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Integrals for (dual) quasi-Hopf algebras. Applications [☆]

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Abstract

A classical result in the theory of Hopf algebras concerns the uniqueness and existence of integrals: for an arbitrary Hopf algebra, the integral space has dimension $\leqslant 1$, and for a finite-dimensional Hopf algebra, this dimension is exactly one. We generalize these results to quasi-Hopf algebras and dual quasi-Hopf algebras. In particular, it will follow that the bijectivity of the antipode follows from the other axioms of a finite-dimensional quasi-Hopf algebra. We give a new version of the Fundamental Theorem for quasi-Hopf algebras. We show that a dual quasi-Hopf algebra is co-Frobenius if and only if it has a non-zero integral. In this case, the space of left or right integrals has dimension one.

0. Introduction

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfel'd in [8], in connection with the Knizhnik–Zamolodchikov system of partial differential equations, cf. [12]. From a categorical point of view, the notion is not so different from classical bialgebras: we consider an algebra H, and we want to make the category of H-modules, equipped with the tensor product of vector spaces, into a monoidal category. If we require that the associativity constraint is the natural associativity condition for vector spaces, then

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we obtain a bialgebra structure on H, in general, we obtain a quasi-bialgebra structure, that is, we have a comultiplication and a counit on H, where the comultiplication is not necessarily coassociative, but only quasi-coassociative.

Of course the theory of quasi-bialgebras and quasi-Hopf algebras is technically more complicated than the classical Hopf algebra theory. A more conceptual difference however, is the fact that the definition of a bialgebra is self-dual, and this symmetry is broken when we pass to quasi-bialgebras. As a consequence, we do not have the notion of comodule or Hopf module over a quasi-Hopf algebra, and results in Hopf algebras that depend on these notions cannot be generalized in a straightforward way. For instance, the classical proof of the uniqueness and existence of integral is based on the Fundamental Theorem for Hopf modules [21].

Hausser and Nill [11] proved that a finite-dimensional quasi-Hopf algebra is a Frobenius algebra, and has a one-dimensional integral space. Independently, Panaite and Van Oystaeyen [16] proved the existence of integrals for finite-dimensional quasi-Hopf algebras, using the approach developed in [23], without using quasi-Hopf bimodules.

For a finite-dimensional Hopf algebra H, it follows from the Fundamental Theorem that $\int_l^H \otimes H^*$ and H are isomorphic as Hopf modules. In Section 2, we will see that the isomorphism survives as a left H-linear isomorphism in the case of a finite-dimensional quasi-Hopf algebra. The method of proof is quite different from the classical one: the isomorphism is constructed explicitly, using the projection of H onto the integral space constructed in [16]. In Drinfel'd's original definition [8], the antipode of a quasi-Hopf algebra is required to be bijective. Actually our proof of $\int_l^H \otimes H^* \cong H$ does not use this bijectivity, and has as a consequence that, for a finite-dimensional quasi-Hopf algebra, the bijectivity of the antipode follows from the other axioms; another consequence is that the integral space is one-dimensional. In a recent preprint [19], Schauenburg gave a different proof of the fact that the antipode of a finite-dimensional quasi-Hopf algebra is bijective.

The infinite-dimensional case is treated as well. We show that a quasi-Hopf algebra (without the assumption that the antipode is bijective) is finite-dimensional if and only if the antipode is bijective and the integral space is non-zero. The integral space of an infinite-dimensional quasi-Hopf algebra with bijective antipode is zero. A semisimple Hopf algebra with bijective antipode is finite-dimensional. Hausser and Nill [11] also introduced cointegrals on a finite-dimensional quasi-Hopf algebra; these cointegrals are elements of the dual space H^*, and, using a Structure Theorem for quasi-Hopf bimodules, Hausser and Nill prove that the space of cointegrals $\mathcal{L}$ is one-dimensional, and that all non-zero integrals are nondegenerate. In Section 3, we further investigate cointegrals. In [11], it is asked whether there is a connection between the projection of H onto the space of integrals from [16], and the projection of H^* onto the space of cointegrals, introduced in [11]. This is done in Lemma 3.2, and, as an application, we give some characterizations of cointegrals, see Proposition 3.4.

In the second part of Section 3, we propose an alternative definition of the space of coinvariants of a quasi-Hopf bimodule. This alternative space of coinvariants is isomorphic to the Hausser–Nill space of coinvariants, and can be used to give a second version of the Structure Theorem. Our alternative has nevertheless two advantages, compared to the Hausser–Nill approach: first, it is invariant under the adjoint action (cf. Lemma 3.6). Secondly, in the finite-dimensional case, it gives rise to an alternative definition of cointegral: we take the alternative coinvariants of H^*. If we write down this formula explicitly, we obtain a formula that still makes sense in the infinite-dimensional case, so we obtain a plausible definition for cointegrals in the infinite-dimensional case.

As we have already pointed out, the definition of quasi-Hopf algebra is not self-dual. Actually, we can introduce dual quasi-Hopf algebras, these are coalgebras, with a multiplication that is not associative, but only quasi-associative. In Section 4, we introduce integrals in dual quasi-Hopf algebras. We were able to prove that the rational dual of a dual quasi-Hopf algebra A is isomorphic as a comodule to the tensor product of A itself and the integral space. This generalizes the classical statement for Hopf algebras (see [21]), but, again, we have to give a direct proof, and cannot deduce the statement from a Structure Theorem. As in the classical case, it then follows immediately that the integral space is zero if and only if the rational dual is zero. Also we can use the integrals to investigate properties of a dual quasi-Hopf algebra A as a coalgebra (Theorem 4.5). The existence of a non-zero integral is equivalent to A being a co-Frobenius coalgebra, a QcF coalgebra, or a left semiperfect coalgebra. Moreover, for a dual quasi-Hopf algebra, all these notions are left–right symmetric. Furthermore the existence of a non-zero integral is equivalent to A being a generator or a projective object in the category of (left or right) comodules. As a first application of this coalgebraic viewpoint, we find that a dual quasi-Hopf subalgebra of a dual quasi-Hopf algebra with non-zero integral has non-zero integrals. Secondly, it follows that non-zero integrals are unique up to multiplication by a scalar. Also we can give the connection between left and right integrals (Proposition 4.9), and this generalizes [3, Proposition 1.3]. We were able to prove that the antipode of a dual quasi-Hopf algebra with a non-zero integral is injective, but it remains open if it is also surjective, as it is the case for a classical Hopf algebra, see [17]. Our final result is Maschke's Theorem for dual quasi-Hopf algebras (Theorem 4.10), stating that a dual quasi-Hopf algebra is cosemisimple if and only there exists an integral T such that $T(1) = 1$.

