A General Theory of Fuzzy Plausibility Measures

ULRICH HÖHLE

Fachbereich Mathematik, Universität-Gesamthochschule-Wuppertal,
Gaußstraße 20, D–5600 Wuppertal 1, West Germany

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The purpose of this paper is to present a fuzzification of probability theory, or more precisely to give a fuzzification of plausibility measures first introduced by Shafer in 1976. Although plausibility measures include probability measures as well as possibility measures, it is a typical result of this theory that only a fuzzification of possibility measures is attainable, while a fuzzification of probability measures seems to be impossible. Moreover with regard to fuzzy plausibility measures we specify a concept of mean values and entropies, which can be considered as a direct generalization of the classical notions of mean value and entropy based upon probability measures.

6. INTRODUCTION

The fundamental notions of the theory of probability are event, probability of an event, and realization of the system of all events. Referring to Los [9, 10 (cf. Sect. 46 in [15]; Sects. I.1 and I.2 in [11]), this concept can be described as follows: The set of all events forms a Boolean algebra \( \mathbb{B} \). The unit (zero) element 1 (0) is said to be the certain (impossible) event. The probability is a mapping from the Boolean algebra \( \mathbb{B} \) to the real unit interval satisfying the following axioms

\[
\begin{align*}
\text{(P1)} & \quad \mu(1) = 1, \ \mu(0) = 0 & \text{(boundary conditions)} \\
\text{(P2)} & \quad 0 < \mu(b) \forall b \neq 0 & \text{ (strict positivity)} \\
\text{(PR)} & \quad b \land c = 0 \Rightarrow \mu(b \lor c) = \mu(b) + \mu(c) & \text{ (additivity)}
\end{align*}
\]

A realization of \( \mathbb{B} \) is a mapping \( \omega \) from \( \mathbb{B} \) to the Boolean algebra \( \{0, 1\} \) consisting of two elements only; and we use the terminology: An event \( b \in \mathbb{B} \) occurs with respect to a realization \( \omega \) if and only if \( \omega(b) = 1 \). The importance of the notion “realization” is due to the fact that the concept of realizations permit a measure-theoretical interpretation of the real number \( \mu(b) \)—the so-called probability of the event \( b \). Precisely, we have the following theorem (cf. [9, 7]): Let \( \tau_n \) be the product topology on the set \( \Omega := \{0, 1\}^\mathbb{B} \) of all realizations of \( \mathbb{B} \) with respect to the discrete topology.
on \(\{0, 1\}\), and \(\mu\) be a probability on \(\mathcal{B}\); then there exists an unique regular Borel measure \(\nu\) on \(\Omega\) (w.r.t. \(\tau\)) satisfying the following conditions:

\[
\nu(\{\omega \in \Omega \mid \omega(b) = 1\}) = \mu(b) \quad \forall b \in \mathcal{B} \quad (R)
\]

\[
\nu(\{\omega \in \Omega \mid \omega \text{ is not a Boolean homomorphism}\}) = 0. \quad (PR')
\]

The aim of this paper is to generalize the probability (measure) \(\mu\) in such a way that for all \(b \in \mathcal{B}\) a measure-theoretical interpretation of \(\mu(b)\) is possible—i.e., we develop a generalized probability theory on the condition that measure-theoretical tools remain applicable. This restriction is essential, in order to have a natural definition of the mean value as well as of the entropy with respect to \((\mathcal{B}, \mu)\).

First we weaken the axiom (PR) and obtain the concept of plausibility measures introduced by Shafer [14 (cf. [13])]. Second, we extend the range of plausibility measures and arrive at the class of fuzzy plausibility measures. The importance of the concept of fuzzy plausibilities is based on the fact that with respect to fuzzy plausibility measures we are in the position to give a precise, mathematical meaning of the following statement: "It is unlikely that the possibility of the event \(b\) is at least \(t\)"—i.e., the probability that the possibility of \(b\) is at least \(t\) is smaller than \(10^{-4}\). Finally we discuss some concrete examples, in which fuzzy plausibility measures appear quite naturally.

1. Plausibility Measures

**Definition 1.1.** Let \(B\) be an abstract Boolean algebra. A map \(\mu : \mathcal{B} \to [0, 1]\) is called a plausibility measure if and only if \(\mu\) satisfies the following axioms:

(P1) \(\mu(\mathbb{1}) = 1, \mu(\mathbb{0}) = 0\)

(P2) \(0 < \mu(b) \forall b \neq 0\)

(PL) For every nonvoid finite subset \(\{b_1, \ldots, b_n\}\) of \(\mathcal{B}\) the subsequent inequality is valid

\[
\sum_{i=1}^{n} (1) \sum_{1 \leq i < j \leq n} \mu(\vee_{1 \leq i \leq n} b_i) \geq \mu(\bigwedge_{1 \leq i \leq n} b_i). \quad \text{(subadditivity)}
\]

**Remark 1.2.** (a) Every plausibility measure \(\mu\) is isotone—i.e., \(\mu\) fulfills the condition \(b \leq c \Rightarrow \mu(b) \leq \mu(c)\).

(b) Since (PR) implies (PL), we obtain that every probability measure is a plausibility measure.

\(^1\omega : \mathcal{B} \to \{0, 1\}\ is a Boolean homomorphism if and only if \(\omega\) is provided with the properties \(\omega(b \lor c) = \max(\omega(b), \omega(c)), \omega(b \land c) = \min(\omega(b), \omega(c)), \omega(\mathbb{1}) = 1, \omega(\mathbb{0}) = 0\).

\(^2\) Similar problems are also investigated in [21].
(c) Every strict positive possibility measure \( \mu \) on an abstract Boolean algebra \( B \) (cf. [20])—i.e., every map \( \mu : B \to [0, 1] \) equipped with the properties:

\[
\begin{align*}
(\text{P1}) & \quad \mu(\top) = 1, \mu(\emptyset) = 0 \\
(\text{P2}) & \quad 0 < \mu(b) \forall b \neq \emptyset \\
(\text{P0}) & \quad \mu(b \lor c) = \text{Max}(\mu(b), \mu(c)) \quad \text{(min-additivity)}
\end{align*}
\]

is a plausibility measure on \( B \).

**Example 1.3.** Let \( X \) be an arbitrary, nonvoid set, \( B = \mathcal{P}(X) \) be the power set of \( X \), and let \( \{C_1, \ldots, C_n\} \) be a finite covering\(^3\) of \( X \). Every \( n \)-tuple \( (A_1, \ldots, A_n) \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) induces a plausibility measure \( \mu_{ij} \) on \( \mathcal{P}(X) \) as follows:

\[
\mu_{ij}(A) = \sum_{i=1}^{n} \{ \lambda_i, A \cap C_i \neq \emptyset \} \quad \forall A \in \mathcal{P}(X).
\]

If \( \{C_1, \ldots, C_n\} \) is a chain—e.g., \( C_1 \subseteq C_2 \subseteq \cdots \subseteq C_n \)—then \( \mu_{ij} \) is a strict possibility measure. Moreover, if \( X \) is finite, and if each \( C_i \) contains exactly one point \((i = 1, \ldots, n)\), then \( \mu_{ij} \) is a probability measure.

