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Probing the arrangement of hyperplanes

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Abstract

In this paper we investigate the combinatorial complexity of an algorithm to determine the geometry and the topology related to an arrangement of hyperplanes in multi-dimensional Euclidean space from the "probing" on the arrangement. The "probing" by a flat means the operation from which we can obtain the intersection of the flat and the arrangement. For a finite set H of hyperplanes in $\boldsymbol{E}^d$, we obtain the worst-case number of fixed direction line probes and that of flat probes to determine a generic line of H and H itself. We also mention the bound for the computational complexity of these algorithms based on the efficient line probing algorithm which uses the dual transform to compute a generic line of H.

We also consider the problem to approximate arrangements by extending the point probing model, which have connections with computational learning theory such as learning a network of threshold functions, and introduce the vertical probing model and the level probing model. It is shown that the former is closely related to the finger probing for a polyhedron and that the latter depends on the dual graph of the arrangement.

The probing for an arrangement can be used to obtain the solution for a given system of algebraic equations by decomposing the u-resultant into linear factors. It also has interesting applications in robotics such as a motion planning using an ultrasonic device that can detect the distances to obstacles along a specified direction.

1. Introduction

An *arrangement* $\mathscr{A}(H)$ of a finite set H of hyperplanes in d-dimensional Euclidean space $\boldsymbol{E}^d$ is a dissection of $\boldsymbol{E}^d$ into connected pieces of various dimensions defined by the hyperplanes, and each k-dimensional connected piece of the dissection is called a *k-face* of the arrangement. A 0-face is called a *vertex*, a 1-face is called an *edge*, a $(d-1)$-face is called a *facet*, and a d-face is called a *cell*.

For any point set $X \subseteq \boldsymbol{E}^d$, let $\mathrm{cl}(X)$ denote the closure of X. A face f is said to be a *subface* of another face g if the dimension of f is one less than the dimension of g and f is contained in the boundary of g. If f is a subface of g, then we also say that f and g are *incident* or that they define an *incidence*. For $0 \leqslant k \leqslant d$, a k-flat in $\boldsymbol{E}^d$ is defined as the affine hull of $k+1$ affinely independent points, that is, a k-flat is an affine subspace

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with dimension k. Clearly, a hyperplane is a $(d-1)$-flat. An arrangement $\mathscr{A}(H)$ is called *simple* if and only if any $d-k$ hyperplanes of H intersect in a common k-flat for $0 \leqslant k < d$.

A line l is called *parallel* to the k-flat f if f contains a line parallel to l. We say l *misses* f if $l \cap f = \emptyset$. For a set of hyperplanes H in $\boldsymbol{E}^d$, if a point $\boldsymbol{p}$ is contained in no hyperplane $h \in H$, $\boldsymbol{p}$ is called *generic* with respect to H. If a point $\boldsymbol{p}$ is contained in only one hyperplane $h \in H$, $\boldsymbol{p}$ is called *proper* with respect to H. A direction vector $\boldsymbol{v}$ is called *generic* with respect to H if $\boldsymbol{v}$ is not parallel to any hyperplane $h \in H$. A line l is called *generic* with respect to H if the direction vector of l is generic with respect to H and the intersecting points $l \cap h$ are proper for each $h \in H$.

In this paper we use standard Cartesian coordinates $x_1, x_2, \ldots, x_d$, to represent points in $\boldsymbol{E}^d$. A k-flat is called *vertical* if it is parallel to the x_d-axis. For every non-vertical hyperplane h, we can choose the normal vector $\boldsymbol{u}$ of h such that the x_d-coordinate value of $\boldsymbol{u}$ is positive, and there is a unique real number α such that h consists of all points $\boldsymbol{x}$ satisfying $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{x} = \alpha$. We say that a point $\boldsymbol{p} \in \boldsymbol{E}^d$ is *above*, *on*, or *below* h if $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{p}$ is greater than, equal to, or less than α, respectively. Let h^+ denote the set of points above h, and h^- denote the set of points below h. Both, h^+ and h^- are open half-spaces. We define $v(\boldsymbol{p}, h)$ for a point $\boldsymbol{p} \in \boldsymbol{E}^d$ as

$$v(\boldsymbol{p}, h) = \begin{cases} +1 & \text{if } \boldsymbol{p} \in h^+, \\ 0 & \text{if } \boldsymbol{p} \in h, \text{ and} \\ -1 & \text{if } \boldsymbol{p} \in h^-. \end{cases}$$

Let H be a finite set of hyperplanes in $\boldsymbol{E}^d$, and let $h_1, h_2, \ldots, h_n$ be the hyperplanes in H. In case that H contains non-vertical hyperplanes, a redefinition of the coordinate system can be used to make all hyperplanes of H non-vertical. For a point $\boldsymbol{p}$, we define the vector

$$\boldsymbol{u}(\boldsymbol{p}) = (v(\boldsymbol{p}, h_1), v(\boldsymbol{p}, h_2), \ldots, v(\boldsymbol{p}, h_n)),$$

and it is called the *sign vector* of the point $\boldsymbol{p}$. Note that points $\boldsymbol{p}$ and $\boldsymbol{q}$ are in the same face f if and only if $\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{u}(\boldsymbol{q})$, and we consider the sign vector of the face f as

$$\boldsymbol{u}(f) = \boldsymbol{u}(\boldsymbol{p})(= \boldsymbol{u}(\boldsymbol{q}));$$

see [7] for detailed definitions concerning arrangements.

We consider the *probing* operation by a flat for an arrangement from which we can obtain the intersection of the flat and the arrangement, and investigate the problem to evaluate the complexity to determine the geometry and the topology related to the arrangement by using them. In this paper we consider the combinatorial complexity on the number of probing operations. We also consider the computational complexity on the number of fundamental operations such as arithmetic operations ($+$, $-$, $\times$, $\div$) where we assume the computational complexity of a probing operation as proportional to the size of the output from the probing, that is, the number of intersecting points on the probing line; see [2] as a reference. Note that these operations are assumed to be performed with infinite precision over the complex numbers. These assumption, similar to that of counting only comparisons in sorting-related problems, can be justified under various circumstances.

One of the motivations to study the probing problem of arrangements lies in the area of computational algebra. For a finite set H of n hyperplanes $h_i = \{x \in \boldsymbol{R}^d \mid \boldsymbol{a}_i^{\mathrm{T}} x = b_i\}$ $(i = 1, \ldots, n)$ in $\boldsymbol{R}^d$, a function $f_H : \boldsymbol{R}^d \to \boldsymbol{R}$ defined by

$$f_H(x) = \prod_{i=1}^{n} (\boldsymbol{a}_i^{\mathrm{T}} x - b_i)$$

is called the *defining polynomial* of the arrangement of H [17]. It also plays an important role in the interior-point method for linear programming [11, 20]. In modern elimination theory for algebraic equations such as Lazard's method [13], some way of computing the u-resultant, which is the defining polynomial of n hyperplanes in $\boldsymbol{C}^d$, of a given system of algebraic equations is presented, and solutions are obtained by decomposing the u-resultant into linear factors, that is, determining the hyperplanes. This factorization is done by probing the hyperplanes by lines [3] or computing the gradient on a generic line to the hyperplanes by partially using probes [12, 18, 15]. The problem of probing hyperplanes, however, does not seem to have been well studied as a combinatorial problem.

Another motivation of the probing problem of arrangements is concerning the study of probing a convex polyhedron which is a well-studied problem [4, 6]. They use the arrangement of hyperplanes containing each face of the polyhedron to compute probing lines as generic lines of the arrangement. However, the computational complexity was not evaluated in their studies. Thus, it is also an important problem to evaluate the computational complexity to determine a generic line of the arrangement as well as the arrangement itself.

The probing problem itself has also interesting practical applications. An example of a probe is a robot arm moving in a fixed direction and reporting the spatial position of a contact points on obstacles. Another example is an ultrasonic device that can detect the distances to other objects along a specified direction.

In Section 2 we describe the results on line probing for arrangements in [1].

In Section 3, we extend these results to the fixed line probing. It is important to consider the restriction on the directions of the probing lines because the probing cost in the real applications can be dependent on its direction. In computational algebra, for example, we usually use the standard Cartesian coordinate system, in which case the number of arithmetic operations to compute the intersection of hyperplanes with the probing line can be reduced by choosing a parallel line to one of the axes as the probing line. We prove that the combinatorial complexity is same even if we restrict the direction of the probing lines to the fixed one. This implies that the total cost to probe arrangements can be reduced by fixing the direction of all probing lines to what minimizes the cost of the probing operation itself. We also show the bound for the computational complexity of these algorithms based on the efficient line probing algorithm which uses the dual transform to compute a generic line of H.

In Section 4, we generalize these results to the flat probing. It is also important to generalize the probing lines to the probing flats because, for example, we have useful probing tools in the real world such as the scanner which can report the cross section of objects by a plane as well as by a line. We prove that the combinatorial complexity is essentially same even if we generalize the dimension of the probing flats. That is, for a finite set H of hyperplanes in $\boldsymbol{E}^d$, we prove that the total dimension of flat probes which are necessary and sufficient to determine a generic line of H is d and that to determine H is $d + 1$, where we show the sufficiency for almost all the dimension sequence of flat probes. The result is significant since it implies that we can combine various dimensional flat probes without increasing the total combinatorial complexity.

Finally, in Section 5, we study the problem to approximate arrangements by extending the point probing model in [1]. We introduce the vertical probing model and the level probing model. In the former model, we give the upper bound of the number of vertical probes required to determine the horizontal section. In the latter model, we prove that, for simple arrangements with the aid of the linear dual graph, it is possible to approximate the level of the points in the level hull with maximum error d. However, it is shown that it is, in general, impossible to approximate the level of the points within the constant independent of n. Also, we obtain the other result under the covering enhanced level probing model. These results are not only interesting from a theoretical point of view but also significant from a practical point of view since these models have essential connections with computational learning theory such as learning a network of threshold functions.

