K, G) + v(G)), \tag{32}$$

where infimum is taken over all bounded open sets G such that $K \subset G$ and $\overline{G} \subset \Omega$.

Proof. Let $\varepsilon > 0$, $f \in C_0^\infty(\Omega)$, $f = 1$ on K, $0 \leqslant f \leqslant 1$ on Ω and

$$\operatorname{cap}_p(K, \Omega, v) + \varepsilon \geqslant \|\operatorname{grad} f\|^p_{L_p} + \int_\Omega |f|^p \, \mathrm{d}v.$$

Then

$$\begin{aligned} \operatorname{cap}_p(K, \Omega, v) + \varepsilon &\geqslant \sum_{k=0}^\infty 2^{-p(k+1)} \int_{M_{2^{-k-1}} \setminus M_{2^{-k}}} |\operatorname{grad}(2^{k+1} f - 1)|^p \, \mathrm{d}x + \int_0^1 v(M_t) \, \mathrm{d}(t^p) \\ &\geqslant c \sum_{k=0}^\infty 2^{-pk} (\operatorname{cap}_p(\overline{M_{2^{-k}}}, M_{2^{-k-1}}) + v(M_{2^{-k-1}})). \end{aligned}$$

Since $\operatorname{cap}_p(\overline{M_{2^{-k}}}, M_{2^{-k-1}}) \geqslant \operatorname{cap}_p(K, M_{2^{-k-1}})$, it follows that

$$\operatorname{cap}_p(K, \Omega, v) + \varepsilon \geqslant c \inf_G \ (\operatorname{cap}_p(K, G) + v(G)).$$

The estimate

$$\operatorname{cap}_p(K, \Omega, v) \leqslant \operatorname{cap}_p(K, G) + v(G)$$

is obvious. The result follows. □

We introduce the capacity minimising function

$$S_p(t) = \inf \ \operatorname{cap}_p(g, \Omega, v),$$

where the infimum is taken over all bounded open sets g, $\overline{g} \subset \Omega$, satisfying $\mu(g) > t$.

By Lemma 3,

$$S_p(t) \sim \inf_{g,G} \ (\operatorname{cap}_p(g, G) + v(G))$$

with the infimum extended over open sets g and G such that $\overline{g} \subset G$, $\overline{G} \subset \Omega$, and $\mu(g) > t$. Obviously, condition (31) is equivalent to

$$\sup \frac{t^{p/q}}{S_p(t)} < \infty.$$

Making trivial changes in the proof of Theorem 1 [25] (see also [26, Theorem 8.5.3]), we arrive at *the condition, necessary and sufficient for* (13) *with* $0 < q < p$, $p \geqslant 1$:

$$\int_0^\infty \left(\frac{t^{p/q}}{S_p(t)}\right)^{q/(p-q)} \frac{\mathrm{d}t}{t} < \infty. \tag{33}$$

It follows from the proof of Theorem 1 that in the one-dimensional case

$$S_p(t) \sim \inf\{\tau : \tau^{1-p} + v(\sigma_{d+\tau}(x))\}$$

with the infimum taken over all x, d, τ such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$ and

$$\mu(\sigma_d(x)) > t. \tag{34}$$

By (21),

$$S_p(t) \sim \inf \ \mathscr{R}(\sigma_d(x))^{1-p},$$

where the infimum is taken over all intervals $\sigma_d(x)$, $\overline{\sigma_d(x)} \subset \Omega$, satisfying (34).

5. Compactness and essential norm

We define the space $\mathring{W}_p^1(v)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_{\mathring{W}_p^1(v)} = \left(\int_\Omega |f'(x)|^p \, \mathrm{d}x + \int_\Omega |f(x)|^p \, \mathrm{d}v\right)^{1/p}.$$

Condition (5) is a criterion of boundedness for the imbedding operator

$$I_{p,q} : \mathring{W}_p^1(\nu) \to L_q(\mu)$$

for $q \geqslant p \geqslant 1$. By Theorem 1,

$$\|I_{p,q}\| \sim \sup_{x,\tau,d} \frac{\mu(\sigma_d(x))^{1/q}}{\{\tau^{1-p} + \nu(\sigma_{d+\tau}(x))\}^{1/p}}, \tag{35}$$

where x, τ and d are subject to $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

In this section we establish a compactness criterion for $I_{p,q}$ with $q \geqslant p \geqslant 1$ and obtain sharp too-sided estimates for the essential norm of $I_{p,q}$. We recall that the essential norm of a bounded linear operator A acting from X into Y, where X and Y are linear normed spaces, is defined by

$$\text{ess } \|A\| = \inf_T \|A - T\|$$

with infimum taken over all compact operators $T : X \to Y$.