When we pass from bialgebras and Hopf algebras to quasi-bialgebras and quasi-Hopf algebras, the appearance of the reassociator and the elements α and β in the definition of the antipode, considerably increase the complexity of computations and proofs. This observation is not new, other authors who have been working on quasi-Hopf algebras experienced this before us. However, the philosophy is basically the same as in the case of usual bialgebras: the idea is to make the category of A-modules into a monoidal category. Recently, Schauenburg proposed an alternative approach to proving results on quasi-bialgebras, exploiting the categorical ideas behind quasi-bialgebras, and replacing the computational arguments using the Sweedler notation by conceptual arguments (see [19] for detail). At this moment is not clear to us whether Schauenburg 's ideas can be used to give alternative and/or more transparent proofs of the results in this paper.

1. Preliminaries

We work over a commutative field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfel'd [8], a quasi-bialgebra is a four-tuple $(H, \Delta, \varepsilon, \Phi)$ where H is an associative algebra with unit, Φ is an invertible element in

$H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

$$(\mathrm{id} \otimes \Delta)\big(\Delta(h)\big) = \Phi(\Delta \otimes \mathrm{id})\big(\Delta(h)\big)\Phi^{-1}, \tag{1.1}$$

$$(\mathrm{id} \otimes \varepsilon)\big(\Delta(h)\big) = h \otimes 1, \qquad (\varepsilon \otimes \mathrm{id})\big(\Delta(h)\big) = 1 \otimes h, \tag{1.2}$$

for all $h \in H$, and Φ has to be a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(\Phi \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi), \tag{1.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1 \otimes 1. \tag{1.4}$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras we denote $\Delta(h) = \sum h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention

$$(\Delta \otimes \mathrm{id})\big(\Delta(h)\big) = \sum h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (\mathrm{id} \otimes \Delta)\big(\Delta(h)\big) = \sum h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely:

$$\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 = \cdots,$$
$$\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum v^1 \otimes v^2 \otimes v^3 = \cdots.$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-automorphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{1.5}$$

$$\sum X^1\beta S\big(X^2\big)\alpha X^3 = 1 \quad \text{and} \quad \sum S\big(x^1\big)\alpha x^2\beta S\big(x^3\big) = 1. \tag{1.6}$$

For a quasi-Hopf algebra the antipode is determined uniquely up to a transformation $\alpha \mapsto U\alpha$, $\beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, where $U \in H$ is invertible. The axioms for a quasi-Hopf algebra imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β, we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$. The identities (1.2), (1.3) and (1.4) also imply that

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1. \tag{1.7}$$

Next we recall that the definition of a quasi-Hopf algebra is "twist coinvariant" in the following sense. An invertible element $F \in H \otimes H$ is called a *gauge transformation* or *twist* if $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$. If H is a quasi-Hopf algebra and $F = \sum F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = \sum G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra H_F by keeping the multiplication, unit, counit and antipode of H and replacing the comultiplication, antipode and the elements α and β by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \tag{1.8}$$

$$\Phi_F = (1\otimes F)(\mathrm{id}\otimes\Delta)(F)\Phi(\Delta\otimes\mathrm{id})\big(F^{-1}\big)\big(F^{-1}\otimes 1\big), \tag{1.9}$$

$$\alpha_F = \sum S\big(G^1\big)\alpha G^2, \qquad \beta_F = \sum F^1\beta S\big(F^2\big). \tag{1.10}$$

It is well known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H\otimes H$ such that

$$f\Delta\big(S(h)\big)f^{-1} = \sum (S\otimes S)\big(\Delta^{op}(h)\big), \quad \text{for all } h\in H. \tag{1.11}$$

f can be computed explicitly. First set

$$\sum A^1\otimes A^2\otimes A^3\otimes A^4 = (\Phi\otimes 1)(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\big(\Phi^{-1}\big), \tag{1.12}$$

$$\sum B^1\otimes B^2\otimes B^3\otimes B^4 = (\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi)\big(\Phi^{-1}\otimes 1\big), \tag{1.13}$$

and then define $\gamma,\delta\in H\otimes H$ by

$$\gamma = \sum S\big(A^2\big)\alpha A^3\otimes S\big(A^1\big)\alpha A^4 \quad \text{and} \quad \delta = \sum B^2\beta S\big(B^4\big)\otimes B^2\beta S\big(B^3\big). \tag{1.14}$$

f and f^{-1} are then given by the formulas

$$f = \sum (S\otimes S)\big(\Delta^{op}\big(x^1\big)\big)\gamma\Delta\big(x^2\beta S\big(x^3\big)\big), \tag{1.15}$$

$$f^{-1} = \sum \Delta\big(S\big(x^1\big)\alpha x^2\big)\delta(S\otimes S)\big(\Delta^{\mathrm{op}}\big(x^3\big)\big), \tag{1.16}$$

where $\Delta^{\mathrm{op}}(h) = \sum h_2\otimes h_1$. f satisfies the following relations:

$$f\Delta(\alpha) = \gamma, \qquad \Delta(\beta)f^{-1} = \delta. \tag{1.17}$$

Furthermore, the corresponding twisted reassociator (see (1.9)) is given by

$$\Phi_f = \sum (S\otimes S\otimes S)\big(X^3\otimes X^2\otimes X^1\big). \tag{1.18}$$