From Theorem 1.12 in [7] (cf. Main theorem 2.3 in [7]), we derive the following important

**Theorem 1.4.** Let \( B \) be an abstract Boolean algebra and \( \tau_p \) be the product topology on \( \{0, 1\}^B \). For every plausibility measure \( \mu \) on \( B \) there exists a unique regular Borel probability measure \( \nu_\mu \) on \( \{0, 1\}^B \) (w.r.t. \( \tau_p \)) satisfying the subsequent conditions

\[
\begin{align*}
\nu_\mu\{ \omega \in \{0, 1\}^B | \omega(b) = 1 \} = \mu(b) \quad \forall b \in B & \quad \text{(R')}

\nu_\mu\{ \omega \in \{0, 1\}^B | \omega(\top) = 1, \omega(\emptyset) = 0, \omega(b \lor c) = \text{Max}(\omega(b), \omega(c)) \forall (b, c) \in B \times B \} = 1. & \quad \text{(PL')}\n\end{align*}
\]

**Remark 1.5.** (Interpretation of plausibility measures). The above theorem in connection with the terminology described in the Introduction permits an interpretation of plausibility measures as follows:

1. The certain event occurs always; the impossible event occurs never.

2. If an event \( b \) occurs, and if \( b \) implies \( c \), then \( c \) also occurs.

3. If an event \( b \) does not occur, then the complement \( b' \) of \( b \) occurs.

\(^3\) i.e., \( C_i \neq \emptyset \forall i, \bigcup_{i=1}^{n} C_i = X \).
If an event \( b \) occurs, then the complement \( b' \) of \( b \) may also occur. This property can be considered as a kind of subjectivity, which is included in the concept of plausibility measures.

Remark 1.6 (Entropy of plausibility measures). Let \( \mathcal{I} (\mathbb{B}) \) be the set of all realizations \( \omega \) of \( \mathbb{B} \) provided with the properties

\[
\omega (1) = 1, \quad \omega (\emptyset) = 0 \quad (\text{boundary conditions})
\]

\[
\omega (b \lor c) = \max (\omega (b), \omega (c)) \quad \forall b, c \in \mathbb{B} \quad (\text{ideal-property})
\]

Obviously \( \mathcal{I} (\mathbb{B}) \) is a compact subset of \( \{0, 1\}^{\mathbb{B}} \) with respect to \( \tau_\mu \). Further let \( \mu \) be a plausibility measure on \( \mathbb{B} \). We define a mapping \( e(\mathbb{B}, \mu) : \mathcal{I} (\mathbb{B}) \to [0, +\infty) \) by

\[
e(\mathbb{B}, \mu) : \mathcal{I} (\mathbb{B}) \to [0, +\infty)
\]

\[
e(B, \mu) (\omega) = \sup_{b \in \mathbb{B}} \left[ -\ln (1 - \mu (b)) \right] \cdot (1 - \omega (b)). \quad (1.2)
\]

The extended real number \( [e(\mathbb{B}, \mu)] (\omega) \) can be interpreted as the information represented by the realization \( \omega \) with respect to \( \mu \). Moreover, the relation

\[
\{ \omega \in \mathcal{I} (\mathbb{B}) \mid r < [e(\mathbb{B}, \mu)] (\omega) \}
\]

\[
= \bigcup \{ \{ \omega \in \mathcal{I} (\mathbb{B}), \omega (h) = 0 \} \mid h \in \mathbb{B}, r < -\ln (1 - \mu (h)) \}
\]

is valid; i.e., \( e(\mathbb{B}, \mu) \) is Borel measurable. Identifying \( \mu \) with the regular Borel probability measure \( v_\mu \) (cf. Theorem 1.4), we are in the position to define the “average of information”—the so-called entropy \( E(\mathbb{B}, \mu) \) of \( (\mathbb{B}, \mu) \) as

\[
E(\mathbb{B}, \mu) = \int_{\mathcal{I} (\mathbb{B})} e(\mathbb{B}, \mu) dv_\mu. \quad (1.3)
\]

(a) Let \( \mu \) be a probability measure on atomless Boolean algebra \( \mathbb{B} \). Then \( [e(\mathbb{B}, \mu)] (\omega) = +\infty \), and therewith the entropy \( E(\mathbb{B}, \mu) \) is infinite.

(b) Let \( \mu \) be a probability measure on an atomic Boolean algebra \( \mathbb{B} \), and let \( \mathcal{A} \) be the set of all atoms of \( \mathbb{B} \). Then \( \mathbb{B} \) is isomorphic to the power set \( \mathcal{P}(\mathcal{A}) \) of \( \mathcal{A} \), and each atom \( a \) can be identified with a Boolean homomorphism \( \omega_a : \mathbb{B} \to \{0, 1\} \) defined by

\[
\omega_a (b) = 1, \quad a \leq b,
\]

\[
= 0, \quad a \not\leq b.
\]

Moreover the mapping \( a \to \omega_a \) is measure preserving, i.e.,

\[
v_\mu \{ \omega \in \{0, 1\}^{\mathbb{B}}, \omega (b) = 1 \} = \mu \{ a \in \mathcal{A}, a \leq b \} = \mu (b);
\]
hence \( v^*_\mu \{ \omega_\alpha, \alpha \in \mathcal{A} \} = 1 \) —i.e., the outer measure of the subset \( \{ \omega_\alpha, \alpha \in \mathcal{A} \} \) of \( \mathcal{I}(\mathcal{B}) \) is equal to 1. Therefore we obtain from (1.2) and (1.3)

\[
E(\mathcal{B}, \mu) = \sum_{\alpha \in \mathcal{A}} -\mu(\alpha) \cdot \ln(\mu(\alpha)).
\] (1.4)

(c) If the plausibility measure \( \mu \) is a probability measure, then the sections (a) and (b) show that the concept of entropies defined in (1.3) is the usual one (cf. [4, 17]). In this section we are interested in the behaviour of the entropies with respect to possibility measures. Therefore let \( X \) be an arbitrary nonvoid set, \( \mathcal{B} = \mathcal{P}(X) \) be the power set of \( X \), \( \{ C_i, i = 1, \ldots, n \} \) be a finite covering of \( X \) with \( \emptyset \neq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{n-1} \supseteq C_n = X \), and let \( (\lambda_i)_{i=1}^n \) be a \( n \)-tuple of real numbers \( \lambda_i \), with \( 0 < \lambda_i \leq 1 \), \( \sum_{i=1}^n \lambda_i = 1 \) (cf. Example 1.3). Referring to (1.1) a possibility measure \( \mu_\lambda \) on \( \mathcal{P}(X) \) is given by

\[
\mu_\lambda(A) = \sum \{ \lambda_i, A \cap C_i \neq \emptyset \}.
\]

From (1.3) we obtain

\[
E(\mathcal{P}(X), \mu_\lambda) = \sum_{i=1}^n \lambda_i \cdot \sup \{ -\ln(1 - \mu_\lambda(A)) | A \cap C_i = \emptyset \}
= \sum_{i=1}^n \lambda_i \cdot \ln(1 - \mu_\lambda(C_i))
= \sum_{i=1}^n \lambda_i \cdot \ln \left( \frac{\sum_{j=1}^i \lambda_j}{\sum_{j=1}^{i-1} \lambda_j} \right).
\] (1.5)