Theorem 5. *If $q \geqslant p \geqslant 1$, then the operator $I_{p,q}$ is compact if and only if*

$$\lim_{M\to\infty} \sup_{x,\tau,d} \frac{\mu(\sigma_d(x)\backslash[-M, M])^{1/q}}{\{\tau^{1-p} + \nu(\sigma_{d+\tau}(x))\}^{1/p}} = 0, \tag{36}$$

where x, τ and d are the same as in (35).

Proof (*Sufficiency*). Let μ' for the restriction of μ to the segment $[-M, M]$ and let $\mu_M = \mu - \mu'_M$. We define the imbedding operators

$$I_M : \mathring{W}_p^1(\nu) \to L_q(\mu_M) \quad \text{and } I'_M : \mathring{W}_p^1(\nu) \to L_q(\mu'_M)$$

as well as the imbedding operators

$$i_M : L_q(\mu_M) \to L_q(\mu) \quad \text{and } i'_M : L_q(\mu'_M) \to L_q(\mu).$$

We have

$$I_{p,q} = i_M \circ I_M + i'_M \circ I'_M. \tag{37}$$

We prove that I'_M is compact. Consider the imbedding operators

$$I_M^C : \mathring{W}_p^1(\nu) \to C([-M, M]),$$

$$i_M^C : C([-M, M]) \to L_q(\mu'_M),$$

where $C([-M, M])$ is the space of continuous functions with the usual norm. Clearly, $I'_M = i_M^C \circ I_M^C$. Since I_M^C is compact for any $M > 0$ by the Arzela theorem, the operator I'_M is compact too. The condition $\|I_M\| \to 0$ as $M \to \infty$ is equivalent to (36) owing to (35), with I_M instead of $I_{p,q}$.

Necessity: Let $I_{p,q}$ be compact and let B denote the unit ball in $\mathring{W}_p^1(v)$. The set $I_{p,q}B$ is relatively compact in $L_q(\mu)$. Therefore, for any $\varepsilon > 0$ there exists a finite ε-net $\{f_j\}_{j=1}^N \subset I_{p,q}B = B$ for the set $I_{p,q}B$. Given any f_j, there exists a number $M_j(\varepsilon)$ such that

$$\int_{|x|>M_j(\varepsilon)} |f_j(x)|^q \, \mathrm{d}\mu(x) < \varepsilon^q.$$

Let $M(\varepsilon)$ be equal to $\sup_j M_j(\varepsilon)$. Then for any $f \in B$ and for some $i \in \{1, N\}$ we have

$$\left(\int_\Omega |f(x)|^q \, \mathrm{d}\mu_{M(\varepsilon)}(x)\right)^{1/q} \leqslant \|f - f_j\|_{L_q(\mu)} + \left(\int_\Omega |f_j(x)|^q \, \mathrm{d}\mu_{M_j(\varepsilon)}(x)\right)^{1/q} < 2\varepsilon.$$

Hence inequality (4) holds with $\mu_{M(\varepsilon)}$ and 2ε instead of μ and C. Now (36) follows from the necessity part in Theorem 1. □

Theorem 6. *Let $q \geqslant p \geqslant 1$ and*

$$E(\mu, v) := \lim_{M\to\infty} \sup_{x,\tau,d} \frac{\mu(\sigma_d(x)\backslash[-M, M])^{1/q}}{\{\tau^{1-p} + v(\sigma_{d+\tau}(x))\}^{1/p}}.$$

There exist positive constants c_1 and c_2 such that

$$c_1 E(\mu, v) \leqslant \operatorname{ess} \|I_{p,q}\| \leqslant c_2 E(\mu, v). \tag{38}$$

Proof. We use the same notation as in the previous theorem. The upper bound in (38) is a consequence of the sufficiency part in the proof of Theorem 3.