In a Hopf algebra H, we obviously have the identity

$$\sum h_1\otimes h_2 S(h_3) = h\otimes 1, \quad \text{for all } h\in H.$$

We will need the generalization of this formula to quasi-Hopf algebras. Following [9,10], we define

$$p_R = \sum p^1 \otimes p^2 = \sum x^1 \otimes x^2 \beta S\bigl(x^3\bigr),$$
$$q_R = \sum q^1 \otimes q^2 = \sum X^1 \otimes S^{-1}\bigl(\alpha X^3\bigr) X^2. \tag{1.19}$$

For all $h \in H$, we then have

$$\sum \Delta(h_1) p_R\bigl[1 \otimes S(h_2)\bigr] = p_R[h \otimes 1],$$
$$\sum \bigl[1 \otimes S^{-1}(h_2)\bigr] q_R \Delta(h_1) = (h \otimes 1) q_R, \tag{1.20}$$

and

$$\sum \Delta\bigl(q^1\bigr) p_R\bigl[1 \otimes S\bigl(q^2\bigr)\bigr] = 1 \otimes 1,$$
$$\sum \bigl[1 \otimes S^{-1}\bigl(p^2\bigr)\bigr] q_R \Delta\bigl(p^1\bigr) = 1 \otimes 1, \tag{1.21}$$

$$(q_R \otimes 1)(\Delta \otimes \mathrm{id})(q_R)\Phi^{-1}$$
$$= \sum \bigl[1 \otimes S^{-1}\bigl(X^3\bigr) \otimes S^{-1}\bigl(X^2\bigr)\bigr]\bigl[1 \otimes S^{-1}\bigl(f^2\bigr) \otimes S^{-1}\bigl(f^1\bigr)\bigr](\mathrm{id} \otimes \Delta)\bigl(q_R \Delta\bigl(X^1\bigr)\bigr), \tag{1.22}$$
$$\Phi(\Delta \otimes \mathrm{id})(p_R)(p_R \otimes \mathrm{id})$$
$$= \sum (\mathrm{id} \otimes \Delta)\bigl(\Delta\bigl(x^1\bigr) p_R\bigr)\bigl(1 \otimes f^{-1}\bigr)\bigl(1 \otimes S\bigl(x^3\bigr) \otimes S\bigl(x^2\bigr)\bigr), \tag{1.23}$$

where $f = \sum f^1 \otimes f^2$ is the twist defined in (1.15). Note that the formulas (1.16)–(1.23) (except (1.22) and the second part of (1.20), (1.21)) can be proved without using the bijectivity of the antipode S.

2. Integrals in quasi-Hopf algebras

Let H be a finite-dimensional quasi-Hopf algebra with an antipode S. In [11], it is shown that H is a Frobenius algebra and, as a consequence, the space of left (right) integrals in H is one-dimensional. The proof in [11] relies on the fact that the antipode S is bijective. Using different arguments independent of the bijectivity of the antipode, we will give another proof of the existence and uniqueness of integrals in H. In fact we will prove that, in the definition of finite-dimensional quasi-Hopf algebra, the bijectivity of the antipode can be dropped, in other words, the bijectivity of the antipode follows from the other axioms. This will generalize a similar result for Hopf algebras, see [13].

Let us make our terminology consistent: by a quasi-Hopf algebra, we will mean a quasi-Hopf algebra as defined in Section 1, but without the assumption that the antipode is bijective. If the antipode is bijective, then we will say that we have a quasi-Hopf algebra in the sense of Drinfel'd. With this convention, our main result is the following: a finite-dimensional quasi-Hopf algebra is a quasi-Hopf algebra in the sense of Drinfel'd.

Recall that $t \in H$ is called a left (respectively right) integral in H if $ht = \varepsilon(h)t$ (respectively $th = \varepsilon(h)t$), $\forall h \in H$. We denote by $\int_l^H$ ($\int_r^H$) the space of left (right) integrals in H. If there exists a non-zero left integral in H which is at the same time a right integral, then H is called unimodular.

In [13], the Fundamental Theorem is used to prove the existence and uniqueness of integrals, and then the bijectivity of the antipode follows. In the quasi-Hopf algebra case, this approach will not work, since we cannot define Hopf modules. Van Daele [23] gave a short and direct proof of the existence and uniqueness of integrals in a finite-dimensional Hopf algebra, and Panaite and Van Oystaeyen [16] generalized Van Daele's argument, proving the existence of left integrals in finite-dimensional quasi-Hopf algebras. More precisely, let H be a finite-dimensional quasi-Hopf algebra, $\{e_i\}_{i=\overline{1,n}}$, a basis of H and $\{e^i\}_{i=\overline{1,n}}$ the dual basis of H^*. Following [16], we define

$$P(h) = \sum_{i=1}^{n} \langle e^i, \beta S\big(S(X^2(e_i)_2)\alpha X^3\big)h\rangle X^1(e_i)_1, \tag{2.1}$$

for all $h \in H$. Then one can show that $P(h) \in \int_l^H$, $\forall h \in H$ and $\sum_{i=1}^n \langle e^i, S(P(e_i)\beta)\rangle = \varepsilon(\beta) = 1$. It follows that at least one of the $P(e_i) \neq 0$, and $\int_l^H \neq 0$.

In order to prove the uniqueness of integrals for finite-dimensional quasi-Hopf algebras we need the following lemma.