Since \( \mu_\lambda \) is determined by the fuzzy subset \( f \) of \( X \) defined by (cf. [20]), \( f(x) = \mu_\lambda \{ x \} \ \forall x \in X \), we conclude from (1.5) that the entropy \( E(\mathcal{P}(X), \mu_\lambda) \) depends only on the differences of the values of \( f \). In particular, if the differences are “constant” i.e., \( \lambda_i = 1/n, i = 1, \ldots, n \)—we derive from (1.5) the relation

\[
E(\mathcal{P}(X), \mu_\lambda) = (1/n) \sum_{i=1}^n -\ln(i/n)
= \ln(n) - (1/n) \ln(n!) = \ln(n/(n!)^{1/n}) \xrightarrow{n \to \infty} 1;
\]

i.e., contrary to the case of probability measures the entropy of possibility measures corresponding to \( \lambda \) with \( \lambda_i = 1/n \) are bounded. This situation is not very surprising, since the information represented by the realization of possibility measures is “smaller” than the information represented by the realizations of probability measures (cf. Introduction and Remark 1.5).
Remark 1.7 (Mean values with respect to plausibility measures). Let $\mathcal{B}$ be an abstract Boolean algebra. A $\mathcal{B}$-fuzzy nonnegative, real number is a $\mathcal{B}$-fuzzy subset $F$ of $\mathbb{R}^+$ (i.e., $F: \mathbb{R}^+ \rightarrow \mathcal{B}$, cf. [3]) satisfying the following conditions:

$$F(0) = 0, \quad \forall \{F(n), n \in \mathbb{N}\} = 1 \quad \text{(boundary conditions)}$$

$$F(r) = \bigvee \{F(r'), r' < r\}. \quad \text{(left-continuity)}$$

With $F$ we associate the Borel measurable map $F^\nu: \{0, 1\}^B \rightarrow [0, +\infty]$ (the so-called quasi-inverse mapping of $F$; cf. Sect. 2 in [6]; Sect. 44 in [15]) defined as follows:

$$F^\nu(\omega) = \sup\{r \in \mathbb{R}^+, \omega(F(r)) = 0\} \forall \omega \in \{0, 1\}^B. \quad (1.6)$$

Further let $\mu$ be a plausibility measure on $\mathcal{B}$, and let $\nu_\mu$ be the regular Borel probability measure on $\{0, 1\}^B$ corresponding to $\mu$ (cf. Theorem 1.4). The mean value of a $\mathcal{B}$-fuzzy nonnegative real number $F$ w.r.t. $\mu$ can be introduced as follows (cf. [8]):

$$\int Fd\mu = \int_{\{0, 1\}^B} F^\nu d\nu_\mu. \quad (1.7)$$

From Theorem 1.4 in connection with (1.7) we obtain the important property (cf. Corollary 2 in [8]):

$$\sum_{i=1}^n (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq n} \int \left( \bigvee_{i=1}^j F_{i-1} \right) d\mu \leq \int \left( \bigwedge_{i=1}^n F_i \right) d\mu$$

if $\int F_i d\mu < +\infty \forall \mu$. In particular, if $\mu$ is a probability measure, then $\int F d\mu$ coincides with the usual Lebesgue integral (cf. [12, 16]).

2. Fuzzy Plausibility Measures in the Case of $L = \{0, \frac{1}{2}, 1\}$

A fuzzy subset $F$ of $\{0, \frac{1}{2}, 1\}$ (i.e., $F: \{0, \frac{1}{2}, 1\} \rightarrow [0, 1]$) is said to be a $\{0, \frac{1}{2}, 1\}$-valued fuzzy quantity (cf. Sect. 2 in [6]) if and only if $F(0) = 0, 0 \leq F(\frac{1}{2}) \leq F(1) \leq 1$. Obviously the set $\mathcal{A}(\{0, \frac{1}{2}, 1\})$ of all $\{0, \frac{1}{2}, 1\}$-valued fuzzy quantities forms a complete lattice; and the unit interval $[0, 1]$ can be embedded into $\mathcal{A}(\{0, \frac{1}{2}, 1\})$ as follows:

$x \preccurlyeq F_x$, where $F_x(0) = 0, F_x(\frac{1}{2}) = F_x(1) = 1 - x$;

and therewith $\mathcal{A}(\{0, \frac{1}{2}, 1\})$ is an extension of the unit interval. Moreover, if $F$ is a $\{0, \frac{1}{2}, 1\}$-valued fuzzy quantity and $\iota$ an element of $[0, 1]$, we
associate with \( \kappa(t, F) = \max\{l \in \{0, \frac{1}{2}, 1\}, 1 - t \geq F(l)\} \) the following interpretation

\[
\kappa(t, F) = 1 \iff F \text{ is certainly greater than } t.
\]

\[
\kappa(t, F) = \frac{1}{2} \iff \text{It is undecidable whether } F \text{ is greater than } t \text{ or not.}
\]

\[
\kappa(t, F) = 0 \iff F \text{ is certainly not greater than } t.
\]

In this sense \( \Delta(\{0, \frac{1}{2}, 1\}) \) can be considered as a fuzzification of the unit interval.

**Notation.** Let \( \mathcal{K}(\overline{\mathcal{K}}) \) be a nonvoid subset of a Boolean algebra \( \mathbb{B} \), and let \( r_{\mathcal{K}}(s_{\overline{\mathcal{K}}}) \) be a mapping from \( \mathcal{K} \) to \( \{\frac{1}{2}, 1\} \) (from \( \overline{\mathcal{K}} \) to \( \{\frac{1}{2}, 1\} \)). Then \( r_{\mathcal{K}} \wedge s_{\overline{\mathcal{K}}} \) is a mapping from \( \mathcal{K} \cup \overline{\mathcal{K}} \) to \( \{\frac{1}{2}, 1\} \) defined by

\[
(r_{\mathcal{K}} \wedge s_{\overline{\mathcal{K}}})(b) = r_{\mathcal{K}}(b), \quad b \in \mathcal{K} \cap \overline{\mathcal{K}}
\]

\[
= \min(r_{\mathcal{K}}(b), s_{\overline{\mathcal{K}}}(b)), \quad b \in \mathcal{K} \cap \overline{\mathcal{K}}
\]

\[
= s_{\overline{\mathcal{K}}}(b), \quad b \in \mathcal{K} \cap \overline{\mathcal{K}}.
\]

**Definition 2.1.** Let \( \mathbb{B} \) be an abstract Boolean algebra, and \( J_0(\mathbb{B}) \) be the set of all nonvoid finite subsets of \( \mathbb{B} \). A mapping \( \mu: \mathbb{B} \rightarrow \Delta(\{0, \frac{1}{2}, 1\}) \) is called a \( \{0, \frac{1}{2}, 1\} \)-plausibility measure if and only if \( \mu \) satisfies the following axioms:

\[
\text{(FP1)} \quad \mu(1) = F_1, \quad \mu(\emptyset) = F_0
\]

\[
\text{(FP2)} \quad \mu(b) \neq F_0 \quad \forall b \neq \emptyset
\]

\[
\text{(FP3)} \quad \exists \lambda: \bigoplus_{\mathcal{H}, \mathcal{L} \in J_0(\mathbb{B})} (\{\frac{1}{2}, 1\})^{11} \rightarrow [0, 1] \text{ s.t.}^4
\]

\[
\forall (\mathcal{H}, \overline{\mathcal{H}}) \in J_0(\mathbb{B}) \times J_0(\mathbb{B}) \forall r_{\mathcal{H}} \in \{\frac{1}{2}, 1\}^{11} \forall s_{\overline{\mathcal{H}}} \{\frac{1}{2}, 1\}^{12}
\]

the subsequent inequality is valid,

\[
\sum_{i=0}^{\#(\mathcal{H})} (-1)^i \sum_{\mathcal{L} \in \mathcal{L} \neq \mathcal{H}} \lambda(r_{\mathcal{H}} \wedge s_{\overline{\mathcal{L}}}) \cdot \mu\left(\bigvee (\mathcal{H} \cup \overline{\mathcal{L}})\right)
\]

\[
\times \left(\min_{h \in \mathcal{H} \cup \overline{\mathcal{L}}} (r_{\mathcal{H}} \wedge s_{\overline{\mathcal{L}}})(b)\right)
\]

\[
+ (1 - \lambda(r_{\mathcal{L}} \wedge s_{\overline{\mathcal{H}}})) \cdot \mu\left(\bigvee (\mathcal{H} \cup \overline{\mathcal{L}})\right) \left(\max_{h \in \mathcal{H} \cup \overline{\mathcal{L}}} (r_{\mathcal{H}} \wedge s_{\overline{\mathcal{L}}})(b)\right)
\]

\[
\geq 0 \quad \text{where } s_{\overline{\mathcal{H}}} \text{ is the restriction of } s_{\overline{\mathcal{H}}} \text{ to } \overline{\mathcal{L}}.
\]

\(^4 \bigoplus \) denotes the direct set-theoretical union.
A characterization of \( \{0, \frac{1}{2}, 1\} \)-fuzzy plausibility measures and therewith an interpretation of condition (FPL) is specified in

**Theorem 2.2.** Let \( \mu \) be a mapping from a Boolean algebra \( \mathbb{B} \) to \( \Delta(\{0, \frac{1}{2}, 1\}) \) provided with the properties (FP1) and (FP2). Then the following assertions are equivalent:

(i) \( \mu \) is a \( \{0, \frac{1}{2}, 1\} \)-fuzzy plausibility measure on \( \mathbb{B} \).

