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A Hadamard Inequality for the Second Immanant*

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Denote by $\mathscr{H}_n$ the convex cone of n-by-n positive semi-definite hermitian matrices. Let $\bar{d}_2$ be the normalized immanant afforded by S_n and the partition $(2, 1^{n-2})$. Then $h(A)+(\Delta-1)\det(A) \geqslant \Delta \bar{d}_2(A)$, $A \in \mathscr{H}_n$, where $h(A)$ is the main diagonal product of A and Δ is approximately $(n-1)/e$.

Denote by $\mathscr{H}_n$ the cone of positive semidefinite hermitian n-by-n matrices. In 1893, J. Hadamard [3] proved that $h(A) \geqslant \det(A)$, for all $A \in \mathscr{H}_n$, where $h(A)$ is the product of the main diagonal entries of A. Let χ_2 be the irreducible character of the symmetric group S_n corresponding to the partition $(2, 1, ..., 1)$. The *second immanant*, d_2, is defined on the n-by-n matrices by

$$d_2(A) = \sum_{\sigma \in S_n} \chi_2(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)},$$

where $A = (a_{ij})$. When $n=2$, $d_2(A) = \operatorname{per}(A)$, the *permanent* of A. In general, χ_2 is a character of degree $n-1$, and it is convenient to define the *normalized* second immanent, $\bar{d}_2$, by $\bar{d}_2(A) = d_2(A)/(n-1)$. In 1918, I. Schur [9] proved (among a great many things) that

$$\bar{d}_2(A) \geqslant \det(A),\ A \in \mathscr{H}_n. \tag{1}$$

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In a recent paper, [2], the present authors showed that

$$h(A) \geqslant \bar{d}_2(A), \qquad A \in \mathscr{H}_n,\ n \geqslant 4, \tag{2}$$

a result which supports the permanental dominance conjecture [4]. It is the purpose of this note to present an improvement of (2) which tends to show that (1) is much tighter than (2) for large values of n.

Before proceeding, we introduce some notation. Define

$$\Delta(n) = \frac{n}{(1 + 1/n)^n},$$

noting that $\Delta(n) \doteq n/e$ for large n. Denote by P_n the n-by-n matrix each of whose diagonal entries is 1 and each of whose off diagonal entries is $-1/(n-1)$.

Suppose $n \geqslant 2$.

THEOREM 1. *Let $A \in \mathscr{H}_n$ and let $\Delta = \Delta(n-1)$. Then*

$$h(A) + (\Delta - 1)\det(A) \geqslant \Delta\, \bar{d}_2(A), \tag{3}$$

with equality if and only if A is diagonal, A has a zero row (and column), or A is diagonally congruent to P_n.

Note that (3) may be rewritten as

$$h(A) \geqslant \bar{d}_2(A) + (\Delta - 1)(\bar{d}_2(A) - \det(A)), \tag{3$'$}$$

which, in view of (1), improves (2) as long as $\Delta > 1$. (Indeed, for $n \geqslant 2$, $\Delta(n-1)$ is an increasing function of n which first exceeds 1 when $n = 4$.)

Another way to write (3) is

$$h(A) \geqslant \Delta(\bar{d}_2(A) - \det(A)) + \det(A). \tag{3*}$$

In this form, one has an improvement of Hadamard's Inequality. Note, also, that the theorem remains valid if Δ is any number less than $\Delta(n-1)$. If, for example, $n \geqslant 6$, then $\Delta(n-1) > 2$. Therefore, for $A \in \mathscr{H}_n$ and $n \geqslant 6$,

$$h(A) + \det(A) \geqslant 2\bar{d}_2(A). \tag{4}$$

Similarly, for $A \in \mathscr{H}_n$,

$$h(A) + 2\det(A) \geqslant 3\bar{d}_2(A), \qquad n \geqslant 9;$$

$$h(A) + 3\det(A) \geqslant 4\bar{d}_2(A), \qquad n \geqslant 12;$$

$$h(A) + 4\det(A) \geqslant 5\bar{d}_2(A), \qquad n \geqslant 15;$$

$$h(A) + 5\det(A) \geqslant 6\bar{d}_2(A), \qquad n \geqslant 17;$$

etc. It follows from (4) and the Hadamard Theorem for Permanents [5] that

$$\operatorname{alt}(A) \geqslant \bar{d}_2(A), \qquad A \in \mathscr{H}_n,\ n \geqslant 6,$$

where alt(A) is the *generalized matrix function*

$$\operatorname{alt}(A) = \sum_{\sigma \in A_n} \prod_{t=1}^{n} a_{t\sigma(t)} = \tfrac{1}{2}(\operatorname{per}(A) + \det(A)).$$