Lemma 2.1. *Let t be a left integral in a quasi-Hopf algebra H. Then for all $h \in H$*

$$\sum hX^1t_1 \otimes S\big(X^2t_2\big)\alpha X^3 = \sum X^1t_1 \otimes S\big(X^2t_2\big)\alpha X^3h \tag{2.2}$$

and

$$\sum t_1 \otimes S(t_2) = \sum X^1t_1 \otimes S\big(X^2t_2\big)\alpha X^3\beta = \sum \beta X^1t_1 \otimes S\big(X^2t_2\big)\alpha X^3. \tag{2.3}$$

Proof. For all $h \in H$ we calculate, using (1.5), (1.1), and $t \in \int_l^H$ that

$$\begin{aligned}
\sum hX^1t_1 \otimes S\big(X^2t_2\big)\alpha X^3 &= \sum h_1X^1t_1 \otimes S\big(h_{(2,1)}X^2t_2\big)\alpha h_{(2,2)}X^3 \\
&= \sum X^1(h_1t)_1 \otimes S\big(X^2(h_1t)_2\big)\alpha X^3h_2 \\
&= \sum X^1t_1 \otimes S\big(X^2t_2\big)\alpha X^3h.
\end{aligned}$$

To prove the first equality in (2.3), we take $\sum X^1t_1 \otimes S(X^2t_2)\alpha X^3\beta$. First we apply the 3-cocycle condition

$$\Phi \otimes 1 = (\mathrm{id} \otimes \Delta \otimes \mathrm{id})\big(\Phi^{-1}\big)\big(1 \otimes \Phi^{-1}\big)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi)$$

and then, successively using the fact that $t \in \int_l^H$, (1. 5), (1. 4), (1. 7), and (1. 6), we find the left-hand side of (2.3). The second equality in (2.3) follows from (2.2). $\square$

For a quasi-Hopf algebra H, we introduce H^* as the dual space of H with its natural multiplication $\langle h^*g^*, h\rangle = \sum h^*(h_1)g^*(h_2)$, where $h^*, g^* \in H^*$ and $h \in H$. If H is finite-dimensional, then H^* is also equipped with a natural coassociative coalgebra structure $(\tilde{\Delta}, \tilde{\varepsilon})$ given by $\langle \tilde{\Delta}(h^*), h \otimes h'\rangle = \langle h^*, hh'\rangle$ and $\tilde{\varepsilon}(h^*) = h^*(1)$, where $h^* \in H^*$, $h, h' \in H$ and $\langle\,,\,\rangle : H^* \otimes H \to k$ denotes the dual pairing. On H^* we have the natural left and right H-actions

$$\langle h \rightharpoonup h^*, h'\rangle = \langle h^*, h'h\rangle, \qquad \langle h^* \leftharpoonup h, h'\rangle = \langle h^*, hh'\rangle, \tag{2.4}$$

where $h, h' \in H$ and $h^* \in H^*$. This makes H^* into a H–H-bimodule.

We also introduce $\overline{S} : H^* \to H^*$ as the anti-coalgebra homomorphism dual to S, i.e., $\langle \overline{S}(h^*), h\rangle = \langle h^*, S(h)\rangle$, $\forall h^* \in H^*, h \in H$.

Theorem 2.2. *Let H be a finite-dimensional quasi-Hopf algebra, $\{e_i\}_{i=\overline{1,n}}$ a basis of H with dual basis $\{e^i\}_{i=\overline{1,n}}$ of H^*, and define $\theta : \int_l^H \otimes H^* \to H$, by*

$$\theta(t \otimes h^*) = \sum h^*\big(S\big(X^2 t_2 p^2\big)\alpha X^3\big) X^1 t_1 p^1, \quad \forall t \in \int_l^H,\ h^* \in H^*, \tag{2.5}$$

where $p_R = \sum p^1 \otimes p^2$ is the element defined in (1.19). *Then the following assertions hold*:

(i) *θ is an isomorphism of left H-modules, where $\int_l^H \otimes H^*$ is a left H-module via $h \cdot (t \otimes h^*) = t \otimes h \rightharpoonup h^*$ $\forall h \in H$, $t \in \int_l^H$, $h^* \in H^*$, and H is a left H-module via left multiplication. Consequently* $\dim_k \int_l^H = 1$. *The inverse of θ is given by*

$$\theta^{-1}(h) = \sum_{i=1}^n P(e_i h) \otimes e^i, \quad \forall h \in H, \tag{2.6}$$

where P is the projection onto the space of left integrals, defined in (2.1).

(ii) *The antipode S is bijective.*

(iii) $S(\int_l^H) = \int_r^H$, $S(\int_r^H) = \int_l^H$, *and* $\dim_k \int_r^H = 1$.

Proof. (i) First we show that θ and θ^{-1} are inverses. Indeed, for all $h \in H$ we have:

$$\theta\big(\theta^{-1}(h)\big) \overset{(2.5)}{\underset{(2.1)}{=}} \sum_{i,j=1}^n \big\langle e^j, \beta S\big(S\big(X^2(e_j)_2\big)\alpha X^3\big)e_i h\big\rangle$$

$$\big\langle e^i, S\big(Y^2 X_2^1 (e_j)_{(1,2)} p^2\big)\alpha Y^3\big\rangle Y^1 X_1^1 (e_j)_{(1,1)} p^1$$

$$\begin{aligned}
\text{(1.20)} \quad &= \sum_{j=1}^{n} \langle e^j, \beta S\big(Y^2 X_2^1 p^2 S\big(X^2\big)\alpha X^3\big)\alpha Y^3 h\rangle Y^1 X_1^1 p^1 e_j \\
\text{(1.19), (1.3)} \quad &= \sum Y^1 x^1 Z^1 \beta S\big(Y^2 x^2 X^1 Z_1^2 \beta S\big(x_1^3 X^2 Z_2^2\big)\alpha x_2^3 X^3 Z^3\big)\alpha Y^3 h \\
\text{(1.5), (1.4), (1.7), (1.6)} \quad &= \sum Y^1 \beta S\big(Y^2 X^1 \beta S\big(X^2\big)\alpha X^3\big)\alpha Y^3 h = h.
\end{aligned}$$