2. Preliminaries on line probing

In this section we describe the results on line probing for arrangements in [1] and preliminaries of the following sections. For a survey of results in geometric probing, see [19].

Let H be a set of hyperplanes in $\boldsymbol{E}^d$ for $d \geqslant 2$. In this paper we consider the *multiplicity* $m(h, H)$ which is an arbitrarily given positive integer value associated with each hyperplane $h \in H$ so that we can represent a multiset as the set H with $m(h, H)$. For a point $\boldsymbol{p} \in \boldsymbol{E}^d$, $m(\boldsymbol{p}, H)$ denotes the multiplicity of $\boldsymbol{p}$ in H, which is the sum of $m(h, H)$ for all $h \in H$, $\boldsymbol{p} \in h$. Note that $m(\boldsymbol{p}, H) = 0$ if and only if there is no hyperplane $h \in H$ that contains $\boldsymbol{p}$.

The line probing model considered in this paper is as follows. At first, we have no information about H. From a line probe L, we obtain the *cross points* $C(L, H)$ that denotes the set of intersecting points of L with the hyperplanes not containing L in H and $m(\boldsymbol{p}, H)$ for all $\boldsymbol{p} \in C(L, H)$. From the line probe L, the cross points in $C(L, H)$ are obtained just as a set, and it is not known which point lies on which hyperplane. We also write $C(L)$, $m(h)$ and $m(\boldsymbol{p})$ for $C(L, H)$, $m(h, H)$, $m(\boldsymbol{p}, H)$, respectively, when H is understood from the context. It should be noted that our definition of the line probe is different from what Dobkin et al. [6] call a line probe. Their result which we use in Section 5.2 is based on the finger probe, not on their line probe.

In our line probing model, when the probing line is contained in some hyperplanes in H, we can still obtain $C(L, H)$ from the line probe L, but we cannot obtain the information whether the probing line is contained in some hyperplanes in H or not. It should be noted that we cannot improve the results in this section even if we can also obtain the multiplicity sum of the hyperplanes containing L in H from the line probe L.

Related to the line probing model, we also consider the point probe $\boldsymbol{p}$ by which we can obtain the information whether the point $\boldsymbol{p}$ contained in some hyperplanes in H or not. Renegar [18] gives a simple solution to determine a generic point from point probes using the moment curve $(t, t^2, t^3, \ldots, t^d)$ $(t \in \boldsymbol{R})$ in $\boldsymbol{E}^d$. He proved that, for an arrangement of n hyperplanes in $\boldsymbol{E}^d$, at least one of $dn + 1$ distinct points on the moment curve is a generic point. Hence with $dn + 1$ point probes, a generic point $\boldsymbol{q}$ is obtained.

Here we describe the result by the author in [1] on the combinatorial complexity of the line probing for arrangements.

Theorem 2.1. *Let H be a set of hyperplanes in $\boldsymbol{E}^d$, then the worst-case number of line probes that are needed to determine a generic line and H is d and $d + 1$, respectively.*

Note that the same statement can be proved for a set of hyperplanes in $\boldsymbol{C}^d$.

To evaluate the computational complexity we introduce the concept of the "unit operation" whose cost is constant to the input size of the problem. The order of the time complexity of the computation is evaluated by counting the number of unit operations in it. We assume that the fundamental arithmetic operations ($+$, $-$, $\times$, $\div$) are the unit operations. Let H be the set of n hyperplanes in $\boldsymbol{E}^d$, and $c_1(H)$ denote the cost of a line probe. We consider that the probing line is specified by the containing point $\boldsymbol{p}$ and the direction vector $\boldsymbol{q}$, and that each cross point on the probing line is returned by the real number t such that $\boldsymbol{p} + t\boldsymbol{q}$ denotes the cross point. The cost to transfer the coordinate values of $\boldsymbol{p}$, $\boldsymbol{q}$, and each cross point $\boldsymbol{p} + t\boldsymbol{q}$ is considered to be included in $c_1(H)$ if we cannot ignore it. Note that all operations above including probing operations are assumed to be performed with infinite precision over the real numbers. It is natural to assume that the cost $c_1(H)$ is at least the number of the cross points on the line, that means at least one unit operation is required to obtain each cross point of the probing line. Since the number of the cross points on the line is at most $n = |H|$, we assume $c_1(H) \geqslant n$. We call a line probe is *sorted* if these cross points on the probing line which we obtained are provided sorted along the direction of the probing line. We consider that line probes are not sorted unless they are specified to be sorted. Since the cost to sort the cross points of each line probe is $O(n \log n)$, we can easily estimate the time complexity of the algorithm in which the line probes are not sorted by replacing $c_1(H)$ in the order of the time complexity of the algorithm using sorted line probes with $c_1(H) + n \log n$. The results on the computational complexity of the line probing algorithms in [1] are as follows.

Theorem 2.2. *A generic line can be determined by $2d-1$ sorted line probes with $\Theta(d \cdot c_1(H))$ time. If a generic point or a generic direction is given, the number of the sorted line probes can be reduced to d.*

Theorem 2.3. *We can determine H by $2d$ line probes with $O(d \cdot (c_1(H) + dn \log n))$ time. If a generic point or a generic direction is given, the number of the line probes can be reduced to $d+1$.*

Theorem 2.4. *We can determine H by $3d-1$ sorted line probes with $\Theta(d \cdot c_1(H))$ time. If a generic point or a generic direction is given, the number of the sorted line probes can be reduced to $2d$.*

These theorems are essentially based on the same efficient algorithm developed in [1] to compute a generic line of H in $O(nd)$ time from the cross points on the d probing lines by using the dual transform.

3. Fixed direction line probing

In this section we consider the fixed direction line probing for a set of hyperplanes H in which the direction of all probing lines must be the same. Clearly, we cannot determine a generic line and H from the finite number of fixed direction line probes if there is a hyperplane in H which is parallel to the fixed direction. Thus, we assume a generic direction is given, which is equal to the direction of all probing lines. Note that this assumption is rather strong because we can determine a generic line at the same cost to determine a generic direction in usual case. The following theorems show that the worst-case number of line probes is the same as in Theorem 2.1 in spite of the additional conditions on line probes.

First we prove the following theorem, whose proof indicates the algorithm to determine a generic line using the minimum number of fixed direction line probes. Note that we do not need multiplicity information of the line probes to determine a generic line.

Theorem 3.1. *Let H be a set of hyperplanes in $\boldsymbol{E}^d$ with the given generic direction, then the worst-case number of fixed direction line probes that are needed to determine a generic line is d.*

Proof. For $i = 1, 2, \ldots, d$, let L_i denote the ith line probe whose direction vector is equal to the given generic direction. First, we prove that the lower bound of the worst-case number is d. To prove it, it is sufficient to show that for any given line probes L_i, $i = 1, 2, \ldots, d-1$, there exist the fixed sets of cross points $C(L_i)$, the multiplicity $m(\boldsymbol{p})$ for each $\boldsymbol{p} \in C(L_i)$, and a set H of hyperplanes defined by an arbitrarily chosen line L_0 whose direction vector is also equal to the given generic direction such that L_0 is not a generic line of H. In such a case, we cannot determine the generic line using $d-1$ line probes because for any line we choose it is possible that the line of our choice is not generic.

In constructing the above example, we can assume that not all probing lines L_i, $i = 0, 1, \ldots, d-1$, are contained in a hyperplane h. Otherwise, the set H containing two hyperplanes whose intersection with h are the same $(d-2)$-flat is the example. Let $\boldsymbol{q}_{1,i}$ and $\boldsymbol{q}_{2,i}$, $1 \leqslant i \leqslant d-1$ be different points on L_i, and let $\boldsymbol{q}_0$ be a point on L_0. Let h_i, $i = 1, 2$, be the hyperplane containing $\boldsymbol{q}_0$ and each $\boldsymbol{q}_{i,j}$ for $j = 1, \ldots, d-1$. Note that h_1 is different from h_2 because not all probing lines L_i, $i = 0, 1, \ldots, d-1$, are contained in a hyperplane. Now $H = \{h_1, h_2\}$ is the example because $\boldsymbol{q}_0 \in h_1 \cap h_2$.

Next, we prove that the upper bound of the worst-case number is d. To prove it, it is sufficient to construct the algorithm to define the generic line L_G of H by using d line probes L_i, $i = 1, 2, \ldots, d$.

We define $\boldsymbol{a}_i$, $i = 0, 1, \ldots, d-1$, as linearly independent vectors in $\boldsymbol{E}^d$ where $\boldsymbol{a}_0$ is the given direction vector, and let $\boldsymbol{a}_d = \boldsymbol{o}$. For $i = 1, 2, \ldots, d$, we consider that the line L_i contains the point whose position vector is $\boldsymbol{a}_i$. Using parameter $\alpha_i \in \boldsymbol{R}$, $i = 1, 2, \ldots, d$, we can denote the line L_i as $\alpha_i \boldsymbol{a}_0 + \boldsymbol{a}_i$.

For any hyperplane $h \in H$, L_i, $i = 1, 2, \ldots, d$, are not parallel to h and intersect h by one point in $C(L_i)$ because the given direction is a generic direction. Clearly, we can assume all $C(L_i)$ are not empty.

We define H' as the set of hyperplanes h such that $h \cap L_i \in C(L_i)$ for $i = 1, 2, \ldots, d$. For each L_i, $i = 1, 2, \ldots, d$, we choose a position vector of a point in $C(L_i)$ and denote it by $\boldsymbol{y}(L_i)$. Let $\boldsymbol{x}(L_i) = \boldsymbol{y}(L_i) - \boldsymbol{y}(L_d)$ for $i = 1, 2, \ldots, d-1$. Then these $d-1$ vectors $\boldsymbol{x}(L_i)$, $i = 1, 2, \ldots, d-1$, defined above are linearly independent because $\boldsymbol{x}(L_i) = \lambda_i \boldsymbol{a}_0 + \boldsymbol{a}_i$, where $\lambda_i \in \boldsymbol{R}$, from the definition of L_i. That is, there is one unique hyperplane which contains each point defined by $\boldsymbol{y}(L_i)$, $i = 1, 2, \ldots, d$.