(ii) There exists a regular Borel probability measure \( \nu_\mu \) on \( \{0, \frac{1}{2}, 1\}^\mathbb{B} \) (w.r.t. the product topology) equipped with the subsequent properties

\[
(FP') \quad \nu_\mu(\omega) = \{0, \frac{1}{2}, 1\}^\mathbb{B}, \quad l \leq \omega(b) \Rightarrow 1 - \mu(b)(l) \quad \forall l, b
\]

\[(FPL') \quad \nu_\mu(\{\omega \in \{0, \frac{1}{2}, 1\}^\mathbb{B} \mid \omega(1) = 1, \omega(\emptyset) = 0, \omega(b \lor c) = \max(\omega(b), \omega(c)) \forall b, c \in \mathbb{B}\} - 1.
\]

**Proof.** (a) (i) \( \Rightarrow \) (ii) Let \( J_0(\mathbb{B}) \) be the set of all nonvoid, finite subsets of \( \mathbb{B} \), \( \Pi_b : \{0, \frac{1}{2}, 1\}^\mathbb{B} \to \{0, \frac{1}{2}, 1\} \) be the projection onto the \( b \)-th coordinate, and let \( \lambda \) be the mapping from \( \bigoplus_{b \in J_0(\mathbb{B})} \{0, \frac{1}{2}, 1\}^{\mathbb{B}} \) to the unit interval, such that \( \mu \) satisfies the inequality in (FPL) with respect to \( \lambda \). From (FPL) we infer that for every element \( \mathbb{H} \in J_0(\mathbb{B}) \) there exists a Borel probability measure \( \nu_{\lambda \mathbb{H}} \) on \( \{0, \frac{1}{2}, 1\}^\mathbb{B} \) satisfying the condition

\[
\nu_{\lambda \mathbb{H}} \left( \bigcap_{b \in \mathbb{H}} \Pi_b^{-1}(\{\omega \in \{0, \frac{1}{2}, 1\}^\mathbb{B} \mid \rho_{\lambda \mathbb{H}}(b) \leq l\}) \right) = \lambda(\rho_{\lambda \mathbb{H}}(1 \lor \mathbb{K}) \cdot \left[ (\min_{b \in \mathbb{H}} \rho_{\lambda \mathbb{H}}(b) + (1 - \lambda(\rho_{\lambda \mathbb{H}}))) \cdot \left[ \mu \left( \bigvee_{b \in \mathbb{H}} \right) \right] \right](\max_{b \in \mathbb{H}} \rho_{\lambda \mathbb{H}}(b))
\]

\( \forall \mathbb{H} \subseteq \mathbb{K}, \forall \rho_{\lambda \mathbb{H}} \in \{0, \frac{1}{2}, 1\}^\mathbb{K} \). Moreover, \( (\nu_{\lambda \mathbb{H}})_{\mathbb{H} \in J_0(\mathbb{B})} \) is a projective system of Borel measures; hence by virtue of the celebrated Kolmogoroff theorem there exists a regular Borel probability measure \( \nu_\mu \) on \( \{0, \frac{1}{2}, 1\}^\mathbb{B} \) provided with the property (cf. [1, 5])

\[
\nu_\mu \left( \bigcap_{b \in \mathbb{H}} \Pi_b^{-1}(\{l \in \{0, \frac{1}{2}, 1\} \mid \rho(b) \leq l\}) \right) = \mu \left( \bigvee_{b \in \mathbb{H}} \right)(\mathbb{H}) \quad \forall \mathbb{H} \in J_0(\mathbb{B}). \tag{2.1}
\]

From (2.1) we obtain

\[
\nu_\mu(\omega) = \{0, \frac{1}{2}, 1\}^\mathbb{B}, \quad l \leq \omega(b) \Rightarrow 1 - \mu(b)(l)
\]

\[
\nu_\mu(\{\omega \in \{0, \frac{1}{2}, 1\}^\mathbb{B} \mid l \leq \omega(b \lor c)\}
\]

\[
\Delta \{\omega \in \{0, \frac{1}{2}, 1\}^\mathbb{B} \mid l \leq \max(\omega(b), \omega(c))\} = 0;
\]

and therewith in account of the regularity of \( \nu_\mu \) the Borel measure \( \nu_\mu \) fulfills the desired properties.
(b) (ii) ⇒ (i) Since every measure is isotone, we are in the position to define a mapping \( \lambda: \bigoplus_{H \in \mathcal{H}} \{0, \frac{1}{2}, 1\}^H \rightarrow [0, 1] \) as follows

\[
\lambda(r_H) = \frac{v_\mu(\bigcap_{h \in H} \{ \omega, (\max_{h \in H} r_H(b)) \leq \omega(b) \})}{v_\mu(\bigcap_{h \in H} \{ \omega, (\max_{h \in H} r_H(b)) > \omega(b) \}) - v_\mu(\bigcap_{h \in H} \{ \omega, (\min_{h \in H} r_H(b)) \leq \omega(b) \})}.
\]

(2.2)

By virtue of the additivity of \( v_\mu \), the relation

\[
0 \leq v_\mu\left( \bigcap_{h \in H} \{ \omega, r_H(b) \leq \omega(b) \} \cap \bigcap_{h \in H} \{ \omega, s_H(b) \leq \omega(b) \} \right)
\]

\[
= \sum_{i=0}^{\#H} \sum_{R \subseteq H, \#R = i} (-1)^i \lambda_{R;H}(\bigcap_{h \in H \cap R} \{ \omega, (\max_{h \in H} r_H(b)) \leq \omega(b) \})
\]

\[
\cap \bigg( \bigcap_{h \in H \cap R} \{ \omega, (\min_{h \in H} r_H(b)) \leq \omega(b) \} \bigg)
\]

holds. From (2.2), (2.3), (FR'), and (FPL') we infer that \( \mu \) satisfies the condition (FPL) with respect to \( \lambda \); hence \( \mu \) is a \{0, \frac{1}{2}, 1\}-fuzzy plausibility measure. Q.E.D.

**Remark 2.3.** Contrary to the case of plausibility measures (cf. Theorem 1.4), the regular Borel probability measure \( v_\mu \) on \{0, \frac{1}{2}, 1\}^H \) (cf. Theorem 2.2) is not uniquely determined by the properties (FR') and (FPL').

**Definition 2.4.** \{0, \frac{1}{2}, 1\}-fuzzy realization. Let \( \mathcal{H} \) be an abstract Boolean algebra; a mapping \( \omega: \mathcal{H} \rightarrow \{0, \frac{1}{2}, 1\} \) is called a \{0, \frac{1}{2}, 1\}-fuzzy realization. Further we use the terminology: The degree that an event \( b \) occurs, is at least \( l \) \((l \in \{0, \frac{1}{2}, 1\})\) if and only if \( l \leq \omega(b) \). In particular, if \( \omega(b) = 1 \) \( (\omega(b) = 0, \omega(b) = \frac{1}{2}) \) then we say: The event \( b \) occurs (does not occur; it is undecidable whether \( b \) occurs or not).