Let T be any compact operator: $\mathring{W}_p^1(v) \to L_q(\mu)$ and let ε be any positive number. We choose T to satisfy

$$\operatorname{ess}\|I_{p,q}\| \geqslant \|I_{p,q} - T\| - \varepsilon. \tag{39}$$

There exists a positive $M(\varepsilon)$ such that for any $f \in B$

$$\int_\Omega |Tf(x)|^q \, \mathrm{d}\mu_{M(\varepsilon)}(x) < \varepsilon^q. \tag{40}$$

We introduce the truncation operator $\tau_M : L_q(\mu) \to L_q(\mu_M)$ by

$$(\tau_M f)(x) = \begin{cases} 0, & |x| < M, \\ f(x), & |x| \geqslant M. \end{cases}$$

Using (39) and (40), we obtain

$$\operatorname{ess} \|I_{p,q}\| \geqslant \|I_{M(\varepsilon)} - \tau_{M(\varepsilon)} \circ T\| - \varepsilon \geqslant \|I_{M(\varepsilon)}\| - 2\varepsilon.$$

By (35) applied to $I_{M(\varepsilon)}$ instead of $I_{p,q}$,

$$\|I_{M(\varepsilon)}\| \geqslant c \sup_{x,\tau,d} \frac{\mu_{M(\varepsilon)}(\sigma_d(x))^{1/q}}{\{\tau^{1-p} + v(\sigma_{d+\tau}(x))\}^{1/p}} \geqslant c_1 E(\mu, v).$$

The result follows. □

Remark 5. Making obvious changes into the [26, proof of Theorem 8.6.2], one can conclude that the imbedding operator $I_{p,q}$ with $0<q<p$, $p \geqslant 1$ is compact and bounded simultaneously. In other words, condition (33) is necessary and sufficient for the compactness of $I_{p,q}$ with these p and q.

6. Inequality (3) with $l \geqslant 2$

From Theorem 1 we deduce a characterisation of inequality (7) for nonnegative functions.

Theorem 7. *Let $n=1$ and $1<p \leqslant q<\infty$. Inequality* (7) *holds for all $f \in C_0^\infty(\Omega)$ and $f \geqslant 0$ on Ω, if and only if there exists a constant $K>0$ such that*

$$\mu(\sigma_d(x))^{1/q} \leqslant K(\tau^{1-2p}+\nu(\sigma_{d+\tau}(x)))^{1/p} \tag{41}$$

for all $x \in \Omega$, $d>0$ and $\tau>0$ satisfying $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

Proof. In order to prove the necessity, we set a function f in (7), which is subject to $f \in C_0^\infty(\Omega)$, $f=1$ on $\sigma_d(x)$, $f=0$ outside $\sigma_{d+\tau}(x)$ and $0 \leqslant f(x) \leqslant 1$ on Ω. Then

$$\mu(\sigma_d(x))^{1/q} \leqslant C\left(\int_{\sigma_{d+\tau}(x)\backslash\sigma_d(x)} |f''(y)|^p \, \mathrm{d}y + \nu(\sigma_{d+\tau}(x))\right)^{1/p}.$$

Clearly, f can be chosen on $\sigma_{d+\tau}(x)\backslash\sigma_d(x)$ so that the integral on the right does not exceed $c(p)\tau^{1-2p}$. Estimate (41) follows.

Let us turn to the proof of sufficiency of (41). Let $f \in C_0^\infty(\Omega)$ satisfy $\operatorname{supp} f=[a,b] \subset \Omega$ and $f>0$ for $x \in (a,b)$. Then $f^{1/2} \in W_{2p}^1(a,b)$ and

$$\int_a^b \frac{|f'|^{2p}}{f^p} \, \mathrm{d}x \leqslant \left(\frac{2p-1}{p-1}\right)^p \int_a^b |f''|^p \, \mathrm{d}x \tag{42}$$

according to [23, Lemma 8.2.1]. (One can easily construct a sequence of functions f showing that the constant factor in the right-hand side of (42) is sharp.) By Theorem 1 we find that the function $u:=f^{1/2}$ satisfies

$$\left(\int_a^b |u|^{2q} \, \mathrm{d}\mu\right)^{1/2q} \leqslant cK^{1/2}\left(\int_a^b |u'|^{2p} \, \mathrm{d}x + \int_a^b |u|^{2p} \, \mathrm{d}\nu\right)^{1/2p}.$$

This inequality, being combined with (42), gives (7).