Thus, for any point set $Q \subseteq \bigcup_{i=1}^{d} C(L_i)$, $|Q \cap C(L_i)| = 1$ for all $i = 1, 2, \ldots, d$, there only exists one unique hyperplane h such that $h \cap L_i = C(L_i) \cap Q$ for each $i = 1, 2, \ldots, d$. Because $C(L_i)$ is finite for each $i = 1, 2, \ldots, d$, H' is also finite from the above discussion.

Note that $H' \supseteq H$, because for any $h \in H$ each L_i, $i = 1, 2, \ldots, d$, intersects h by one point in $C(L_i)$. That is, H' means the set of "possible" hyperplanes as a hyperplane in H. Since $H \subseteq H'$, the generic line of H' is also the generic line of H. It is only the problem of computations to determine the generic line for known H'. Thus, we can define the generic line L_G of H as the generic line for H'. $\square$

Next we prove the following theorem, whose proof indicates the algorithm to determine H using the minimum number of fixed direction line probes. Note that we do not need multiplicity information of the line probes except the last one line probe to determine H.

Theorem 3.2. *Let H be a set of hyperplanes in $\boldsymbol{E}^d$ with the given generic direction, then the worst-case number of fixed direction line probes that are needed to determine H is $d+1$.*

Proof. For $i = 1, 2, \ldots, d+1$, let L_i denote the ith line probe whose direction vector is equal to the given generic direction. First, we prove that the lower bound of the worst-case number is $d+1$. To prove it, it is sufficient to show that for any given line probes L_i, $i = 1, 2, \ldots, d$, there exist sets H_1 and H_2 of hyperplanes such that $C(L_i, H_1) = C(L_i, H_2)$ and $m(L_i, H_1) = m(L_i, H_2)$ for $i = 1, 2, \ldots, d$ because this means we cannot determine H using d line probes.

In constructing the above example, we can assume that not all probing lines L_i, $i = 1, 2, \ldots, d$, are contained in a hyperplane h as in the proof of Theorem 3.1. Otherwise, let h_1 and h_2 be the two different hyperplanes whose intersection with h are the same $(d-2)$-flat, and let $H_1 = \{h_1\}$ and $H_2 = \{h_2\}$. Then this example satisfies the requirements.

Now we choose two different points $q_{1,i}$ and $q_{2,i}$ on L_i for $i = 1, 2, \ldots, d$. Let $h_{1,i}$, $i = 1, 2$ be the hyperplane which contains each $q_{i,j}$ for $1 \leqslant j \leqslant d$. Let $h_{2,i}$, $i = 1, 2$, be the hyperplane which contains $q_{3-i,1}$ and each $q_{i,j}$ for $j = 2, 3, \ldots, d$. Then the example $H_1 = \{h_{1,i} | i = 1, 2\}$ and $H_2 = \{h_{2,i} | i = 1, 2\}$ satisfies the requirement.

Next, we prove that the upper bound of the worst-case number is $d+1$. To prove it, it is sufficient to construct the algorithm to define H by using $d+1$ fixed direction line probes L_i, $i = 1, 2, \ldots, d+1$. We define d line probes L_i, $i = 1, 2, \ldots, d$, and the set H' of "possible" hyperplanes as in the upper bound proof of Theorem 3.1 and let L_{d+1} be the generic line of H' defined by d line probes L_i, $i = 1, 2, \ldots, d$, as in the proof of Theorem 3.1.

For each point $\boldsymbol{p} \in C(L_{d+1})$, there is one unique hyperplane $h \in H'$ which intersects L_{d+1} at the point $\boldsymbol{p}$. Clearly, each hyperplane $h \in H \subseteq H'$ intersects L_{d+1} at a point in $C(L_{d+1})$ because of the definition of the generic line. Therefore,

$$H = \{h \in H' | h \cap L_{d+1} \in C(L_{d+1})\},$$

and the multiplicity $m(h) = m(\boldsymbol{p})$ where $\boldsymbol{p} = h \cap L_{d+1}, \boldsymbol{p} \in C(L_{d+1})$. Thus, we can determine H with each multiplicity using $d+1$ line probes. $\square$

Here we only mention the results on the bounds for the computational complexity of the algorithms in the upper bound proofs of Theorems 3.1 and 3.2 based on the efficient line probing computation algorithm developed in [1] to determine a generic line of H in $O(nd)$ time from the cross points on the d probing lines by using the dual transform in $\boldsymbol{E}^d$. We can prove the following theorem from a slight modification of the proofs of Theorems 2.2, 2.3 and 2.4 in [1] according to the algorithms in the upper bound proofs of Theorems 3.1 and 3.2.

Theorem 3.3. *If a generic direction is given, with using the fixed direction line probes, we can determine a generic line of H by d sorted probes with $\Theta(d \cdot c_1(H))$ time, we can* *determine* H *by* $d+1$ *probes with* $O(d \cdot (c_1(H) + dn\log n))$ *time, and we can also determine* H *by* $2d$ *sorted probes with* $\Theta(d \cdot c_1(H))$ *time.*

4. The flat probing model

In this section we consider the flat probing model for a set of hyperplanes H, which is the generalization of the line probing model explained in Section 2. Let F be a k-flat in $\boldsymbol{E}^d$, that is, F be an affine hull with dimension k. Let the *multiplicity* $m(F, H)$ of F in H denote the sum of $m(h, H)$ for all $h \in H$, $F \subseteq h$. The *cross flats* $C(F, H)$ of F with H denotes the set of the intersecting $(k-1)$-flats of F with the hyperplanes not containing F in H. We also write $C(F)$ and $m(F)$ for $C(F, H)$ and $m(F, H)$, respectively, when H is understood from the context.

Now we extend the line probing model to the k-flat probing model. For a k-flat $F \in \boldsymbol{E}^d$, a *k-flat probe* F for H reports the set $C(F, H)$ with $m(f, H)$ for all $(k-1)$-flats $f \in C(F, H)$. A 1-flat probe corresponds to a line probe, and a 0-flat probe corresponds to a point probe.

In this section, we consider the *non-uniform* flat probing that specifies a series of flat probes with various dimensions greater than 0. We define the *rank* of the non-uniform flat probing as the total number of the dimensions of all flats in it. In case of the line probing, its rank is equal to the number of line probes because the dimension of a line is always 1. The rank of the dimension sequence of flats in a non-uniform flat probing is similarly defined as the total number. A positive integer sequence is called *generic line feasible* if there is an algorithm to determine the generic line of H by using non-uniform flat probes in which the dimension sequence of flats is equal to it.

We can use a k-flat probe instead of any k line probes in case that there is a k-flat containing these k lines. Since there are k lines which are not contained in any k-flat, it is not clear that we can always replace k line probes with a k-flat probe. Note that, in the following theorem, we do not need multiplicity information of the flat probes to determine the generic line.

Theorem 4.1. *Let* H *be a set of hyperplanes in* $\boldsymbol{E}^d$, *then the minimum rank of non-uniform flat probes that are needed to determine a generic line is* d, *and any positive integer sequence with rank* d *is generic line feasible.*

Proof. First, we prove that the lower bound of the minimum rank is d. For a flat $F \in \boldsymbol{E}^d$, let $\dim(F)$ denote the dimension of F. To prove it, it is sufficient to show that for any given non-uniform flat probes F_i, $i = 1, 2, \dots, r$, such that $\sum_{i=1}^{r} \dim(F_i) = d-1$, there exist the fixed sets of cross flats $C(F_i)$, the multiplicity $m(f)$ for each $f \in C(F_i)$, and a set H of hyperplanes defined by an arbitrarily chosen line L_0 such that L_0 is not a generic line of H. In such a case, we cannot determine the generic line using non-uniform flat probes with rank $d-1$ because for any line we choose it is possible that the line of our choice is not generic.

In constructing the above example, we can assume that the generating vectors of all probing flats F_i, $i = 1, 2, \ldots, r$, and L_0 are linearly independent. Otherwise, the set H containing the hyperplane which misses all probing flats F_i, $i = 1, 2, \ldots, r$, and L_0 is the example.

Because the generating vectors of all probing flats F_i, $i = 1, 2, \ldots, r$, and L_0 are linearly independent (that is, there is no hyperplane containing all flats F_i and L_0), we can choose $2\dim(F_i)$ points $\boldsymbol{q}_{i,1}, \boldsymbol{q}_{i,2}, \ldots, \boldsymbol{q}_{i,\dim(F_i)}, \boldsymbol{q}'_{i,1}, \boldsymbol{q}'_{i,2}, \ldots, \boldsymbol{q}'_{i,\dim(F_i)}$ on F_i, for $i = 1, 2, \ldots, r$, and a point $\boldsymbol{q}_0$ on L_0 such that the affine hull of $\boldsymbol{q}_0$ and $d - 1$ points $\boldsymbol{q}_{i,j}$, $i = 1, \ldots, r, j = 1, 2, \ldots, \dim(F_i)$, is a hyperplane which we denote by h_1 and that the affine hull of $\boldsymbol{q}_0$ and $d - 1$ points $\boldsymbol{q}'_{i,j}$ for $i = 1, \ldots, r, j = 1, 2, \ldots, \dim(F_i)$ is a different hyperplane which we denote by h_2. Now $H = \{h_1, h_2\}$ is the example because $\boldsymbol{q}_0 \in h_1 \cap h_2$, which is a point on L_0.

Next, we prove that the upper bound of the minimum rank is also d and that any positive integer sequence with rank d is generic line feasible. To prove it, it is sufficient to construct the algorithm to define a generic line of H by using non-uniform flat probes F_i, $i = 1, 2, \ldots, r$, whose dimension sequence is equal to the given one with rank d.