**Remark 2.5.** (Entropy with respect to \{0, \frac{1}{2}, 1\}-fuzzy plausibility measures). Let \( \mathcal{H} \) be a Boolean algebra, and \( \mathfrak{I}(\mathcal{H}, \{0, \frac{1}{2}, 1\}) \) be the set of all \{0, \frac{1}{2}, 1\}-fuzzy realizations \( \omega \) provided with the properties

\[
\omega(1) = 1, \quad \omega(\emptyset) = 0 \quad (\text{boundary conditions})
\]

\[
\omega(b \lor c) = \max(\omega(b), \omega(c)) \quad (\text{ideal-property})
\]
Further let $p$ be a $\{0, \frac{1}{2}, 1\}$-fuzzy plausibility measure on $B$ and $v_\mu$ a regular Borel probability measure on $\{0, \frac{1}{2}, 1\}^B$ satisfying the conditions (FR') and (FPL') (cf. Theorem 2.2). Extending the definition of the entropy of plausibility measures (cf. Remark 1.6) to $\mu$ we have to pay a price for the "fuzziness" involved in $\mu$. We consider two fundamentally different situations:

(a) ($\{0, \frac{1}{2}, 1\}$-fuzzy entropy) for every $l \in \{0, \frac{1}{2}, 1\}$ the mapping $e(B, \mu, l): \mathcal{F}(B, \{0, \frac{1}{2}, 1\}) \to [0, +\infty]$ defined by

$$
e(B, \mu, l)(\omega) = \sup \{ -\ln [\mu(b)](1-l)^+, l \notin \omega(b) \}$$

is Borel measurable. The extended real number $[e(B, \mu, l)](\omega)$ can be interpreted as the information at the level $l$ represented by the $\{0, \frac{1}{2}, 1\}$-fuzzy realization $\omega$ with respect to $\mu$. The "average of information at the level $l$" is given by

$$[E(B, \mu)](l) = \int_{\mathcal{F}(B, \{0, \frac{1}{2}, 1\})} e(B, \mu, l) \, dv_\mu,$$

From the relation

$$\{ \omega, r < [e(B, \mu, l)](\omega) \} = \bigcup \{ \omega, l \notin \omega(h) \} \mid r < -\ln \mu(h)(1-l)^+ \}$$

in connection with (FR') and (FPL') we infer that $[E(B, \mu)](l)$ is independent of $v_\mu$—i.e., $[E(B, \mu)](l)$ depends only on $\mu$. Therefore we call the quantity $E(B, \mu)$ defined by (2.4) the $\{0, \frac{1}{2}, 1\}$-fuzzy entropy of $(B, \mu)$. Using the terminology of [6], $E(B, \mu)$ can be considered as a $\{0, \frac{1}{2}, 1\}$-valued $[0, +\infty]$-fuzzy quantity.

(b) (Set of all entropies with respect to $\{0, \frac{1}{2}, 1\}$-fuzzy plausibility measures.) We introduce a Borel measurable mapping $e(B, \mu, l): \mathcal{F}(B, \{0, \frac{1}{2}, 1\}) \to [0, +\infty]$ by

$$[e(B, \mu, l)](\omega) = \sup_{h \in B} \{ -\ln [\mu(h)](1-l)^+ \cdot (1-\omega(h)) \}.$$

The extended real number $[e(B, \mu, l)](\omega)$ can be interpreted as the information represented by the $\{0, \frac{1}{2}, 1\}$-fuzzy realization $\omega$ with respect to $\mu(\cdot)(l)$. From (2.5) we derive the "average of information" as follows:

$$\bar{e}_\nu(B, \mu) = \int_{\mathcal{F}(B, \{0, \frac{1}{2}, 1\})} \left( \frac{1}{2} \cdot [e(B, \mu, \frac{1}{2}) + e(B, \mu, 1)] \right) \, dv_\mu.$$
Obviously \( \delta_{\nu}(\mathcal{B}, \mu) \) is not independent of the Borel measure \( \nu \) associated with \( \mu \) (cf. Remark 2.3). Therefore let \( \mathcal{M}_\mu \) be the set of all regular Borel probability measures \( \nu \) on \( \{0, \frac{1}{2}, 1\}^\mathbb{B} \) satisfying the conditions \((\text{FR'})\) and \((\text{FPL'})\). Then \( \Sigma_\lambda(\mathbb{B}, \mu) = \{ \delta_{\nu}(\mathbb{B}, \mu), \nu \in \mathcal{M}_\mu \} \) is said to be the set of all entropies of \((\mathbb{B}, \mu)\). In particular, if \( \Sigma_\lambda(\mathbb{B}, \mu) \cap \mathcal{C}\{+\infty\} \) is nonvoid, then \( \Sigma_\lambda(\mathbb{B}, \mu) \cap \mathcal{C}\{+\infty\} \) is a convex subset of \( \mathbb{R}^+ \).

**Remark 2.6** (Mean value with respect to \( \{0, \frac{1}{2}, 1\} \)-fuzzy plausibility measures). Corresponding to the foregoing remark we specify two different methods for the construction of the mean value of \( \mathbb{B} \)-fuzzy nonnegative real numbers with respect to \( \{0, \frac{1}{2}, 1\} \)-fuzzy plausibility measures.

(a) \((\{0, \frac{1}{2}, 1\} \)-fuzzy mean value). Let \( F \) be a \( \mathbb{B} \)-fuzzy nonnegative real number. For every element \( l \in \{0, \frac{1}{2}, 1\} \) the mapping \( F'(\cdot, l) : \mathcal{H}(\mathbb{B}, \{0, \frac{1}{2}, 1\}) \to [0, +\infty] \) defined by

\[
F'(\omega, l) = \sup\{r \in \mathbb{R}^+, l \leq \omega(F(r))\}
\]

is Borel measurable. Further let \( \nu \) be a regular Borel probability measure on \( \{0, \frac{1}{2}, 1\}^\mathbb{B} \) equipped with the properties \((\text{FR'})\) and \((\text{FPL'})\). Then the extended real number

\[
\left( \int F d\mu \right)(l) = \int_{\mathcal{H}(\mathbb{B}, \{0, 1/2, 1\})} F'(\cdot, l) d\nu
\]  

is independent of \( \nu \) — i.e. \( \left( \int F d\mu \right)(l) \) depends on \( (F, \mu, l) \) only. The quantity \( \int F d\mu \) defined by \((2.6)\) is called the \( \{0, \frac{1}{2}, 1\} \)-fuzzy mean value of \( F \) with respect to \( \mu \). In particular \( \int F d\mu \) can be considered as a \( \{0, \frac{1}{2}, 1\} \)-valued, \([0, +\infty]\)-fuzzy quantity.