Let f be an arbitrary nonnegative function in $C_0^\infty(\Omega)$. Representing Ω as the union of nonoverlapping intervals with the same properties as (a,b) we complete the proof. □

An alternative proof of Theorem 7 relies upon the following conductor inequality whose proof is based upon the *smooth level truncation* introduced in [21] (see also [23, Section 8.2.1]).

Proposition 2. *Let $n \geqslant 1$, $f \in C_0^\infty(\Omega)$, $f \geqslant 0$, $a=const>1$, and $p>1$. Then*

$$\int_0^\infty \operatorname{cap}_{p,2}^+(\overline{M_{at}}, M_t) \, \mathrm{d}(t^p) \leqslant c(p,a) \int_\Omega |\operatorname{grad}_2 f|^p \, \mathrm{d}x, \tag{43}$$

where $\mathrm{grad}_2 = \{\partial^2/\partial x_i \partial x_j\}_{i,j=1}^n$ *and*

$$\mathrm{cap}^+_{p,2}(\overline{g}, G) = \inf \left\{ \int_G |\mathrm{grad}_2 \varphi(x)|^p \, \mathrm{d}x : \varphi \in C_0^\infty(G),\ 1 \geqslant \varphi \geqslant 0 \text{ on } G, \right.$$
$$\left. \varphi = 1 \text{ in a neighborhood of } \overline{g} \right\}. \tag{44}$$

(Concerning the measurability of the function $t \to \mathrm{cap}^+_{p,2}(\overline{M_{at}}, M_t)$ see the beginning of the proof of Proposition 1.)

Proof. Let $H \in C^2(\mathbb{R})$,

$$H(x) = \begin{cases} 0, & \text{for } x < \varepsilon, \\ 1, & \text{for } x > 1 - \varepsilon, \end{cases}$$

where ε is an arbitrary number in (0, 1). By (44),

$$\mathrm{cap}^+_{p,2}(\overline{M_{at}}, M_t) \leqslant \int_\Omega \left| \mathrm{grad}_2 \left(H \left(\frac{f(x) - t}{(a-1)t} \right) \right) \right|^p \mathrm{d}x$$
$$\leqslant \frac{c(a)}{t^p} \int_{M_t \setminus M_{at}} \left(\frac{|\mathrm{grad}\, f|^{2p}}{f^p} + |\mathrm{grad}_2 f|^p \right) \mathrm{d}x.$$

Hence the left-hand side in (43) is dominated by

$$p\, c(a) \int_0^\infty \int_{M_t \setminus M_{at}} \left(\frac{|\mathrm{grad}\, f|^{2p}}{f^p} + |\mathrm{grad}_2 f|^p \right) \mathrm{d}x \, \frac{\mathrm{d}t}{t}.$$

Owing to (12), this can be written as

$$p\, c(a) \log a \int_\Omega \left(\frac{|\mathrm{grad}\, f|^{2p}}{f^p} + |\mathrm{grad}_2 f|^p \right) \mathrm{d}x,$$

which does not exceed the right-hand side of inequality (43) in view of (42). The result follows. □

Example 2. Let us show that condition (41) is not sufficient for (7) with $p = 1$. Let ν be Dirac's measure concentrated at $x = 0$ and let $\mathrm{d}\mu(x) = (1 + x^2)^{-1}\, \mathrm{d}x$. Obviously, condition (41) holds. We construct a sequence of nonnegative functions $\eta_m \in C_0^\infty(\mathbb{R})$, $m = 1, 2, \ldots$, defined by $\eta_m(x) = \varphi_m(x - m - 1)$, where φ_m is a smooth, nonnegative, even function on $\mathbb{R}$, vanishing for $x \geqslant m + 1$ and such that $\varphi_m(x) = m + 1 - x$ for $1 \leqslant x \leqslant m$. Then $\eta_m(0) = 0$,

$$\int_\mathbb{R} |\eta_m''|\, \mathrm{d}x = const, \qquad \int_\mathbb{R} \eta_m^q \, \mathrm{d}\mu \to \infty,$$

i.e. inequality (7) with $p = 1$ fails.