We define $\boldsymbol{a}_i$, $i = 1, 2, \ldots, d$, as linearly independent vectors in $\boldsymbol{R}_d$. Here we consider the corresponding d line probes L_i for $i = 1, 2, \ldots, d$ to the flat probes F_j for $j = 1, 2, \ldots, r$. Let L_1 be the line whose direction vector is $\boldsymbol{a}_1$. For $j = 1, 2, \ldots, r$, let $m_j = 1 + \sum_{i=1}^{j} \dim(F_i)$. For $j = 1, 2, \ldots, r$, we also define F_j such that the generating vectors of F_j are $\boldsymbol{a}_i$ for $i = m, m + 1, \ldots, m + \dim(F_j) - 1$ and that F_j contains the line L_m, and define each L_i for $i = m + 1, \ldots, m + \dim(F_j)$ such that the direction vector of L_i is $\boldsymbol{a}_i$ and that L_i contains the point $\boldsymbol{p}_{i-1} \in L_{i-1} - C(F_j) \cap L_{i-1}$ for $i \leqslant d$. Thus, for $j = 1, 2, \ldots, r$, we can generate the outputs $C(L_i)$ of the corresponding line probes for $i = m, m + 1, \ldots, m + \dim(F_j) - 1$ by defining $C(L_i) = L_i \cap F_j$.

Let $U = \{L_i | i = 1, 2, \ldots, d\}$. Using parameter $\alpha_i \in \boldsymbol{R}$, $i = 1, 2, \ldots, d$, and constants $\beta_i \in \boldsymbol{R}$, $i = 1, 2, \ldots, d - 1$, we can denote the line L_i as $\alpha_i \boldsymbol{a}_i + \sum_{j=1}^{i-1} \beta_j \boldsymbol{a}_j$, where $\alpha_i = 0$ means the point $\boldsymbol{p}_{i-1}$ for $i \geqslant 2$. Note that for any point in $C(L_i)$, $i = 1, 2, \ldots, d - 1$, $\alpha_i \neq \beta_i$ holds from the definition of L_i.

If a hyperplane $h \in H$ contains $\boldsymbol{p}_i \in L_i$, h also contains L_i because $\boldsymbol{p}_i \notin C(L_i)$. Clearly, it follows that h contains $\boldsymbol{p}_{i-1}$. By using it recursively, we can conclude that h contains L_j for $j = 1, 2, \ldots, i$. Therefore, for $i = 1, 2, \ldots, d - 1$, all hyperplanes $h \in H$ containing $\boldsymbol{p}_i$ are parallel to L_j, $j = 1, 2, \ldots, i$.

For any hyperplane $h \in H$, there is at least one line probe $L_i \in U$ that is not parallel to h because all direction vectors of the line probes in U are linearly independent. That is, at least one line $L_i \in U$ intersects h by one point in $C(L_i)$. Let $U' = \{L_i \in U | C(L_i) \neq \emptyset\}$. If U' is empty then H must be empty and clearly we can determine the generic line. Thus, we can assume U' is not empty.

Let X be a non-empty subset of U', and let $k = \max_{L_i \in X} i$. We define $H(X)$ as the set of hyperplanes h such that $h \cap L_i \in C(L_i)$ for all $L_i \in X$ and that h is parallel to L_i for all $L_i \in U - X$. For each $L_i \in X$, we choose a position vector of a point in $C(L_i)$ and denote it by $\boldsymbol{y}(L_i)$. Let $\boldsymbol{x}(L_i) = \boldsymbol{y}(L_i) - \boldsymbol{y}(L_k)$ for $L_i \in X - \{L_k\}$. For each $L_i \in U - X$, we choose a direction vector $\boldsymbol{l}_i$ of L_i and let $\boldsymbol{x}(L_i) = \boldsymbol{l}_i$. These $d-1$ vectors $\boldsymbol{x}(L_i)$, $L_i \in U - \{L_k\}$, defined above can be written as

$$\boldsymbol{x}(L_i) = \begin{cases} (\alpha_i - \beta_i)\boldsymbol{a}_i - \sum_{j=i+1}^{k} \beta_j \boldsymbol{a}_j & \text{for } L_i \in X - \{L_k\}, \\ \alpha_i \boldsymbol{a}_i & \text{for } L_i \in U - X, \end{cases}$$

where $\alpha_i \neq 0$ and $\alpha_i - \beta_i \neq 0$ for $i \neq k$. From the easy computation of the rank of the matrix defined by these $d-1$ vectors, we can show that these $d-1$ vectors are linearly independent. This means that there is one unique hyperplane parallel to all $L \in U - X$ which contains each point $y(L)$, $L \in X$.

Thus, for any point set $Q \subseteq \bigcup_{L \in U'} C(L), |Q \cap C(L)| = 1$ for all $L \in U'$, there only exists one unique hyperplane h such that $h \cap L = C(L) \cap Q$ for all $L \in X$ and that h is parallel to L for all $L \in U - X$. Because $C(L)$ is finite for all $L \in U'$, $H(X)$ is also finite from the above discussion.

Now we define the finite set H' of hyperplanes as

$$H' = \bigcup_{X \subseteq U', X \neq \emptyset} H(X).$$

Note that $H' \supseteq H$, because for any $h \in H$ there exist a line $L \in U$ which intersects h and each $L \in U$ is either parallel to h or intersects h. That is, H' means the set of "possible" hyperplanes as a hyperplane in H. Since $H \subseteq H'$, the generic line of H' is also the generic line of H. It is only the problem of computations to determine the generic line for known H'. Thus, we can define the generic line L_G of H as the generic line for H'. That means, we can determine a generic line of H by using non-uniform flat probes whose dimension sequence is equal to the given one with rank d. □

Although a k-flat with $k \geqslant 2$ covers infinite number of lines, Theorem 4.1 means we cannot essentially improve the complexity of the algorithm to determine a generic line by using k-flat probes with $k \geqslant 2$ instead of line probes. That is, the line probing is essential to determine a generic line with regard to the combinatorial complexity of the algorithm.

A positive integer sequence is called *proper* if the last element of it is equal to 1. A positive integer sequence is also called *arrangement feasible* if there is an algorithm to determine H by using non-uniform flat probes in which the dimension sequence of flats is equal to it.

Note that, in the following theorem, we do not need multiplicity information of the flat probes except the last one line probe to determine H.

Theorem 4.2. *Let H be a set of hyperplanes in E^d, then the minimum rank of non-uniform flat probes that are needed to determine H is $d+1$, and any proper positive integer sequence with rank $d+1$ is arrangement feasible.*

Proof. First, we prove that the lower bound of the minimum rank is $d+1$. To prove it, it is sufficient to show that for any given non-uniform flat probes F_i, $i = 1, 2, \ldots, r$, such that $\sum_{i=1}^{r} \dim(F_i) = d$, there exist the set H_1 and the set of H_2 of hyperplanes such that $C(F_i, H_1) = C(F_i, H_2)$ and $m(F_i, H_1) = m(F_i, H_2)$ for $i = 1, 2, \ldots, r$ because this means we cannot determine H by using non-uniform flat probes with rank d.

In constructing the above example, we can assume that the generating vectors of all probing flats F_i, $i = 1, 2, \ldots, r$, are linearly independent as in the proof of Theorem 4.1. Otherwise, let $H_1 = \emptyset$ and $H_2 = \{h\}$ where the hyperplane h misses all F_i, $i = 1, 2, \ldots, r$. Then this example satisfies the requirements.

For $i = 1, 2, \ldots, r$, we can choose $\dim(F_i)$ points $\boldsymbol{q}_{i,1}, \boldsymbol{q}_{i,2}, \ldots, \boldsymbol{q}_{i,\dim(F_i)}$ on F_i such that all d points $q_{i,j}$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, \dim(F_i)$, are affinely independent because the generating vectors of all probing flats F_i, $i = 1, 2, \ldots, r$, are linearly independent. Let h_i, $1 \leqslant i \leqslant r$, be the hyperplane containing $\boldsymbol{q}_{i,1}, \boldsymbol{q}_{i,2}, \ldots, \boldsymbol{q}_{i,\dim(F_i)}$ which is parallel to all flat probes F_j for $1 \leqslant j \leqslant r$, $j \neq i$. Let h_0 be the hyperplane which contains all d points $q_{i,j}$, $i = 1, 2, \ldots, r, j = 1, 2, \ldots, \dim(F_i)$. Then the example $H_1 = \{h_i \mid i = 1, 2, \ldots, r\}$ and $H_2 = \{h_0\}$ satisfies the requirement.

Next, we prove that the upper bound of the minimum rank is also $d + 1$ and that any proper positive integer sequence with rank d is arrangement feasible. To prove it, it is sufficient to construct the algorithm to define H by using non-uniform flat probes F_i, $i = 1, 2, \ldots, r$, whose dimension sequence is equal to the given proper one with rank $d + 1$. We can determine the corresponding d line probes L_i with its outputs $C(L_i)$ and $m(\boldsymbol{p})$, $\boldsymbol{p} \in C(L_i)$, for $i = 1, 2, \ldots, d$ by using non-uniform flat probes whose dimension sequence is equal to $\dim(F_1), \dim(F_2), \ldots, \dim(F_{r-1})$ with rank d as in the upper bound proof of Theorem 4.1. Let H' be the set of "possible" hyperplanes and let L_{d+1} be the generic line of H' defined by d line probes L_i, $i = 1, 2, \ldots, d$, as in the upper bound proof of Theorem 4.1. We define the last flat probe F_r as L_{d+1}, and let $C(L_{d+1}) = C(F_r)$. For each point $\boldsymbol{p} \in C(L_{d+1})$, there is one unique hyperplane $h \in H'$ which intersects L_{d+1} at the point $\boldsymbol{p}$, and for each hyperplane $h \in H \subseteq H'$, h intersects L_{d+1} at a point in $C(L_{d+1})$ because of the definition of the generic line. Therefore,

$$H = \{h \in H' \mid h \cap L_{d+1} \in C(L_{d+1})\},$$

and the multiplicity $m(h) = m(\boldsymbol{p})$ where $\boldsymbol{p} = h \cap L_{d+1}$, $\boldsymbol{p} \in C(L_{d+1})$. Thus, we can determine H by using non-uniform flat probes whose dimension sequence is equal to the given proper one with rank $d + 1$. $\square$

Theorem 4.2 means that the line probing is still essential to determine H with regard to the combinatorial complexity of the algorithm.