Proof of Theorem 1. It is known (see, e.g., [8]) that

$$d_2(A) = \sum_{t=1}^{n} a_{tt} \det(A(t)) - \det(A), \tag{5}$$

where $A(t)$ is the $(n-1)$-by-$(n-1)$ principal submatrix of $A = (a_{ij})$ obtained by deleting row and column t. If $A \in \mathscr{H}_n$ has a zero on the main diagonal, then both sides of (3) equal 0, accounting for one case of equality. Otherwise, since both sides of (3) are similarly affected by a diagonal congruence, we may assume that A is a *correlation matrix*, i.e., $a_{11} = a_{22} = \cdots = a_{nn} = 1$. Putting (5) into (3) under these conditions transforms the desired inequality into

$$n - 1 \geqslant \Delta \sum_{t=1}^{n} \det(A(t)) - (n\Delta - n + 1) \det(A). \tag{6}$$

Note that (6) can be written in terms of elementary symmetric functions of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, the eigenvalues of A:

$$n - 1 \geqslant \Delta E_{n-1}(\lambda) - (n\Delta - n + 1) E_n(\lambda). \tag{7}$$

Our problem now is to maximize the right-hand side of (7) subject to the side conditions $\lambda_1, \lambda_2, ..., \lambda_n \geqslant 0$ and $\lambda_1 + \cdots + \lambda_n = n$. By the method of Lagrange multipliers, the maximum will occur either on the boundary (where some $\lambda_i = 0$) or at a critical value (where $\lambda_1 = \lambda_2 = \cdots = \lambda_n$). (See, e.g., [1].) This means we have two cases to consider:

Case 1. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then their common value is 1 (and $A = I_n$). In this case, the right-hand side of (7) becomes $\Delta n - (n\Delta - n + 1) = n - 1$, our first case of equality. (A nonsingular matrix in $\mathscr{H}_n$ is diagonally congruent to I_n if and only if it is diagonal.)

Case 2. We now suppose some $\lambda_i = 0$. Since we are dealing with symmetric functions, there is no loss of generality if we assume $\lambda_n = 0$. Let

$\hat{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_{n-1})$. Then our problem is to maximize the function $\Delta E_{n-1}(\hat{\lambda})$ subject to the side conditions $\lambda_1, \lambda_2, ..., \lambda_{n-1} \geqslant 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} = n$. We may argue as before or simply recall that the elementary symmetric functions are "Schur concave" [6]. In any case, we see that $E_{n-1}(\hat{\lambda})$ is maximized when $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = n/(n-1)$, in which case $\Delta E_{n-1}(\hat{\lambda}) = n-1$ by our choice of Δ. It remains to establish the third case of equality. This happens when our original matrix is diagonally congruent to a correlation matrix C with spectrum $\lambda_1 = \cdots = \lambda_{n-1} = n/n-1$ and $\lambda_n = 0$. In this case $C = (n/(n-1))I_n - xx^*$, where x is a complex n-tuple and xx^* a rank 1 positive semidefinite matrix. Since C is a correlation matrix, $|x_1|^2 = \cdots = |x_n|^2 = 1/(n-1)$. From this we can see that xx^* is unitarily diagonally similar to J, that C is unitarily diagonally similar to P_n, and that the original matrix is diagonally congruent to P_n. ∎

It was conjectured in [7] that the "single hook" immanants satisfy the following ordering for $A \in \mathscr{H}_n$:

$$\operatorname{per}(A) = \bar{d}_n(A) \geqslant \bar{d}_{n-1}(A) \geqslant \cdots \geqslant \bar{d}_2(A) \geqslant \bar{d}_1(A) = \det(A).$$

Our result shows that $\bar{d}_2(A)$ is much closer to $\det(A)$ than to $\operatorname{per}(A)$, which tends to suggest that a counterexample to the conjectured inequalities probably will not involve $\bar{d}_2$.

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