Example 3. We shall check that (41) does not suffice for (7) to be valid for all $f \in C_0^\infty(\mathbb{R})$ if $p \geqslant 1$. Let ν and μ be Dirac's measures concentrated at 0 and 1, respectively. Consider the function $\varphi_0 \in C_0^\infty(\mathbb{R})$

such that $\varphi_0(x) = x$ for $x \in [-1, 1]$. We set $\varphi_m(x) = \varphi_0(x/m)$. Then

$$\left(\int_{\mathbb{R}} |\varphi_m|^q \, d\mu\right)^{1/q} = m^{-1}, \quad \left(\int_{\mathbb{R}} |\varphi_m''|^p \, dx\right)^{1/p} = c\, m^{-2+1/p}$$

and inequality (7) fails for $p > 1$. The case $p = 1$ was treated in Example 2.

Example 4. Now we consider the case of the derivative of order $l \geqslant 3$ in inequality (3) for all $f \in C_0^\infty(\Omega)$ such that $f(x) \geqslant 0$ on Ω. By the obvious relation

$$\inf\left\{\int_a^b |f^{(l)}(x)|^p \, dx : f \in C^\infty[a, b],\, f(x) \geqslant 0,\, f(a) = 0,\, f(b) = 1\right\} = c_{l,p}(b-a)^{1-lp},$$

we obtain the following necessary condition for (3)

$$\sup_{x \in \Omega, d > 0} \mu(\sigma_d(x))^{1/q} \left(\inf_{\sigma_{d+\tau}(x) \subset \Omega} \left(\tau^{1-pl} + \nu(\sigma_{d+\tau}(x))\right)\right)^{-1/p} < \infty. \tag{45}$$

We shall verify that this condition is not sufficient for (3) when $l \geqslant 3$ and $p \geqslant 1$.

Suppose first that $p > 1$. Let ν and μ be Dirac's measures concentrated at 0 and 1, respectively. Then (45) holds. Let φ_0 be a nonnegative function in $C_0^\infty(\mathbb{R})$ such that $\varphi_0(x) = x^2$ for $|x| \leqslant 1$. We put $\varphi_m(x) = \varphi_0(x/m)$, $m = 1, 2, \ldots$. Then

$$\int_{\mathbb{R}} |\varphi_m|^q \, d\mu = m^{-2q}, \quad \int_{\mathbb{R}} |\varphi^{(l)}(x)|^p \, dx = c\, m^{1-pl}$$

and inequality (3) fails.

Consider the remaining case $l = 3$, $p = 1$. Let ν be Dirac's measure at O. Then condition (1) has the form

$$\sup_x (1 + x^2) \mu((x, \infty))^{1/q} < \text{const}.$$

For $d\mu(x) = (1 + |x|)^{-2q-1}$ the last condition holds.

We introduce the sequence $\{\Gamma_m(x)\}_{m \geqslant 1}$ by

$$\Gamma_m(x) = \int_0^x \eta_m(t) \, dt \quad \text{for } |x| \leqslant 2m + 2,$$

where η_m is the same as in Example 2. For $|x| \geqslant 2m + 2$ we define Γ_m so that $\Gamma_m \geqslant 0$ and

$$\sup_m \int_{2m+2}^\infty |\Gamma_m^{(3)}(t)| \, dt < \infty.$$

We see that

$$\int_{\mathbb{R}} |\Gamma_m^{(3)}| \, dt = \int_{-\infty}^{2m+2} |\varphi_m''| \, dt + \int_{2m+2}^\infty |\Gamma_m^{(3)}| \, dt < \infty$$

and inequality (3) with $p = 1$, $l = 3$ does not hold.