For all $t \in \int_l^H$ and $h^* \in H^*$ we compute

$$\begin{aligned}
\theta^{-1}\big(\theta(t \otimes h^*)\big) &= \sum_{i=1}^{n} h^*\big(S\big(X^2 t_2 p^2\big)\alpha X^3\big) P\big(e_i X^1 t_1 p^1\big) \otimes e^i \\
\text{(2.1), (2.2)} \quad &= \sum_{i,j=1}^{n} h^*\big(S\big(X^2 t_2 p^2\big)\alpha X^3 \beta S\big(S\big(Y^2 (e_j)_2\big)\alpha Y^3\big) e_i\big) \\
&\qquad \langle e^j, X^1 t_1 p^1\rangle Y^1 (e_j)_1 \otimes e^i \\
\text{(2.3)} \quad &= \sum_{i=1}^{n} h^*\big(S\big(S\big(Y^2 t_{(1,2)} p_2^1\big)\alpha Y^3 t_2 p^2\big) e_i\big) Y^1 t_{(1,1)} p_1^1 \otimes e^i \\
\text{(1.1), (1.5)} \quad &= \sum_{i=1}^{n} h^*\big(S\big(S\big(Y^2 p_2^1\big)\alpha Y^3 p^2\big) e_i\big) t Y^1 p_1^1 \otimes e^i \\
\text{(1.19), (1.3)} \quad &= \sum_{i=1}^{n} h^*\big(S\big(S\big(y_1^2 x^1 X^2\big)\alpha y_2^2 x^2 X_1^3 \beta S\big(y^3 x^3 X_2^3\big)\big) e_i\big) t y^1 X^1 \otimes e^i \\
\text{(1.5), (1.4), (1.7)} \quad &= \sum_{i=1}^{n} h^*\big(S\big(S\big(x^1\big)\alpha x^2 \beta S\big(x^3\big)\big) e_i\big) t \otimes e^i \\
\text{(1.6)} \quad &= \sum_{i=1}^{n} h^*(e_i) t \otimes e^i = t \otimes h^*.
\end{aligned}$$

Since θ is a bijection and $\dim_k H = \dim_k H^*$ is finite, is follows that $\dim_k \int_l^H = 1$. We are left to show that θ is H-linear. For all $h \in H$, $t \in \int_l^H$ and $h^* \in H^*$ we have:

$$\begin{aligned}
h\theta\big(t \otimes h^*\big) &= \sum h^*\big(S\big(X^2 t_2 p^2\big)\alpha X^3\big) h X^1 t_1 p^1 \\
\text{(2.2)} \quad &= \langle h \rightharpoonup h^*, S\big(X^2 t_2 p^2\big)\alpha X^3\rangle X^1 t_1 p^1 \\
&= \theta\big(t \otimes h \rightharpoonup h^*\big).
\end{aligned}$$

(ii) First we prove that $\overline{S}$ is bijective. H^* is finite-dimensional, so it suffices to show that $\overline{S}$ is injective. Let $h^* \in H^*$ be such that $\overline{S}(h^*) = 0$, and take $0 \neq t \in \int_l^H$. For all $h \in H$ we have

$$\begin{aligned}
\theta\big(t \otimes \beta S(h) \rightharpoonup h^*\big) &= \big\langle \beta S(h) \rightharpoonup h^*, S\big(X^2 t_2 p^2\big)\alpha X^3\big\rangle X^1 t_1 p^1 \\
&= \sum \big\langle h^*, S\big(X^2 t_2 p^2\big)\alpha X^3 \beta S(h)\big\rangle X^1 t_1 p^1 \\
\stackrel{(2.3)}{} &= \sum \big\langle h^*, S\big(h t_2 p^2\big)\big\rangle t_1 p^1 \\
&= \sum \big\langle \overline{S}\big(h^*\big), h t_2 p^2\big\rangle t_1 p^1 = 0.
\end{aligned}$$

Since θ is bijective, we obtain that $t \otimes \beta S(h) \rightharpoonup h^* = 0$. Now, since $t \neq 0$ and $\dim_k \int_l^H = 1$, it follows that $\beta S(h) \rightharpoonup h^* = 0$, for all $h \in H$. Therefore, by (1.6), for all $h' \in H$ we have

$$h^*\big(h'\big) = \sum \big\langle h^*, h' S\big(x^1\big)\alpha x^2 \beta S\big(x^3\big)\big\rangle = \sum \big\langle \beta S\big(x^3\big) \rightharpoonup h^*, h' S\big(x^1\big)\alpha x^2\big\rangle = 0.$$

It is not hard to see that $\overline{S}^* : H^{**} \to H^{**}$, $\overline{S}^*(h^{**}) = h^{**} \circ \overline{S}$, $\forall h^{**} \in H^{**}$, is a bijective map. If we define $\xi : H \to H^{**}$ by $\xi(h)(h^*) = h^*(h)$, $\forall h \in H$, $h^* \in H^*$, then we can easily show that $\{\xi(e_i)\}_{i=\overline{1,n}}$ is a basis of H^{**} dual to the basis $\{e^i\}_{i=\overline{1,n}}$ of H^*, and it follows that ξ is bijective. Moreover, ξ^{-1} is given by $\xi^{-1}(h^{**}) = \sum_{i=1}^n h^{**}(e^i)e_i$, $\forall h^{**} \in H^{**}$. In addition, $\xi^{-1} \circ \overline{S}^* \circ \xi = S$ so S is bijective.

(iii) We have already seen that S is an anti-algebra automorphism of H and $\dim_k \int_l^H = 1$. The rest of the proof is identical to the proof for classical Hopf algebras. $\square$

Remark 2.3. We cannot deduce the isomorphism θ in Theorem 2.2 from a Structure Theorem for dual quasi-Hopf bicomodules. If A is a dual quasi-Hopf algebra (for the complete definition see the last section) then the category of A-bicomodules ${}^A\mathcal{M}^A$ is monoidal and A is in a canonical way an algebra in ${}^A\mathcal{M}^A$. Thus, it makes sense to define a right dual quasi-Hopf A-bicomodule M as being a right A-module in ${}^A\mathcal{M}^A$. Denote by ${}^A\mathcal{M}_A^A$ the category whose objects are right dual quasi-Hopf A-bicomodules and morphisms A-bicomodule maps which are also right A-linear (for more details see [19]). This definition is dual to the one given by Hausser and Nill [11] for quasi-Hopf bimodules. So, using their Structure Theorem, by duality, we can prove a Structure Theorem for dual quasi-Hopf A-bicomodules and then we can apply it in the particular case $M = H$ (here H is a finite-dimensional quasi-Hopf algebra and $A = H^*$, the linear dual space of H). But, on one hand, in order to obtain H as an object in ${}^{H^*}\mathcal{M}_{H^*}^{H^*}$ we need the antipode S of H to be bijective and, on the other hand, using the definition of Hausser and Nill for coinvariants, in the dual case we do not obtain the space of integrals in H. As a consequence, if the isomorphism θ is derived from a Structure Theorem for dual quasi-Hopf bicomodules then it is not the dual case of Hausser and Nill result.