(b) Maintaining the notations of Section (a) we introduce a Borel measurable mapping \( F'^\prime : \mathcal{H}(\mathbb{B}, \{0, \frac{1}{2}, 1\}) \to [0, +\infty] \) as follows:

\[
F'^\prime(\omega) = \sup_{r \in \mathbb{R}^+} r \cdot (1 - \omega(F(r))).
\]

Further let \( \mathcal{M}_\mu \) be the set of all regular Borel probability measures \( \nu \) on \( \{0, \frac{1}{2}, 1\}^\mathbb{B} \) satisfying \((\text{FR'})\) and \((\text{FPL'})\). Then \( \int F d\mu := \{ \int F'^\prime d\nu, \nu \in \mathcal{M}_\mu \} \) is said to be the set of all mean values of \( F \) with respect to \( \mu \). In particular, if \( \int F d\mu \cap \mathcal{C}\{+\infty\} \) is nonvoid, then \( \int F d\mu \cap \mathcal{C}\{+\infty\} \) is a convex subset of \( \mathbb{R}^+ \).
3. Fuzzy Plausibility Measures (General Case)

The aim of this section is to extend the results of Section 2 to the class of complete chains \( L \) containing a countable subset \( C \) provided with the following property

\[
\forall l \exists A \subseteq C \text{ s.t. } l = \bigvee A.
\]

For the sake of simplicity we restrict ourselves to the case of the real unit interval \( L = [0, 1] \). A fuzzy subset \( F \) of \([0, 1]\) (i.e., \( F: [0, 1] \rightarrow \mathbb{R} \)) is said to be a \([0, 1]-\text{valued fuzzy quantity}\) (cf. Sect. 2 in [6]) if and only if \( F(0) = 0 \), \( \sup \{ F(x) \mid x \in A \} = F(\sup A) \quad \forall A \subseteq [0, 1] \). Obviously the set \( \mathcal{A}([0, 1]) \) of all \([0, 1]-\text{valued fuzzy quantities}\) coincides with the set of all fuzzy nonnegative, real numbers \( F \) with \( F(1^+)=1 \) (cf. [6]). Moreover \( \mathcal{A}([0, 1]) \) forms a complete lattice; and the unit interval \([0, 1]\) can be embedded into \( \mathcal{A}([0, 1]) \) as follows (cf. Sect. 2):

\[
x \preceq F_x , \quad \text{where } F_x(x) = 1 - x , \quad 0 < x \leq 1 , \quad x = 0 , \quad x = 0.
\]

In particular \( \kappa(t, F) := \sup \{ x \in [0, 1], F(x) \leq 1 - t \} \) can be considered as the degree that the \([0, 1]-\text{valued fuzzy quantity}\) \( F \) is greater than \( t \).

**Notation and Definition.** (a) Let \( \mathbb{K} (\hat{\mathbb{K}}) \) be a nonvoid subset of a Boolean algebra \( \mathbb{B} \), and let \( r_{\mathbb{K}} (s_{\hat{\mathbb{K}}}) \) be a mapping from \( \mathbb{K} \) to \( [0, 1] \) (from \( \hat{\mathbb{K}} \) to \( [0, 1] \)). Then \( r_{\mathbb{K}} \circ s_{\hat{\mathbb{K}}} \) is a mapping from \( \mathbb{K} \cap \hat{\mathbb{K}} \) to \( [0, 1] \) defined as follows

\[
(r_{\mathbb{K}} \circ s_{\hat{\mathbb{K}}})(b) = r_{\mathbb{K}}(b), \quad b \in \mathbb{K} \cap \mathbb{K},
\]

\[
= \min(r_{\mathbb{K}}(b), s_{\hat{\mathbb{K}}}(b)), \quad b \in \mathbb{K} \cap \hat{\mathbb{K}},
\]

\[
= s_{\hat{\mathbb{K}}}(b), \quad b \in \hat{\mathbb{K}} \cap \mathbb{K}.
\]

(b) Let \( J_0(\mathbb{B}) \) be the set of all nonvoid finite subsets of \( \mathbb{B} \). A mapping \( \hat{\lambda}: \bigoplus_{\mathbb{B} \in J_0(\mathbb{B})} [0, 1]^{|H|} \rightarrow [0, 1] \) is called left-continuous if and only if for all \( H \in J_0(\mathbb{B}) \) and for all sequences \( \{r^n_{\mathbb{B}}\}_{n \in \mathbb{N}} \) in \( [0, 1]^{|H|} \) with \( r^n_{\mathbb{B}}(b) \leq r^{n+1}_{\mathbb{B}}(b) \) \( \forall b \in H \), the relation

\[
\lim_{n \to \infty} \hat{\lambda}(r^n_{\mathbb{B}}) = \hat{\lambda}(r^0_{\mathbb{B}}), \quad \text{where } r^0_{\mathbb{B}}(b) = \sup_{n \in \mathbb{N}} r^n_{\mathbb{B}}(b) \quad \forall b \in H
\]

is valid.

**Definition 3.1 (Fuzzy plausibility measures).** Let \( \mathbb{B} \) be an abstract Boolean algebra and \( J_0(\mathbb{B}) \) be the set of all nonvoid finite subsets of \( \mathbb{B} \). A
mapping $\mu: B \to A([0, 1])$ is called a fuzzy plausibility measure if and only if $\mu$ satisfies the following axioms:

(FP1) $\mu(\emptyset) = F_1$, $\mu(\emptyset) = F_0$

(FP2) $\mu(b) \neq F_0 \forall b \neq \emptyset$

(FPL) There exists a left-continuous mapping

$$\lambda: \bigoplus_{b \in J_0(B)} [0, 1]^\mathfrak{B} \to [0, 1] \text{ s.t. } \forall (H, \mathfrak{B}) \in J_0(B) \times J_0(B) \forall r_\mathfrak{B} \in [0, 1]^\mathfrak{B} \forall s_\mathfrak{R} \in [0, 1]^\mathfrak{R} \text{ the subsequent inequality holds:}$$

$$\sum_{i=0}^{\#(\mathfrak{R})} (-1)^i \sum_{\mathfrak{R} \subseteq \mathfrak{R}, \#(\mathfrak{R}) = i} \lambda(r_{\mathfrak{R}} \wedge s_{\mathfrak{R}})$$

$$\times \left[ \mu \left( \bigvee \left( \bigcup_{H \cup \mathfrak{R}} \mathfrak{R} \right) \right) \right] \left( \operatorname{Min}_{b \in H \cup \mathfrak{R}} (r_{\mathfrak{R}} \wedge s_{\mathfrak{R}})(b) \right)$$

$$+ (1 - \lambda(r_{\mathfrak{R}} \wedge s_{\mathfrak{R}}))$$

$$\times \left[ \mu \left( \bigvee \left( \bigcup_{H \cup \mathfrak{R}} \mathfrak{R} \right) \right) \right] \left( \operatorname{Max}_{b \in H \cup \mathfrak{R}} (r_{\mathfrak{R}} \wedge s_{\mathfrak{R}})(b) \right)$$

$$\geq 0,$$

where $s_\mathfrak{R}$ is the restriction of $s_\mathfrak{R}$ to $\mathfrak{R}$.

**Proposition 3.2.** Let $\theta$ an ordinary plausibility measure on $B$ (cf. Sect. 1). Then $\mu: B \to A([0, 1])$ defined by

$$\mu(b) = F_{\theta(b)} \quad \forall b \in B$$

is a fuzzy plausibility measure on $B$.

**Proof.** Since (PL) implies (FPL), the assertion is obvious.

An analogue of Theorem 2.2 is

**Theorem 3.3.** Let $\mu$ be a mapping from a Boolean algebra $B$ to $A([0, 1])$ provided with the properties (FP1) and (FP2). Then the following assertions are equivalent:

(i) $\mu$ is a fuzzy plausibility measure.

(ii) There exists a regular Borel probability measure $v_\mu$ on $[0, 1]^B$ (w.r.t. the product topology) equipped with the subsequent properties

(FR') $v_\mu \{ \omega \in [0, 1]^B \mid \omega(b) \leq l \} = 1 - \mu(b)(l) \forall l, b$

(FPL') $v_\mu \{ \omega \in [0, 1]^B \mid \omega(\emptyset) = 0, \omega(b) = 0, \omega(b \lor c) = \operatorname{Max}(\omega(b), \omega(c)) \forall b, c \in B \} = 1$.