7. Two-weight inequalities involving fractional Sobolev norms

Let us consider the inequality

$$\left(\int_{\mathbb{R}^n} |f|^q \, \mathrm{d}\mu\right)^{1/q} \leqslant c\left(\langle f\rangle_{p,l}^p + \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}v\right)^{1/p}, \tag{46}$$

where $f \in C_0^\infty(\mathbb{R}^n)$, $p \geqslant 1$, $0 < l < 1$, and

$$\langle f\rangle_{p,l}^p = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+pl}} \, \mathrm{d}x \, \mathrm{d}y.$$

As is well known [36], any smooth extension of f onto $\mathbb{R}^{n+1}$ admits the estimate

$$\langle f\rangle_{p,l}^p \leqslant c \int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\mathrm{grad}\, F|^p \, \mathrm{d}x \, \mathrm{d}x_{n+1} \tag{47}$$

and there exists a linear extension operator $f \to F \in C^\infty(\mathbb{R}^{n+1})$ decaying to 0 at infinity and such that

$$\int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\mathrm{grad}\, F|^p \, \mathrm{d}x \, \mathrm{d}x_{n+1} \leqslant c\langle f\rangle_{p,l}^p. \tag{48}$$

The same argument as in Proposition 1 leads to a conductor inequality, similar to (1), for the integral

$$\int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\mathrm{grad}\, F|^p \, \mathrm{d}x \, \mathrm{d}x_{n+1}, \tag{49}$$

with the left-hand side involving the conductor capacitance generated by (49) (compare with (10)).

Minimizing (49) over all extensions of f and using (47) and (48), we arrive at the fractional conductor inequality

$$\int_0^\infty \mathrm{cap}_{p,l}(\overline{M_{at}}, M_t) \, \mathrm{d}(t^p) \leqslant c(l,p,a)\langle f\rangle_{p,l}^p, \tag{50}$$

where $a > 1$ and

$$\mathrm{cap}_{p,l}(\overline{g}, G) = \inf \langle \varphi\rangle_{p,l}^p \tag{51}$$

with the infimum taken over all $\varphi \in C_0^\infty(G)$ subject to $\varphi = 1$ on $\overline{g}$, $\varphi = 0$ outside G, and $1 \geqslant \varphi \geqslant 0$ on G. This infimum does not change if one requires $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geqslant 1$ on $\overline{g}$ and $\varphi \leqslant 0$ outside G.

By (50) we obtain the following criterion for (46).

Theorem 8. *Let $1 \leqslant p \leqslant q$. Inequality (46) holds for all $f \in C_0^\infty(\mathbb{R}^n)$ if and only if there exists a constant K such that for all open bounded sets g and G subject to $\overline{g} \subset G$ there holds*

$$\mu(g)^{1/q} \leqslant K\,(\mathrm{cap}_{p,l}(\overline{g}, G) + v(G))^{1/p}. \tag{52}$$

The proof does not differ from that of Lemma 1 (see [21, Lemma 4] or [23, Lemma 2.2.2/2]).

Remark 6. The last criterion can be simplified for $p=1$, $q\geqslant 1$, as follows

$$\mu(g)^{1/q}\leqslant K\left(\int_g\int_{\mathbb{R}^n\setminus g}\frac{\mathrm{d}x\ \mathrm{d}y}{|x-y|^{n-pl}}+v(g)\right)$$

for all open bounded sets g. In fact, the necessity results by setting the characteristic function of g into (46). The sufficiency follows from

$$\langle u\rangle_{1,l}=2\int\int_{|u(x)|\leqslant|u(y)|}\int_{|u(x)|}^{|u(y)|}\mathrm{d}t\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+l}}=2\int_0^\infty\int_{M_t}\int_{\mathbb{R}^n\setminus M_t}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+l}}\,\mathrm{d}t$$

combined with (14) where $p=1$.

We turn to the inequality

$$\left(\int_{\mathbb{R}^n}|f|^q\,\mathrm{d}\mu\right)^{1/q}\leqslant c\left(\langle\operatorname{grad} f\rangle_{p,1+l}^p+\int_{\mathbb{R}^n}|f|^p\,\mathrm{d}v\right)^{1/p},\tag{53}$$

where $f\in C_0^\infty(\mathbb{R}^n)$, $f\geqslant 0$, and $0<l<1$.