Now, let $M \in {}^A\mathcal{M}_A^A$. First, since A is a coassociative coalgebra we can define the set of right coinvariants $M^{\mathrm{co}(A)}$ of M as being $M^{\mathrm{co}(A)} := \{m \in M \mid \rho_r(m) = m \otimes 1\}$, where $\rho_r : M \to M \otimes A$, $\rho_r(m) = \sum m_{(0)} \otimes m_{(1)}$, $m \in M$, is the right coaction of A on M. Since $\rho_r(m \cdot a) = \rho_r(m)\Delta(a)$ for all $m \in M$ and $a \in A$ (we denote by $M \otimes A \ni m \otimes A \to m \cdot a \in M$, $m \in M$, $a \in A$, the right action of A on M), as in [4, Theorem 2.11] we can show that $\overline{P} : M \to M^{\mathrm{co}(A)}$ given by $\overline{P}(m) := \sum m_{(0)} \cdot \beta(m_{(1)})S(m_{(2)})$, $m \in M$, is well defined and a surjection. Thus, the map $\bar{\theta}^{-1} : M \to M^{\mathrm{co}(A)} \otimes A$, $\bar{\theta}^{-1}(m) := \sum \overline{P}(m_{(0)}) \otimes m_{(1)}$,

$m \in M$, is well defined. It is not hard to see that $M^{\mathrm{co}(A)} \otimes A \in {}^A\mathcal{M}_A^A$ via the structures $(m \otimes a) \cdot b := m \otimes ab$, and $M^{\mathrm{co}(A)} \otimes A \ni m \otimes a \to \sum a_1 \otimes m \otimes a_2 \otimes a_3 \in A \otimes M^{\mathrm{co}(A)} \otimes A \otimes A$, $m \in M^{\mathrm{co}(A)}$, $a, b \in A$. In this way $\bar{\theta}^{-1}$ becomes a right A-colinear map but not a morphism in ${}^A\mathcal{M}_A^A$; moreover, we do not know if, in general, $\bar{\theta}^{-1}$ is bijective. Under these circumstances, if we take H a finite-dimensional quasi-Hopf algebra then $H \in {}^{H^*}\mathcal{M}^{H^*}$ via

$$H \ni h \to \sum_{i,j=1}^{n} e^i \otimes e_j h e_i \otimes e^j \in H^* \otimes H \otimes H^*,$$

where $\{e_i\}_{i=\overline{1,n}}$ is a basis of H with dual basis $\{e^i\}_{i=\overline{1,n}}$ of H^*. Moreover, if we define

$$h \cdot h^* := \sum h^*\big(S\big(X^2 h_2\big)\alpha X^3\big) X^1 h_1, \quad \forall h \in H,\ h^* \in H^*,$$

then with this structures H is not an object in ${}^{H^*}\mathcal{M}_{H^*}^{H^*}$ but $\rho_r(h \cdot h^*) = \rho_r(h)\tilde{\Delta}(h^*)$, $H^{\mathrm{co}(H^*)} = \int_l^H$ and the projection $\overline{P} : H \to \int_l^H$ is just the projection P defined in (2.1). Therefore, in this case $\bar{\theta}^{-1}$ coincides with θ^{-1} defined in (2.6), so it is bijective. We notice that, in the proof of the fact that θ^{-1} defined by (2.6) is the inverse of θ, a key role is played by the relation (2.3) in Lemma 2.1. Since the equality (2.3) involve all the structures of H as a quasi-Hopf algebra, it cannot be generalized for the coinvariants $M^{\mathrm{co}(A)}$ of a dual quasi-Hopf A-bicomodule M. In conclusion, in order to obtain θ from a Structure Theorem for dual quasi-Hopf bicomodules we need the above context but it does not provide a "suggestion" for the general case. Also, the same kind of problems occur when we work with relative Hopf modules [4], instead of dual quasi-Hopf bicomodules.

Let H be a quasi-Hopf algebra and t a left integral in H. Using the fact that H is an associative algebra, we find that th is also a left integral in H, for all $h \in H$, hence the space of left (right) integrals in H is a two-sided ideal. Moreover, if H is finite-dimensional, then it follows from the uniqueness of the integral in H, that there exists $\mu \in H^*$ such that

$$th = \mu(h)t, \quad \forall t \in \textstyle\int_l^H \text{ and } h \in H. \tag{2.7}$$

More precisely, $\mu \in \mathrm{Alg}(H, k)$. It was noted in [11] that $\mathrm{Alg}(H, k)$ is a group with multiplication given by $\nu\xi = (\nu \otimes \xi) \circ \Delta$, unit ε, and inverse $\mu^{-1} = \mu \circ S$. In [11], μ is called the modulus of H, but, following the classical terminology for Hopf algebras, we will call μ the distinguished group-like element. Observe that $\mu = \varepsilon$ if and only if H is unimodular. As in the case of a Hopf algebra, it follows from the bijectivity of the antipode that

$$hr = \mu^{-1}(h)r = \mu\big(S(h)\big)r, \quad \forall r \in \textstyle\int_r^H \text{ and } h \in H. \tag{2.8}$$

For infinite-dimensional Hopf algebras it is well-known that the space of left (right) integrals in H is zero [21, p. 107]. In order to prove a similar result for quasi-Hopf algebras we first need a lemma.