Since we can assume that the computational complexity of a k-flat probing is at least that of k line probes in considering the computational complexity to determine a generic line of H and H itself by using non-uniform flat probes, we cannot improve the bounds for the computational complexity in Theorems 2.2, 2.3 and 2.4 to determine a generic line of H and H itself by using flat probes. If the computational complexity of a k-flat probing is equal to that of k line probes, the computational complexity to determine a generic line of H and H itself by using non-uniform flat probes whose dimension sequence is equal to the given one is the same as in Theorem 2.2, 2.3 and 2.4, provided that we can compute the cross point of a k-flat with a line contained in a $(k + 1)$-flat with the constant time.

5. Extended point probing models

In this section we introduce the extended point probing models, which are the vertical probing and the level probing, in order to approximate the arrangement of hyperplanes, and consider their relation to the neural network models in connection with computational learning theory to learn a network of threshold functions. We also consider the combinatorial complexity of the vertical probing and the approximation error of the level probing. All arrangements should be considered as the arrangements of non-vertical hyperplanes in this section.

5.1. Approximation of hyperplane arrangements

For a set of hyperplanes H in $\boldsymbol{E}^d$, the point probing which is introduced in Section 2 only tells us whether the specified point is contained in some hyperplane in H or not. Clearly, it is possible that we cannot obtain any combinatorial information on the arrangement of H by using finite number of the point probes. In this section we extend this point probing model to provide additional information on H concerning the specified point, that is, to provide not only whether the point is in H or not but also the minimum height of intersecting points of hyperplanes in H with the vertical line containing the specified point or the number of hyperplanes below the point. We call the former *vertical probing* and the latter *level probing.*

We define these two probing models in detail. For an input vector $\boldsymbol{x} \in \boldsymbol{E}^d$, let $t_{\boldsymbol{w},\theta}(\boldsymbol{x})$ be the threshold function defined as follows:

$$t_{\boldsymbol{w},\theta}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{w}^{\mathrm{T}}\boldsymbol{x} - \theta \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{w} \in \boldsymbol{E}^d$ is the weight vector and $\theta \in \boldsymbol{R}$ is the threshold value. In many cases, the input vector to the threshold function is restricted to a 0–1 vector, but here we do not impose such restriction. The value of above linear threshold function is determined by the value of $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{x} - \theta$, which is called the linear value of the function. Let the real vectors $\boldsymbol{w}_i \in \boldsymbol{E}^d$ and the real numbers $\theta_i \in \boldsymbol{R}$ for $i = 1, 2, \ldots, n$. Let H be the set of n hyperplanes:

$$h_i = \{(\boldsymbol{x}, y) \in \boldsymbol{E}^{d+1} \mid y = \boldsymbol{w}_i^{\mathrm{T}}\boldsymbol{x} - \theta_i\}$$

for $i = 1, 2, \ldots, n$ in $\boldsymbol{E}^{d+1}$, and let H' be the set of n hyperplanes:

$$h_i' = \{\boldsymbol{x} \in \boldsymbol{E}^d \mid \boldsymbol{w}_i^{\mathrm{T}}\boldsymbol{x} = \theta_i\}$$

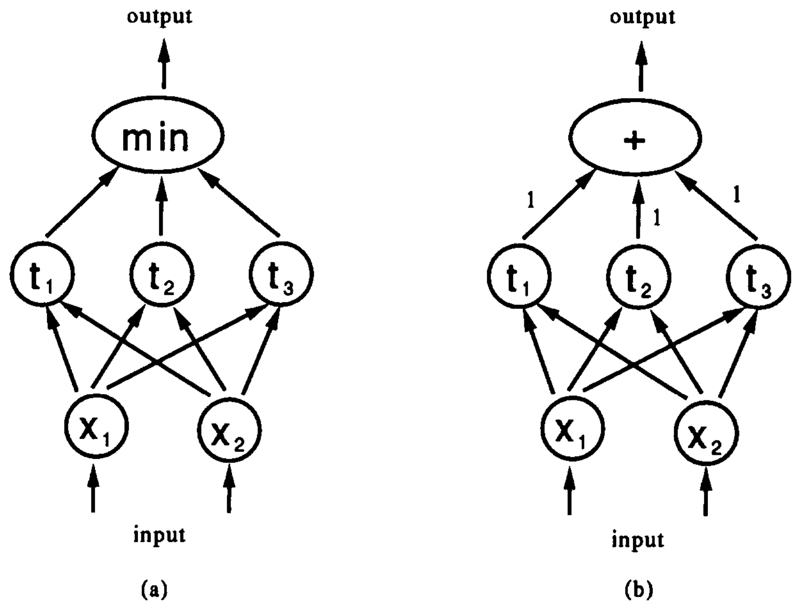


Fig. 1. (a) Function η and (b) function ξ.

for $i = 1, 2, \ldots, n$ in $\boldsymbol{E}^d$. We define a function $\eta : \boldsymbol{E}^d \to \boldsymbol{R}$ as

$$\eta(\boldsymbol{x}) = \min_{i=1,\ldots,n} \{\boldsymbol{w}_i^{\mathrm{T}} \boldsymbol{x} - \theta_i\}.$$

The *vertical probing* for H by the point $(\boldsymbol{x}, y) \in \boldsymbol{E}^{d+1}$ where $\boldsymbol{x} \in \boldsymbol{E}^d$ and $y \in \boldsymbol{R}$ reports $\eta(\boldsymbol{x})$. We also define a function $\xi : \boldsymbol{E}^d \to \boldsymbol{R}$ as

$$\xi(\boldsymbol{x}) = \sum_{i=1}^{n} \{t_{\boldsymbol{w}_i,\theta_i}(\boldsymbol{x})\}.$$

The *level probing* for H' by the point $\boldsymbol{x} \in \boldsymbol{E}^d$ reports $\xi(\boldsymbol{x})$.

The problem to approximate the 0–1 output function such as the above-mentioned threshold function $t_{\boldsymbol{w},\theta}(\boldsymbol{x})$ with only the information of its 0–1 outputs for a set of example input vectors is studied in the PAC learning model of the learning theory. However, if the learner receives some additional information about the example besides the 0–1 output, that is, real values which reflects the linear value of the function, the number of necessary examples may be reduced. Although a single threshold function can be trivially learned with the information of its linear value, a network of linear threshold-like functions such as a neural network with d input and one output, is by no means trivial. Here we consider two types of simply structured networks as target functions which correspond to vertical probing and level probing. Fig. 1(a) depicts the network representation of the function η as a three-layered network with $d = 2$ and $n = 3$. Note that the units in the hidden layer are not real

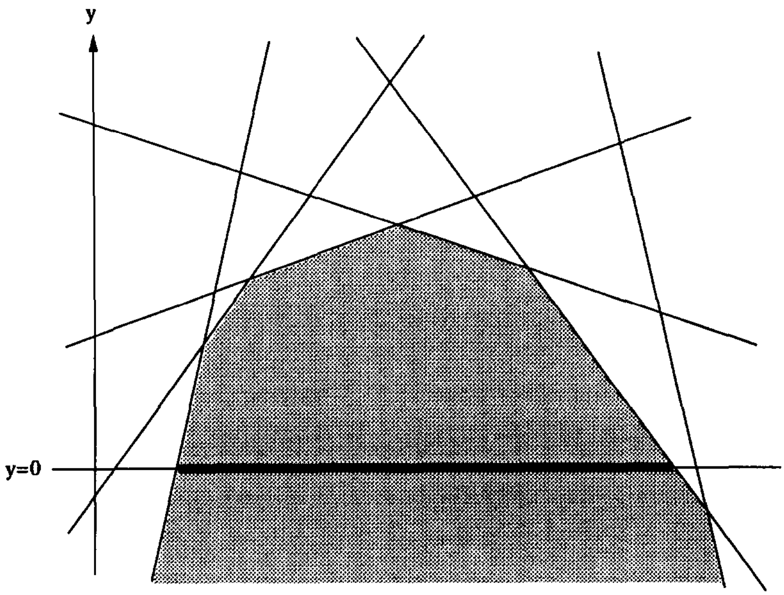


Fig. 2. A lower-unbounded polyhedron, and its horizontal section in the plane.

threshold functions but they output the linear values. Also, note that in this network weights from the hidden layer to the output unit can be regularized to one by transferring them to the weights for the input of the hidden units. Fig. 1(b) shows the function ξ as a network. In this case, the output threshold function is restricted to a simple one, i.e., non-weighted summation of outputs of the hidden units.

5.2. *The vertical probing model*

The function η can be regarded as describing the d-dimensional convex polyhedron $\mathscr{P} = \{x \in E^d \mid w_i^{\mathrm{T}} x - \theta_i \leqslant 0\}$ since $x \in \mathscr{P}$ iff $\eta(x) \leqslant 0$. For the n threshold functions in the hidden layer of η, we consider the intersection $\mathscr{P}_0$ of half-spaces in the $(d+1)$-dimensional space naturally determined by them:

$$\mathscr{P}_0 = \bigcap_{i=1}^{n} \{(x, y) \in E^{d+1} \mid x \in E^d,\ y \leqslant w_i^{\mathrm{T}} x - \theta_i\}.$$

The d-dimensional polyhedron $\mathscr{P}$ is the intersection of this polyhedron $\mathscr{P}_0$ with hyperplane $y = 0$ in E^{d+1}. The convex polyhedron in E^d that is the lowermost cell in the arrangement such as $\mathscr{P}_0$ is called *lower-unbounded*, and the intersection of the lower-unbounded polyhedron with the hyperplane $y = 0$ such as $\mathscr{P}$ is called the *horizontal section* of the polyhedron; see Fig. 2 which illustrates them. In the figure, the shaded area shows a lower-unbounded polyhedron, and the thick line segment is its horizontal section.