Lemma 4. *Let $F\in C^\infty(\mathbb{R}^{n+1})$ and $F\geqslant 0$. Then there exists a positive constant $c=c(n,p,l)$ such that*

$$\int_{\mathbb{R}^{n+1}}|x_{n+1}|^{p(1-l)-1}\frac{|\operatorname{grad} F|^{2p}}{F^p}\,\mathrm{d}x\,\mathrm{d}x_{n+1}\leqslant c\int_{\mathbb{R}^{n+1}}|x_{n+1}|^{p(1-l)-1}|\operatorname{grad}_2 F|^p\,\mathrm{d}x\,\mathrm{d}x_{n+1}.\tag{54}$$

Proof. Estimate (54) with $(\partial F/\partial x_1,\ldots,\partial F/\partial x_n)$ instead of grad F in the left-hand side follows immediately from (42). In order to estimate the integral involving only the derivative $\partial F/\partial x_{n+1}$ we need the next inequality for nonnegative functions of one variable

$$\int_{\mathbb{R}}|t|^{p(1-l)-1}\frac{|f'(t)|^{2p}}{f(t)^p}\,\mathrm{d}t\leqslant c\int_{\mathbb{R}}|t|^{p(1-l)-1}|f''(t)|^p\,\mathrm{d}t,\tag{55}$$

which can be proved as follows. According to [24],

$$\frac{f'(t)^2}{f(t)}\leqslant c\,Mf''(t),$$

where M is the Hardy–Littlewood maximal operator. Since the weight $|t|^{p(1-l)-1}$ belongs to the Muckenhoupt class A_p, inequality (55) results from the boundedness of M in $L_p(\mathbb{R};|t|^{p(1-l)-1}\mathrm{d}t)$. The proof of (54) is complete. □

We state a direct corollary of Lemma 4.

Corollary 2. *Let F be the same as in Lemma* 4 *and let h be a function in $C^{1,1}(0,\infty)$ such that $C:=\sup\{t>0:|h'(t)|+t\,|h''(t)|<\infty\}$. Then*

$$\|\,|x_{n+1}|^{1-l-1/p}\operatorname{grad}_2 h(F)\|_{L_p(\mathbb{R}^{n+1})}\leqslant c\,C\,\|\,|x_{n+1}|^{1-l-1/p}\operatorname{grad}_2 F\|_{L_p(\mathbb{R}^{n+1})}.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$, $f \geqslant 0$. The standard extension operator with nonnegative radial kernel gives a nonnegative extension $F \in C^\infty(\mathbb{R}^{n+1})$ of f satisfying

$$\| \, |x_{n+1}|^{1-l-1/p} \mathrm{grad}_2 F \|_{L_p(\mathbb{R}^{n+1})} \leqslant c \, \langle f \rangle_{p,1+l}.$$

Therefore, arguing as in the proof of Proposition 2 and using the last inequality and the trace inequality (47), we arrive at the conductor inequality

$$\int_0^\infty \mathrm{cap}^+_{p,1+l}(\overline{M_{at}}, M_t) \mathrm{d}(t^p) \leqslant c(l, p, a) \langle f \rangle^p_{p,1+l}, \tag{56}$$

where

$$\mathrm{cap}^+_{p,1+l}(\overline{g}, G) = \inf\{\langle \varphi \rangle^p_{p,1+l} : \varphi \in C_0^\infty(G),\ 1 \geqslant \varphi \geqslant 0 \text{ on } G, \text{ and } \varphi = 1 \text{ on a neighbourhood of } \overline{g}\}.$$

Repeating the proof of Lemma 1 and using (56) instead of (1), we arrive at the following criterion.

Theorem 9. *Let* $1 \leqslant p \leqslant q$. *Inequality*

$$\left(\int_{\mathbb{R}^n} |f|^q \, d\mu \right)^{1/q} \leqslant c \left(\langle f \rangle^p_{p,1+l} + \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}v \right)^{1/p}$$

holds for all nonnegative $f \in C_0^\infty(\mathbb{R}^n)$ *if and only if there exists a constant* K *such that*

$$\mu(g)^{1/q} \leqslant K (\mathrm{cap}^+_{p,1+l}(\overline{g}, G) + v(G))^{1/p}$$

for all open bounded sets g *and* G *subject to* $\overline{g} \subset G$.

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