Lemma 2.4. *Let H be a quasi-Hopf algebra in the sense of Drinfel'd and define*

$$\underline{\Delta}: H \to H \otimes H, \quad \underline{\Delta}(h) = \sum h_{\underline{1}} \otimes h_{\underline{2}} := \sum q^1 h_1 p^1 \otimes q^2 h_2 p^2, \quad \forall h \in H, \tag{2.9}$$

where $p_R = \sum p^1 \otimes p^2$ and $q_R = \sum q^1 \otimes q^2$ are defined by (1.19). *If J is a non-zero two-sided ideal of H such that $\underline{\Delta}(J) \subseteq J \otimes H$, then $J = H$.*

Proof. From (1.21), we easily deduce that

$$\sum \big(1 \otimes S^{-1}(p^2)\big)\underline{\Delta}\big(p^1 h q^1\big)\big(1 \otimes S(q^2)\big) = \Delta(h), \quad \forall h \in H.$$

This implies that $\Delta(J) \subseteq J \otimes H$, since J is a two-sided ideal of H and $\underline{\Delta}(J) \subseteq J \otimes H$. Now, if $\varepsilon(J) = 0$, then for any $h \in H$ we have $h = \sum \varepsilon(h_1)h_2 \in \varepsilon(J)H = 0$, so $J = 0$, a contradiction. Thus $\varepsilon(J) \neq 0$, and there exists $a \in J$ with $\varepsilon(a) = 1$. Using (1.5), we obtain that $\beta = \varepsilon(a)\beta = \sum a_1 \beta S(a_2) \in JH \subseteq J$, so $\beta \in J$. Using (1.6) and the fact that J is a two-sided ideal of H, we find that $1 = \sum X^1 \beta S\big(X^2\big)\alpha X^3 \in J$, and $J = H$. □

For $h \in H$ and $h^* \in H^*$, we define $h^* \to h = \sum h^*(h_{\underline{2}})h_{\underline{1}}$. For a two-sided ideal I of H, we let $H^* \to I$ be the subspace of I generated by all the elements of the form $h^* \to a$, with $h^* \in H^*$ and $a \in I$.

Theorem 2.5. *Let H be a quasi-Hopf algebra in the sense of Drinfel'd and I a non-zero two-sided ideal of H. Then*

$$J = H^* \to I = H.$$

As a consequence, we obtain

(i) *If H is a quasi-Hopf algebra with an antipode S, then H is finite-dimensional if and only if S is bijective and $\int_l^H \neq 0$.*
(ii) *If a quasi-Hopf algebra in the sense of Drinfel'd is semisimple as an algebra, then it is finite-dimensional.*

Proof. The statement follows from Lemma 2.4 if we can show that J is a non-zero two-sided ideal of H such that $\underline{\Delta}(J) \subseteq J \otimes H$.

Obviously $\varepsilon \to h = h$, and therefore $I \subseteq J$. For all $h \in H$, $h^* \in H^*$ and $a \in I$ we have

$$\begin{aligned} \big(h^* \to a\big)h &= \sum h^*\big(q^2 a_2 p^2\big)q^1 a_1 p^1 h \\ &\overset{(1.20)}{=} \sum h^*\big(q^2 (ah_1)_2 p^2 S(h_2)\big)q^1 (ah_1)_1 p^1 \\ &= \sum \big(S(h_2) \rightharpoonup h^*\big) \to (ah_1) \end{aligned}$$

and J is a right ideal. J is also a left ideal, since

$$\begin{aligned} h\big(h^* \rightarrow a\big) &= \sum h^*\big(q^2 a_2 p^2\big) h q^1 a_1 p^1 \\ \overset{(1.20)}{} &= \sum h^*\big(S^{-1}(h_2) q^2 (h_1 a)_2 p^2\big) q^1 (h_1 a)_1 p^1 \\ &= \sum \big(h^* \leftharpoonup S^{-1}(h_2)\big) \rightarrow (h_1 a). \end{aligned}$$

Write $f = \sum f^1 \otimes f^2$. Using (1.22), (1.1) and (1.23) we can show that

$$\begin{aligned} &h^* \rightarrow \big(g^* \rightarrow h\big) \\ &\quad = \sum \big[\big(g^1 S\big(x^3\big) \rightharpoonup h^* \leftharpoonup S^{-1}\big(f^2 X^3\big)\big)\big(g^2 S\big(x^2\big) \rightharpoonup g^* \leftharpoonup S^{-1}\big(f^1 X^2\big)\big)\big] \rightarrow \big(X^1 a x^1\big) \end{aligned}$$

for all $h^*, g^* \in H^*$ and $h \in H$. I is a two-sided ideal of H, so the above equality shows that $H^* \rightarrow J \subseteq J$. To prove that $\underline{\Delta}(J) \subseteq J \otimes H$, we use the same arguments as in [15, p. 12]. Take $a \in J$, and write $\underline{\Delta}(a) = \sum_{i=1}^n a_i \otimes a_i'$, where $a_1, \ldots, a_m \in J$ and $a_{m+1}, \ldots, a_n$ are linearly independent modulo J. For any $h^* \in H^*$, $h^* \rightarrow a = \sum_{i=1}^n h^*(a_i') a_i \in J$. The linear independence of $a_{m+1}, \ldots, a_n$ modulo J implies that $h^*(a_i') = 0$, and therefore $a_i' = 0$ (h^* is arbitrary), for all $i > m$. We find that $\underline{\Delta}(a) \in J \otimes H$, as needed.

(i) One implication follows from Theorem 2.2. Conversely, assume that S is bijective and $I = \int_l^H \neq 0$. Then I is a non-zero two-sided ideal of H and the first part of our theorem tells us that $H^* \rightarrow I = H$. Thus there exist $\{h_i^*\}_{i=\overline{1,n}} \subseteq H^*$ and $\{t_i\}_{i=\overline{1,n}} \subseteq \int_l^H$ such that $1 = \sum_{i=1}^n h_i^* \rightarrow t_i$. For any $i = \overline{1,n}$ we have $\underline{\Delta}(t_i) = \sum_{j=1}^{n_i} a_j^i \otimes b_j^i$, for some $\{a_j^i\}_{j=\overline{1,n_i}} \subseteq H$ and $\{b_j^i\}_{j=\overline{1,n_i}} \subseteq H$. Therefore, for any $h^* \in H^*$ and $i = \overline{1,n}$ we have $h^* \rightarrow t_i = \sum_{j=1}^{n_i} h^*(b_j^i) a_j^i$. For all $h \in H$ we obtain that

$$\begin{aligned} h = \sum_{i=1}^n h\big(h_i^* \rightarrow t_i\big) &= \sum_{i=1}^n \big(h^* \leftharpoonup S^{-1}(h_2)\big) \rightarrow h_1 t_i \\ &= \sum_{i=1}^n \big(h^* \leftharpoonup S^{-1}(h)\big) \rightarrow t_i \quad \text{since } t_i \in \textstyle\int_l^H,\ \forall i = \overline{1,n} \\ &= \sum_{i=1}^n \sum_{j=1}^{n_i} h^*\big(S^{-1}(h) b_j^i\big) a_j^i. \end{aligned}$$

This shows that H is finite-dimensional, since it is a subspace of the span of $\{a_j^i \mid i = \overline{1,n}, j = \overline{1,n_i}\}$.