We can consider the vertical probing as the conditioned version of the finger probing for the convex polytopes studied by Dobkin et al. [6]. That is, the finger probe reports the contact point for a specified line with arbitrary direction while the vertical probing reports the contact point for only a vertical (fixed direction) line. However, it should be noted that the finite number of vertical probes by $(\boldsymbol{x}_j, y_j) \in \boldsymbol{E}^{d+1}$ for $j = 1, 2, \ldots, k$ cannot tell us the information of

$$\mathscr{P}_0 \cap \overline{\{(\boldsymbol{x}, y) \in \boldsymbol{E}^{d+1} \mid \boldsymbol{x} \in \mathrm{conv}(\{\boldsymbol{x}_j \mid j = 1, 2, \ldots, k\}), y \in \boldsymbol{R}\}},$$

that is, the outside of the probed area for $\mathscr{P}_0$. Thus, we cannot determine $\mathscr{P}_0 \in \boldsymbol{E}^{d+1}$ by the finite number of vertical probes. A bounded area in the hyperplane $y = 0$ with which the image of each facet of the lower-unbounded polyhedron by the orthogonal projection onto the hyperplane $y = 0$ intersects is called the *defining area* of the lower-unbounded polyhedron. From the viewpoint of the learning theory, we can assume that the defining area of the lower-unbounded polyhedron is given and consider the goal of the vertical probing for $\mathscr{P}_0$ is not to determine all hyperplanes $y = \boldsymbol{w}_i^{\mathrm{T}} \boldsymbol{x} - \theta_i$ for $i = 1, 2, \ldots, n$ but to determine the horizontal section $\mathscr{P}$ of $\mathscr{P}_0$. In this sense, from a slight modification of the proof of the bound for the worst-case number of the finger probes to determine $\mathscr{P}_0$ by Dobkin et al. [6], we can prove the bound for the worst-case number of the vertical probes for $\mathscr{P}$. The following theorem shows this bound. Note that $\mathscr{P}_0$ has at most n facets.

Theorem 5.1 (Dobkin et al. [6]). *Let $\mathscr{P}_0$ be a lower-unbounded polyhedron in $\boldsymbol{E}^d$ and let the defining area of $\mathscr{P}_0$ be given. Then, to determine the horizontal section of $\mathscr{P}_0$,* $\deg_0(\mathscr{P}_0) + \deg_{d-1}(\mathscr{P}_0)$ *vertical probes are necessary and* $\deg_0(\mathscr{P}_0) + (d+2)(\deg_{d-1}(\mathscr{P}_0) - 1)$ *vertical probes are sufficient.*

The famous upper bound theorem proved by McMullen [14] on the *simplicial* polytope (i.e. a polytope of which each facet is a simplex) says that the cyclic polytope with n vertices in $\boldsymbol{E}^d$ attains the maximum number of k-faces among simplicial polytopes with n vertices in $\boldsymbol{E}^d$. The dual polytope of the simplicial polytope is called a *simple* polytope. Obviously, a cell of the simple arrangement is a simple polytope. We define the function $\boldsymbol{\Phi}_k(d, n)$ for $0 \leqslant k \leqslant d - 2$ as

$$\boldsymbol{\Phi}_k(d, n) = \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{i}{k} \binom{n-d+i-1}{i} + \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{d-i}{k} \binom{n-d+i-1}{i}.$$

From the result of the upper bound theorem in the dual space, we can obtain the following theorem.

Theorem 5.2 (McMullen [14]). *For any simple d-polytope P with n facets we have*

$$\deg_k(P) \leqslant \Phi_k(d,n),$$

for $0 \leqslant k \leqslant d-2$.

From the computation of the value of $\Phi_0(d,n)$, we can easily obtain the following corollary.

Corollary 5.3 (McMullen [14]). *For any simple d-polytope P with n facets we have*

$$\deg_0(P) \leqslant \Phi_0(d,n) = \binom{n-\lfloor (d+1)/2 \rfloor}{\lfloor d/2 \rfloor}\binom{n-\lfloor d/2 \rfloor - 1}{\lfloor (d-1)/2 \rfloor}.$$

Note that $\Phi_0(d,n) = \mathrm{O}(n^{\min\{d-k, \lfloor d/2 \rfloor\}})$. Using a straightforward perturbation argument, it can be shown that the same bound holds for arbitrary convex polytopes with n facets.

Since $\mathscr{P}_0$ has at most n facets, we can bound the worst-case number of the above vertical probes from Corollary 5.3 as stated in the following.

Corollary 5.4. *Let $\mathscr{P}_0$ be a lower-unbounded polyhedron defined by the n hyperplanes in $\boldsymbol{E}^d$ and let the defining area of $\mathscr{P}_0$ be given. Then, to determine the horizontal section of $\mathscr{P}_0$, $\Phi_0(d,n) - 1 + (d+2)n = \mathrm{O}(n^{\lfloor d/2 \rfloor})$ vertical probes are sufficient, where*

$$\Phi_0(d,n) = \binom{n-\lfloor (d+1)/2 \rfloor}{\lfloor d/2 \rfloor}\binom{n-\lfloor d/2 \rfloor - 1}{\lfloor (d-1)/2 \rfloor}.$$

5.3. The level probing model

The n threshold functions of d inputs in the hidden layer of ξ naturally induces the arrangement $\mathscr{A}(H')$ of the set H' of n hyperplanes in the d-dimensional Euclidean space. Any point $\boldsymbol{x}$ in a cell of the arrangement $\mathscr{A}(H')$ has the same value of $\xi(\boldsymbol{x})$, and this reported value of $\xi(\boldsymbol{x})$ is called the *level* of the point $\boldsymbol{x}$. For the arrangement $\mathscr{A} \in \boldsymbol{E}^d$, a set S of points is called the *covering* of $\mathscr{A}$ if each cell of $\mathscr{A}$ has exactly one point in the set S. In this case, we also say that the arrangement $\mathscr{A}$ is *covered* by S. The level probes by the covering set of points for the arrangement is called the *covering level probes*. Then, another arrangement $\mathscr{A}'$ of n hyperplanes is said to be *consistent* with respect to S if $\mathscr{A}'$ induces the same level as induced by $\mathscr{A}$ for all points in S.

We can draw the dual graph G of the arrangement $\mathscr{A}$ which is covered by the point set S such that each vertex v in $V(G)$ which corresponds to a cell c in $\mathscr{A}$ is drawn as the unique point in S which is contained in the cell c and that each edge xy in $E(G)$ is

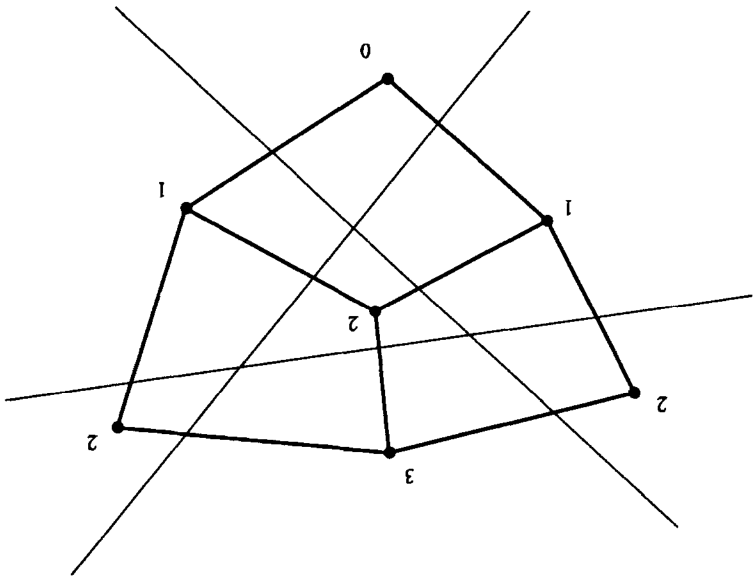
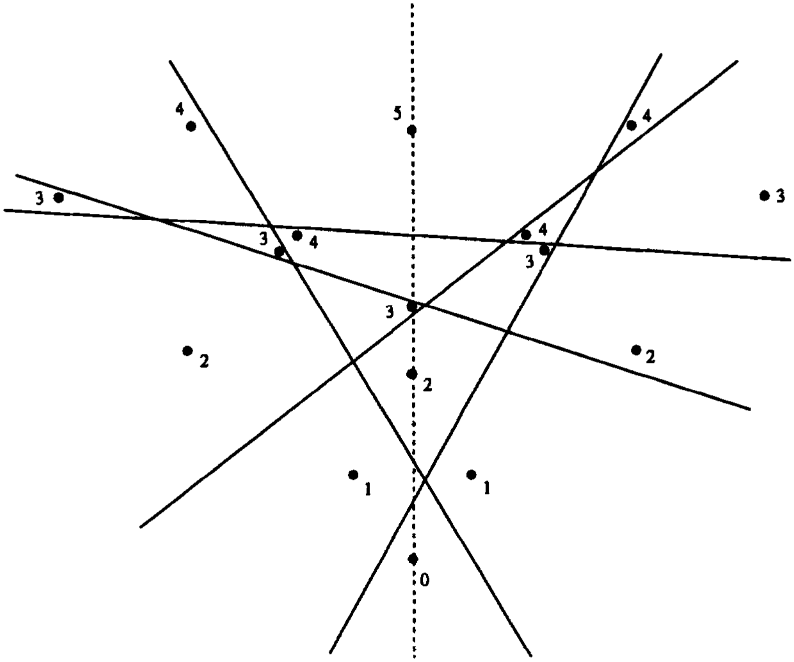


Fig. 3. A linear dual graph of an arrangement.