**Proof.** If we replace $\{0, \frac{1}{2}, 1\}$ by $[0, 1]$, then the proof of Theorem 2.2 can be repeated verbatim.
Remark 3.4. (a) A fuzzy realization of a Boolean algebra $\mathbb{B}$ is a mapping $\omega: \mathbb{B} \to [0, 1]$. A fuzzy realization $\omega$ is called a possibility measure on $\mathbb{B}$ (cf. Remark 1.2(c)) if and only if $\omega$ is provided with the properties (P1) and (P0). Referring to Theorem 3.3, we are in the position to associate with every fuzzy plausibility measure $\mu$ a probability measure $\nu_\mu$, which is supported by the set $\mathcal{P}(\mathbb{B})$ of all possibility measures on $\mathbb{B}$. In this context we interpret the real number $1 - \mu(b)(x)$ as the probability that the possibility of $b$ is at least $x$. Moreover $\nu_\mu$ is not uniquely determined by the properties (FR') and (FPL') (cf. Remark 2.3).

(b) For every fuzzy plausibility measure $\mu$ on $\mathbb{B}$ the relation

$$\sum_{i=1}^{n} (-1)^{i-1} \sum_{j_1 \prec \cdots \prec j_n} \mu \left( \bigvee_{i=1}^{j_i} b_i \right) (x) \leq \mu \left( \bigwedge_{i=1}^{n} b_i \right) (x) \quad \forall x$$

is valid; and therewith every fuzzy plausibility measure $\mu$ is isotone—i.e., $\mu$ fulfills the property

$$b \leq c \Rightarrow \mu(c)(x) \leq \mu(b)(x) \quad \forall x \in [0, 1].$$

(c) The set of all fuzzy plausibility measures on an abstract Boolean algebra is convex (cf. Theorem 3.3).

An important nontrivial class of fuzzy plausibility measures consists of the so-called strict positive fuzzy possibility measures specified as follows.

**Definition 3.5.** Let $\mathbb{B}$ be an abstract Boolean algebra. A mapping $\mu: \mathbb{B} \to [0, 1]$ is said to be a strict positive fuzzy possibility measure if and only if $\mu$ satisfies the axioms

- (FP1) $\mu(\top) = F_1$, $\mu(\bot) = F_0$
- (FP2) $\mu(b) \neq F_0 \ \forall b \neq \emptyset$
- (FP0) $[\mu(b \land c)](x) = \min (\mu(b)(x), \mu(c)(x)) \ \forall x \in [0, 1]$.

**Corollary 3.6.** Every strict positive fuzzy possibility measure $\mu$ is a fuzzy plausibility measure.

**Proof.** Let $\mu$ be a strict positive fuzzy possibility measure. It is well known that there exists an unique regular Borel probability measure $v_\mu$ on $[0, 1]^\mathbb{B}$ provided with the property

$$\nu_\mu \left( \bigcap_{b \in \mathbb{B}} \{ \omega \in [0, 1]^\mathbb{B}, \omega(b) < x_b \} \right) = \min_{b \in \mathbb{B}} \mu(b)(x_b) \quad \forall \mathbb{B} \in J_0(\mathbb{B}).$$

From (FP0) we obtain

$$\nu_\mu \left( \bigcap_{b \in \mathbb{B}} \{ \omega, \omega(b) < x \} \right) \triangleq \left\{ \omega, \omega \left( \bigvee_{b \in \mathbb{B}} \right) < x \right\} = 0 \quad \forall \mathbb{B} \in J_0(\mathbb{B});$$
and therewith \( \nu_\mu \) satisfies (FR') and (FPL'). By virtue of Theorem 3.3, \( \mu \) is a plausibility measure.

**Remark 3.7.** Since every \([0, 1]\)-valued fuzzy quantity is monotone, every fuzzy plausibility measure \( \mu \) equipped with the property

\[
[\mu(b \lor c)](x) + [\mu(b \land c)](x) = \mu(b)(x) + \mu(c)(x) \quad \text{(Additivity)}
\]

is an ordinary (finite additive) probability measure—i.e., there exists a probability measure \( \theta \) on \( \mathcal{B} \) such that \( \mu(b) = F_{\theta(b)} \forall b \in \mathcal{B} \). Therefore the class of "fuzzy probability measures" coincides with the class of ordinary probability measures.

**Remark 3.8 (Entropy of fuzzy plausibility measures).** Let \( \mu \) be a fuzzy plausibility measure on \( \mathcal{B} \), and let \( \nu_\mu \) be a regular Borel probability measure on \([0, 1]^3\) satisfying the conditions (FR') and (FPL'):

(a) (Fuzzy entropy) The mapping \( e(\mathbb{B}, \mu, \alpha): \mathcal{P}(\mathcal{B}) \to [0, +\infty] \) defined by

\[
[e(\mathbb{B}, \mu, \alpha)](\omega) = \sup \{-\ln[\mu(b)](1 - \alpha) \mid b \in \mathbb{B}, \omega(b) < \alpha\}
\]

is Borel measurable. The extended real number \( [e(\mathbb{B}, \mu, \alpha)](\omega) \) can be interpreted as the *information at the level* \( \alpha \) represented by the fuzzy realization \( \omega \) w.r.t. \( \mu \); and the "average of information at the level \( \alpha \)" is given by

\[
[E(\mathbb{B}, \mu)](\alpha) = \int_{\mathcal{P}(\mathcal{B})} e(\mathbb{B}, \mu, \alpha) \, dv_\mu \quad \forall \alpha \in [0, 1]. \quad (3.2)
\]

From (FPL') we conclude that \( [E(\mathbb{B}, \mu)](\alpha) \) is independent of \( \nu_\mu \)—i.e., \( [E(\mathbb{B}, \mu)](\alpha) \) depends on \( \mu \) only. Therefore the quantity \( E(\mathbb{B}, \mu) \) defined by (3.2) is called the *fuzzy entropy* of \( (\mathbb{B}, \mu) \). In particular, by virtue of the Beppo-Levi lemma the fuzzy entropy is a \([0, 1]\]-valued, \([0, +\infty]\]-fuzzy quantity (cf. [6]).

(b) (Set of all entropies w.r.t. fuzzy plausibility measures) The mapping \( e(\mathbb{B}, \mu): [0, 1] \times \mathcal{P}(\mathcal{B}) \to [0, +\infty] \) defined by

\[
[e(\mathbb{B}, \mu)](\alpha, \omega) = \sup_{b \in \mathbb{B}} \{-\ln[\mu(b)](\alpha) \cdot (1 - \omega(b))\} \quad (3.3)
\]

is measurable. From (3.3) we derive the "average of information" as

\[
\bar{e}_{\alpha}(\mathbb{B}, \mu) = \int_0^1 \left( \int_{\mathcal{P}(\mathbb{B})} [e(\mathbb{B}, \mu)](\alpha, \omega) \, dv_\mu \right) \, d\alpha.
\]
Obviously \( \tilde{\varepsilon}_{\mu}(\mathbb{B}, \mu) \) is not independent of the Borel measure \( \nu_{\mu} \) associated with \( \mu \). If \( \mathcal{M}_{\mu} \) is the set of all regular Borel probability measures on \([0, 1]^\mathbb{B}\) provided with \((\text{FR})'\) and \((\text{FPL})'\), then \( \Sigma_{\xi}(\mathbb{B}, \mu) := \{ \tilde{\varepsilon}_{\nu}(\mathbb{B}, \mu), \nu_{\mu} \in \mathcal{M}_{\mu} \} \) is said to be the set of all entropies of \((\mathbb{B}, \mu)\).