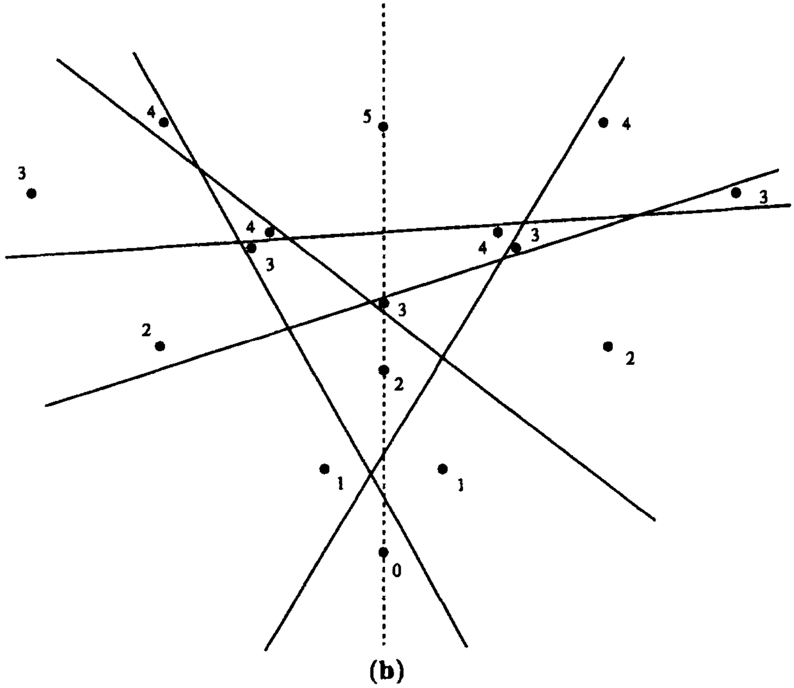
drawn as the straight line segment connecting its two endvertices x and y in S. We call this drawing of G the linear dual graph of $\mathscr{A}$ with respect to S; see Fig. 3 for a two-dimensional example. In the figure, the dots show a set of points S, and an arrangement of lines drawn by the thin lines is covered by S. The thick line segments show edges of the linear dual graph of the arrangement with respect to S. Since each cell of $\mathscr{A}$ forms a convex polyhedron, each edge of the linear dual graph intersects with hyperplanes of $\mathscr{A}$ by exactly one point which lies on the facet between two cells corresponding to two endvertices of the edge.

There is a question about the linear dual graph G_l of an arrangement $\mathscr{A}$ with respect to a point set S whether the linear dual graph of any consistent arrangement of $\mathscr{A}$ is the same as G_l with respect to S or not. This question is negatively proved by the counterexample in which two consistent arrangements have different linear dual graph with respect to a given point set S. In the two-dimensional case, we illustrate such a counterexample in Fig. 4(a) and (b). In the figure, the dots show the point set S with their levels, and two consistent arrangements of lines drawn as the solid lines in (a) and (b) form the counterexample. The point set S is invariable by the horizontal flip (symmetric) around the dashed vertical line, and two consistent arrangements in (a) and (b) can be mapped one another by the same horizontal flip. But the linear dual graphs are clearly different because they are not invariable by the same horizontal flip (asymmetric).

Thus, we introduce the concept of *dual-equivalent* for two arrangements of hyperplanes with respect to a covering point set. Specifically, two arrangements of



(a)



(b)

Fig. 4. A counterexample of consistent arrangements whose linear dual graphs are different in the plane.

hyperplanes are called *dual-equivalent* with respect to a covering point set S if and only if their linear dual graphs with respect to S are the same one. Obviously, dual-equivalent arrangements are always consistent arrangements, but not vice versa from the above counterexample. In the following section, we consider the approximation of the arrangement by the covering level probes combined with the information on its linear dual graph.

Since the level probing for an arrangement does not report the information on the exact point in a hyperplane of the arrangement, it is clear that we cannot determine the arrangement of hyperplanes by the finite number of level probes. However, there is a problem whether we can approximate the outputs of the function $\xi(\boldsymbol{x})$ which correspond to the results of the level probes from the covering level probes for the arrangement provided that we have the information on its linear dual graph. Let $\xi'(\boldsymbol{x})$ be the approximated function of $\xi(\boldsymbol{x})$, and let n be the number of hyperplanes in the arrangement. Specifically, the problem is interpreted to imply whether we can bound the approximation error $|\xi'(\boldsymbol{x}) - \xi(\boldsymbol{x})|$ by the constant to n or not if underlying arrangements are dual-equivalent. Unfortunately, we also have the counterexample for this problem. Fig. 5 shows a two-dimensional example. In the figure, the dots show a set of points S with their levels. One arrangement is illustrated by the set of the thin solid lines and the thick solid lines, and another arrangement is illustrated by the set of the dashed lines and thick solid lines. These two arrangements are dual-equivalent with respect to S. The approximation error in the shaded area can become greater than any number by adding the parallel thin solid lines below the lowermost one and extending the points of S to cover the newly generated cells. Thus, this example shows that the approximation error $|\xi'(\boldsymbol{x}) - \xi(\boldsymbol{x})|$ is not bounded by the constant to the number of hyperplanes in the arrangement. It should be noted that we can also construct another counterexample of the non-simple arrangement that does not contain parallel lines in $\boldsymbol{E}^2$ as in Fig. 6.

However, we can bound the approximation error for simple arrangements. Now we introduce additional terminologies. Let S be a point set covering the simple arrangement $\mathscr{A}$ of n hyperplanes in $\boldsymbol{E}^d$, and let G_l be the linear dual graph of $\mathscr{A}$ with respect to S. For each vertex v of $\mathscr{A}$, there are 2^d cells and $d \cdot 2^{d-1}$ facets whose closures are containing v. These cells and facets around v correspond to a set of vertices $V(v)$ and a set of edges $E(v)$ in G_l, and we call the subgraph $G(v) = (V(v), E(v))$ the *cell subgraph* of G_l around v. Note that all cell subgraphs of G_l are isomorphic to the d-dimensional hypercube. By using this property of the cell subgraphs, we can easily determine all cell subgraphs of G_l without the information on the underlying arrangement $\mathscr{A}$. Since the maximum difference among the levels of $V(v)$ is d which is the diameter of the cell subgraph $G(v)$, we can assume the set of levels of $V(v)$ as $\{k, k+1, \ldots, k+d\}$. We define the *cell* $c(v)$ of the graph G_l around v as $\mathrm{cl}(\mathrm{conv}(V(v)))$; see Fig. 7 which shows a cell subgraph and its cell in the two-dimensional case. We also define the *level* of the cell $c(v)$ as the minimum level of the points in $V(v)$. The (not necessarily disjoint) union of all cells of G_l is equal to the convex hull $\mathrm{conv}(S)$ and it is called the *level hull* for the level probes. Then, we can prove the following theorem.

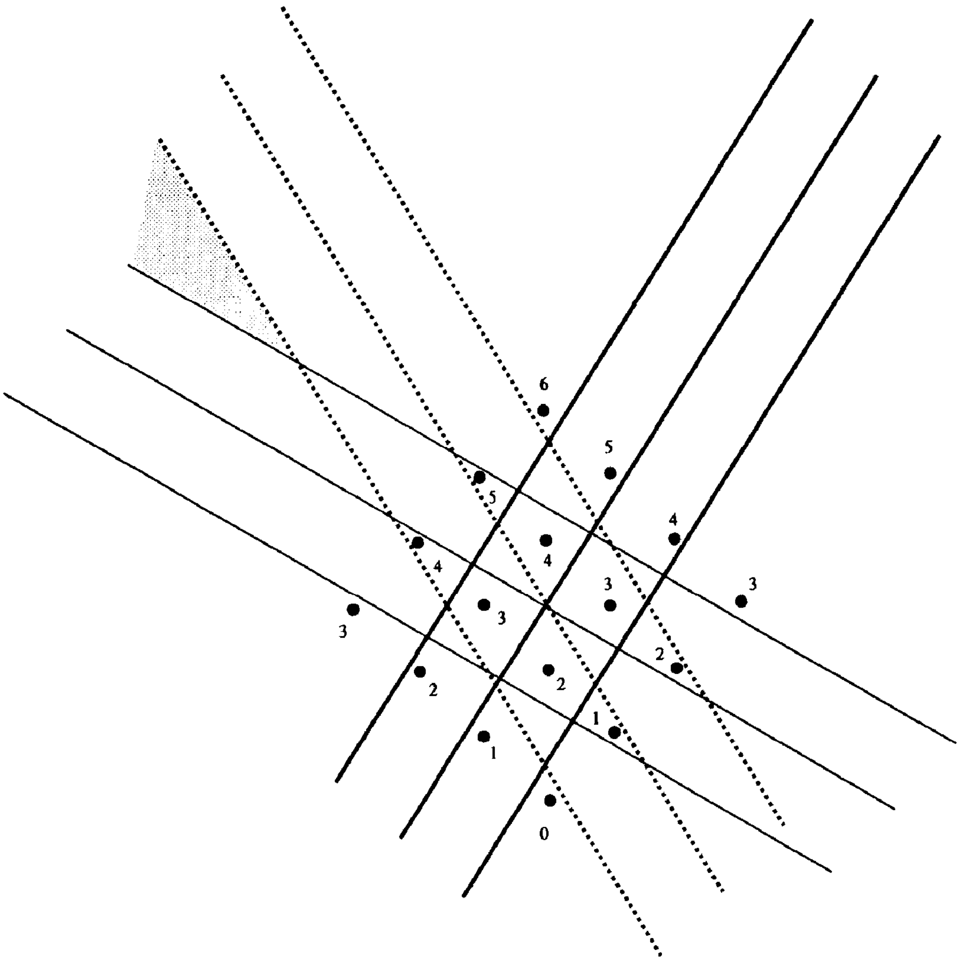


Fig. 5. An example arrangement with unbounded approximation error.

Theorem 5.5. *From the covering level probes for a simple arrangement in $\boldsymbol{E}^d$ with the linear dual graph, we can approximate the level of the points in the level hull with the maximum error d.*

Proof. For each cell $c(v)$ of a vertex v in the arrangement $\mathscr{A}$ in $\boldsymbol{R}^d$, there exist exactly d hyperplanes that intersect with $c(v)$, and each hyperplane which intersects with $c(v)$ contains exactly 2^{d-1} facets whose closures intersect with $c(v)$. On the other hand, each cell subgraph $G(v)$ of the linear dual graph G_l of $\mathscr{A}$ has exactly $d \cdot 2^{d-1}$ edges, and each edge in $G(v)$, which is contained in $c(v)$, intersects with exactly one facet in $\mathscr{A}$. Thus, we can conclude every facet in $\mathscr{A}$ whose closure has non-empty intersection with $c(v)$ also intersects with an edge of $G(v)$.

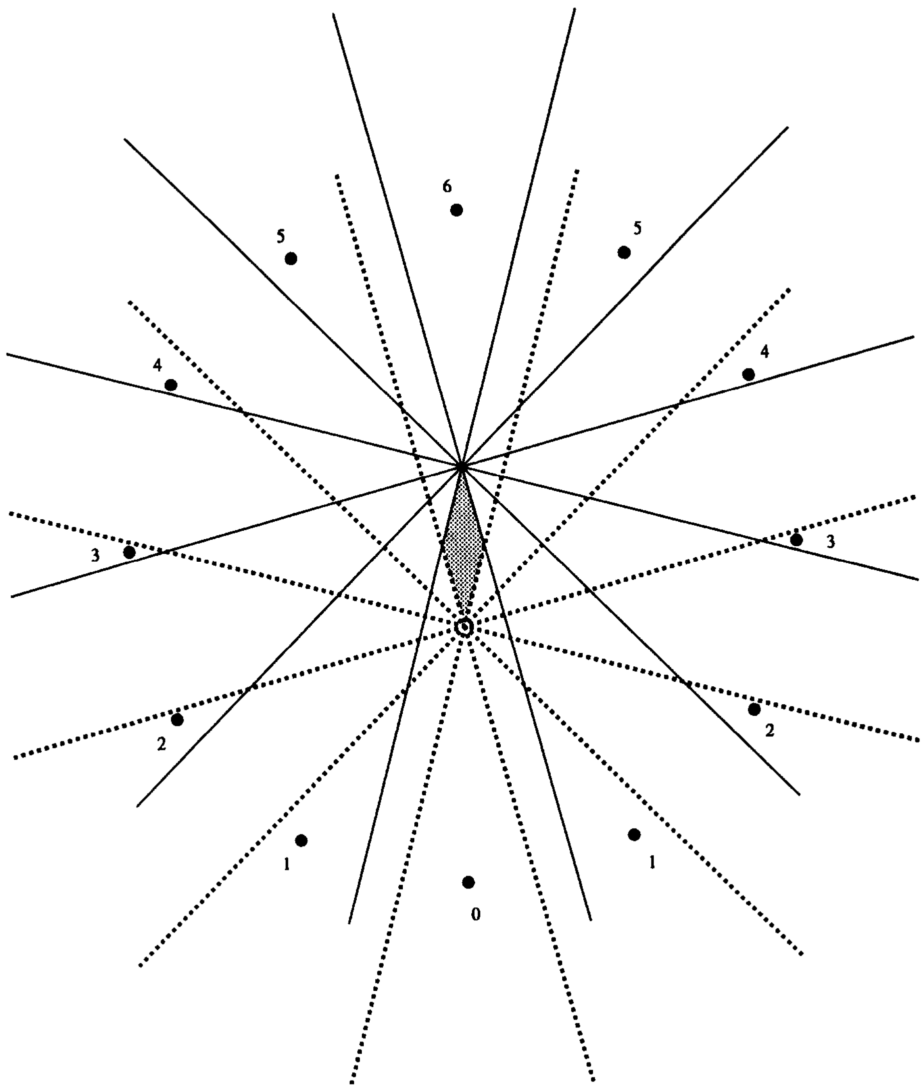


Fig. 6. Another example arrangement with unbounded approximation error.

Let n be the number of hyperplanes of the arrangement $\mathscr{A}$, that is, the maximum level of the points in the probed point set S. Let k-belt for $k = 0, 1, \ldots, n$ be the collection of the points whose levels are equal to k, and let k-envelope for $k = 0, 1, \ldots, n-1$ be the upper boundary of the k-belt. Note that each envelope that intersects with the cell $c(v)$ also intersects with some edge of the corresponding cell subgraph $G(v) \subseteq G_l$ because each envelope can be represented as the union of some facets in $\mathscr{A}$. Since the k-envelope intersects only the edges which are connecting the

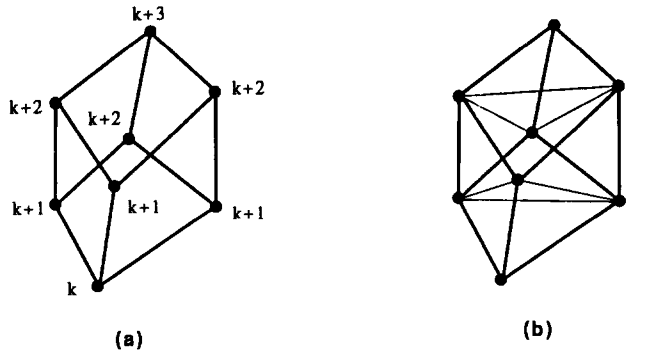


Fig. 7. (a) A cell subgraph and (b) its cell in the three-dimensional case.

points with level k and the points with level $k + 1$, each cell whose level is k can contains from k-envelope to $(k + d - 1)$-envelope, that is, from k-belt to $(k + d)$-belt. Therefore, for any point x in the cell with level k, the possible levels of x are $\{k, k + 1, \dots, k + d\}$. Since every point in the level hull is contained in some cell, the approximated level function ξ' of any dual-equivalent arrangement, the approximation error $|\xi'(x) - \xi(x)|$ is bounded by d. □

Here we consider the *enhanced level probing* for the arrangement of hyperplanes H in $\boldsymbol{E}^d$ as the extended version of the level probing that reports the identical "names" of hyperplanes in H which are below the specified point instead of the number of them. Recall that the sign vector of each face of the arrangement of n hyperplanes is a member of $\{ + 1, 0, 1\}^n$ which represents the relative positions of the face to the hyperplanes, that is, the face is above, on, or below the hyperplanes. Clearly, the sign vector of a cell in the arrangement is a member of $\{ + 1, \ - 1\}^n$, and two cells of the arrangement are adjacent if and only if the sign vectors of these two cells are different in exactly one element. It is easy to show that the sign vector of each cell of the arrangement is uniquely determined using the enhanced level probes by the covering set of points for all cells in the arrangement, which is also called the *covering enhanced level probes*. Thus, we can determine the linear dual graph of the arrangement uniquely from the covering enhanced level probes for the arrangement. Since the information from the covering enhanced level probes includes the information from the covering level probes, we can show the following corollary of Theorem 5.5.

Corollary 5.6. *From the covering enhanced level probes for a simple arrangement in* $\boldsymbol{E}^d$ *we can approximate the level of all points in the level hull with the maximum error* d.

By using the results by Cordovil and Fukuda [5] we can show that the structure of the simple arrangement of hyperplanes can be determined uniquely in the sense of combinatorial equivalence from the covering enhanced level probes for the arrangement. For studies related to [5], see [8–10].

For $i = 0, 1, \ldots, d$, let F_i be the set of i-faces of the arrangement $\mathscr{A}(H)$ of the n hyperplanes H in $\boldsymbol{E}^d$, where we denote each face by its sign vector in $\{+1, 0, -1\}^n$. We can determine the set of cells F_d with their sign vectors by the covering enhanced level probes. For two sign vectors x and y, we call x and y are *adjacent* if and only if these two vectors are different in exactly one element where the different element is non-zero both in x and in y, and the sign vector which is defined by replacing one different element of x (and y) with 0 is called the *common subvector* of x and y. For $i = d-1, d-2, \ldots, 0$, we can also determine F_i and the incidences between F_{i+1} and F_i from F_{i+1} as follows:

- For each two faces x and y in F_{i+1} whose sign vectors are adjacent, append a face f whose sign vector is the common subvector of x and y to F_i unless it is contained in F_i, and make incidences between x and f and between y and f.

Thus, we can determine the structure of the arrangement.

Theorem 5.7. *From the covering enhanced level probes for a simple arrangement in the multidimensional Euclidean space we can uniquely determine the structure* (*in the sense of the combinatorial equivalence*) *of the arrangement.*

6. Concluding remarks

In this paper, the combinatorial complexity to determine the geometry and the topology related to arrangements by the probing is investigated. For a finite set H of hyperplanes in $\boldsymbol{E}^d$, we have obtained the number of flat probes which are necessary and sufficient to determine a generic line of H and to determine H. We have shown that the line probing was the most essential probing among arbitrary dimensional flat probing to determine a generic line of H and H itself in the sense of combinatorial complexity, and have evaluated the time complexity of them based on the efficient line probing algorithm using the dual transform to compute H. We have also extended the point probing in order to approximate H. We have discussed these extensions in relation to the neural network models, and have shown their connections with computational learning theory such as learning a network of threshold functions.

The probing problem concerning arrangements is not well studied yet, and there remains many interesting questions. Here we itemize some of them.

1. Concerning the proper condition of Theorem 4.2, it is an open problem whether all positive integer sequences are arrangement feasible or not.
2. The main probing algorithm in this paper is very sensitive to the probing error. It is an open problem to develop the robust probing algorithm with efficiency.
3. The problem whether we can bound the error in the approximation of the level of each point in the level hull by the covering level probes for the single arrangement

in E^d is still open. Here we conjecture that there is an example in E^2 whose maximum error can become greater than any number.

4. The target of the probing in this paper is the arrangement of hyperplanes, and the polytope is also studied well as the target. The study of the probing for other extended targets such as the non-convex objects is still open.

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