**Remark 3.9.** Replacing \( \{0, \frac{1}{2}, 1\} \) by the unit interval \([0, 1]\) we can repeat verbatim Remark 2.6, and therewith we obtain a natural concept of mean values with respect to fuzzy plausibility measures.

### 4. A General Construction of Fuzzy Plausibility Measures

Let \( \Omega = \{1, 2, \ldots, n\} \) be a finite, nonvoid set, \( \mathcal{G} = \mathcal{P}(\Omega) \) be the power set of \( \Omega \) and \( \lambda: \mathcal{G} \to [0, 1] \) be a strictly positive probability measure. Further let \( X \) be an arbitrary, nonvoid set and \( \{ f_1, \ldots, f_n \} \) be a family of fuzzy subsets of \( X \) provided with the subsequent properties

\[
\sup_{x \in X} f_i(x) = 1 \quad \forall i = 1, \ldots, n \quad (\text{normalization})
\]

\[
0 < \max_{x \in X} \{ f_i(x) \} \cup \{ i = 1, \ldots, n \} \quad \forall X \in X. \quad (\text{covering property})
\]

We introduce a mapping \( \mu: \mathcal{P}(X) \to A([0, 1]), \)

\[
\mu(A)(x) = \lambda(\{ i \in \{1, \ldots, n\} | \sup_{x \in A} f_i(x) < \alpha \}) \quad \forall A \in \mathcal{P}(X).
\]

By virtue of Theorem 3.3, \( \mu \) is a fuzzy plausibility measure; and in addition, if \( \{ f_1, \ldots, f_n \} \) is a chain—e.g., \( f_1 \leq f_2 \leq \cdots \leq f_n \)—\( \mu \) is a fuzzy possibility measure (cf. example 1.3).

An explanation of the above situation is given in the following

**Example.** An urn \( \Omega \) contains exactly three balls coloured white, red, and green, respectively. Further let \( X \) be the set consisting of the colours white, red, and green. A ball \( B \) is drawn from the urn at random. In order to obtain a decision concerning the colour of \( B \), this experiment is observed by two persons \( P_1 \) and \( P_2 \):

(a) (crisp case) **Decision.** A subset \( A \) of \( X \) (i.e., \( A \in \mathcal{P}(X) \)) occurs if and only if \( P_1 \) or \( P_2 \) asserts that the colour of \( B \) is contained in \( A \).

We ask the question: *How great is the uncertainty or more precisely the plausibility that the colour "red"—i.e., \( A = \{ \text{red} \} \)—occurs?*

**Solution.** Let \( B_W \) (\( B_R, B_G \)) be the white (red, green) ball, and let \( W(R, G) \) be the white (red, green) colour; i.e., \( \Omega = \{ B_W, B_R, B_G \}, X = \{ W, R, G \} \). Since the balls are well shaken-up, we assume: \( \lambda(\{ B_W \}) = \)

---

*This section appeared in "Fuzzy Sets and Decision Analysis," TIMS 20, pp. 93–95, North-Holland, 1984. Reproduced by permission of the publisher.*
For each outcome $B$ of $\Omega$ the above decision determines a crisp subset $f_B$ of $X$ as follows:

**Case 1.** $P_1$ and $P_2$ are sound; then we obtain: $f_{Bw} = \{W\}$, $f_{Br} = \{R\}$, $f_{Bg} = \{G\}$. Hence the (crisp) plausibility $\mu(A)$ that the subset $A$ of $X$ occurs, is given by the formula

$$\mu(A) = \lambda\{B \in \Omega, f_B \cap A \neq \emptyset\} \quad \forall A \in \mathcal{P}(X).$$

Obviously $\mu$ is a probability measure on $\mathcal{P}(X)$ and, as usual, the probability that the colour "red" occurs is equal to $\frac{2}{3}$.

**Case 2.** $P_1$ is sound, but $P_2$ is unsound, e.g., $P_2$ has a visual error, such that $P_2$ recognizes white as white, but red as green and, vice versa, green as red. Then we obtain

$$f_{Bw} = \{W\}, \quad f_{Br} = f_{Bg} = \{R, G\}.$$  

Again the (crisp) plausibility $\mu(A)$ that the subset $A$ of $X$ occurs is given by the formula

$$\mu(A) = \lambda\{B \in \Omega, f_B \cap A \neq \emptyset\}$$

and contrary to Case 1 the plausibility that "red" occurs, is equal to $\frac{2}{3}$.

(b) (noncrisp case) $P_1$ is sound, and $P_2$ is colour-blind, e.g., $P_2$ recognizes white as white, but red and green as grey. Therefore in general, $P_2$ is not able to bring about a crisp decision concerning the colour of the ball $B$.

**Fuzzy decision.** (1) The degree that a subset $A$ of $X$ occurs, is 1 (i.e., $A$ occurs) if and only if $P_1$ or $P_2$ asserts that the colour of $B$ is contained in $A$. The degree that a subset $A$ of $X$ occurs, is 0 (i.e., $A$ does not occur) if and only if $P_1$ and $P_2$ asserts that the colour of $B$ is not contained in $A$. (2) The occurrence of a subset $A$ of $X$ is *undecidable*, i.e., the degree that $A$ occurs, is $\frac{1}{2}$, if and only if $P_1$ asserts that the colour of $B$ is *not* contained in $A$, and $P_2$ asserts that he is unable to decide whether the colour of $B$ is or is not contained in $A$.

We ask the question: How great is the uncertainty or, more precisely, the plausibility that the degree of the occurrence of "red" is 1 (or at least $\frac{1}{2}$)?

**Solution.** For each outcome $B$ of $\Omega$ the above fuzzy decision determines a fuzzy subset $f_B$ of $X$ as follows:

$$f_{Bw}(W) = 1, \quad f_{Bw}(R) = f_{Bw}(G) = 0,$$
$$f_{Br}(W) = 0, \quad f_{Br}(R) = 1, \quad f_{Br}(G) = \frac{1}{2},$$
$$f_{Bg}(W) = 0, \quad f_{Bg}(R) = \frac{1}{2}, \quad f_{Bg}(G) = 1.$$
Therefore the fuzzy plausibility $\mu(A)$ of $A$—i.e., the plausibility that the subset $A$ of $X$ occurs to a certain degree—is given by the following formula

$$\mu(A)(x) = \lambda \left\{ B \in \Omega, \max_{x \in A} f_B(x) < x \right\} \quad \forall x \in [0, 1].$$

In particular $\frac{1}{3} (\frac{4}{7})$ is the plausibility or more precisely the probability that the degree of the occurrence of “red” is 1 (at least $\frac{1}{3}$).

Remark. The above construction of fuzzy plausibility measures shows that fuzzy plausibility measures can be used as a natural, mathematical tool in problems involving some specific kind of subjectivity. Obviously the foregoing example deals with two fundamentally different kinds of subjectivity caused by the illness of $P_2$, the first kind can still be described by a crisp model (cf. Case 2), while the second kind of subjectivity requires a fuzzy model, which leads direct to the concept of fuzzy plausibility measures.

5. CONCLUSIONS

The axioms of fuzzy plausibility measures are certain rules, according to which the two fundamental notions of uncertainty, probability, and possibility, can be composed. Referring to the work of Feron [2] fuzzy plausibility measures $\mu$ on an abstract Boolean algebra $\mathbb{B}$ can be considered as the probability distributions of a specific class of random fuzzy subsets of $\mathbb{B}$ (cf. Theorem 1.4, 2.2, and 3.3). Moreover, the concept of mean values and entropies with respect to $(\mathbb{B}, \mu)$ shows that an information theory based upon fuzzy plausibility measures is possible. Finally a concrete example (cf. Sect. 4) emphasizes the applicability of fuzzy plausibility measures to real-world problems.

REFERENCES