(ii) Let H be a semisimple quasi-Hopf algebra with bijective antipode. Then $\mathrm{Ker}(\varepsilon)$ is a two-sided ideal of H. Since H is a semisimple left H-module, there exists a left ideal I of H such that $H = I \oplus \mathrm{Ker}(\varepsilon)$. $\mathrm{Ker}(\varepsilon)$ has codimension 1 in H, hence I has dimension 1. Write $1 = t + h$, with $t \in I$, $h \in \mathrm{Ker}(\varepsilon)$. $t \neq 0$, because $1 \notin \mathrm{Ker}(\varepsilon)$. It follows that $I = kt$, since $\dim_k(I) = 1$. For all $h' \in H$, we have $h't \in I$, and also $h't = \varepsilon(h')t + (h' - \varepsilon(h')1)t$ with $\varepsilon(h')t \in I$ and $(h' - \varepsilon(h')1)t \in \mathrm{Ker}(\varepsilon)$. Since we have a direct sum, it follows that

$h't = \varepsilon(h')t$, and t is a non-zero left integral in H. From (i), it then follows that H is finite-dimensional. $\square$

Remarks 2.6. (i) Let H be a quasi-Hopf algebra in the sense of Drinfel'd. Then $S(\int_l^H) = \int_r^H$ and $\dim_k \int_l^H = \dim_k \int_r^H$. Therefore, if H is infinite-dimensional then $\int_l^H = \int_r^H = 0$.

(ii) Let H be a finite-dimensional quasi-Hopf algebra and t a non-zero left integral in H. Theorem 2.5 implies that $H = H^* \rightarrow t$ so the map

$$\bar{\theta} : H^* \to H, \quad \bar{\theta}\big(h^*\big) = h^* \rightarrow t = \sum h^*\big(q^2 t_2 p^2\big) q^1 t_1 p^1 \quad \forall h^* \in H^*, \tag{2.10}$$

is bijective. Moreover, $\bar{\theta}$ is left H-linear, where the left H-action on H^* is given by the formula

$$h \cdot h^* = h^* \leftharpoonup S^{-1}(h)$$

for $h \in H$ and $h^* \in H^*$. If H is a classical Hopf algebra, then $\bar{\theta}$ is also left H^*-linear where the H^*-actions on H^* and H are given by convolution and $h^* \rightharpoonup h = \sum h^*(h_2)h_1$, respectively. This means that H is a left cyclic H^*-module generated by a left non-zero integral.

3. Cointegrals on quasi-Hopf algebras

In the first part of this section we study the cointegrals on a finite-dimensional quasi-Hopf algebra (so the antipode of H is bijective).

Definition 3.1 [11]. Let H be a quasi-Hopf algebra, M an H-bimodule and $\rho : M \to M \otimes H$ an H-bimodule map. Then (M, ρ) is called a right quasi-Hopf H-bimodule if the following relations hold:

$$(\mathrm{id}_M \otimes \varepsilon) \circ \rho = \mathrm{id}_M, \tag{3.1}$$

$$\Phi \cdot (\rho \otimes \mathrm{id}_M)\big(\rho(m)\big) = (\mathrm{id}_M \otimes \Delta)\big(\rho(m)\big) \cdot \Phi, \quad \forall m \in M. \tag{3.2}$$

A morphism between two right quasi-Hopf H-bimodules is an H-bimodule map $f : M \to M'$ satisfying $\rho' \circ f = (f \otimes \mathrm{id}) \circ \rho$. ${}_H\mathcal{M}_H^H$ is the category of right quasi-Hopf H-bimodules and morphisms of right quasi-Hopf H-bimodules.

We will use the Sweedler type notation

$$\rho(m) = \sum m_{(0)} \otimes m_{(1)}, \qquad (\rho \otimes \mathrm{id}_M)\big(\rho(m)\big) = m_{(0,0)} \otimes m_{(0,1)} \otimes m_{(1)}, \qquad \text{etc.}$$

Let H be a quasi-Hopf algebra and M a right quasi-Hopf H-bimodule. We define $E : M \to M$, by

$$E(m) = \sum X^1 \cdot m_{(0)} \cdot \beta S\big(X^2 m_{(1)}\big)\alpha X^3, \tag{3.3}$$

for all $m \in M$.

$$M^{\mathrm{co}H} = \big\{n \in M \mid E(n) = n\big\}$$

is called the space of coinvariants of M. We also have (cf. [11, Corollary 3.9])

$$M^{\mathrm{co}H} = \Big\{n \in M \;\Big|\; \rho(n) = \sum E\big(x^1 \cdot n\big) \cdot x^2 \otimes x^3\Big\}. \tag{3.4}$$

We have the following Structure Theorem for right quasi-Hopf H-bimodules (see [11, Theorem 3.8]). The map

$$\nu : M^{\mathrm{co}H} \otimes H \to M, \quad \nu(n \otimes h) = n \cdot h, \quad \forall n \in M^{\mathrm{co}H},\ h \in H, \tag{3.5}$$

is an isomorphism of right quasi-Hopf H-bimodules. Here $M^{\mathrm{co}H} \otimes H$ is a right quasi-Hopf H-bimodule via the structures $a \cdot (n \otimes h) \cdot b = \sum E(a_1 \cdot n) \otimes a_2 hb$ and $\rho(n \otimes h) = \sum E(x^1 \cdot n) \otimes x^2 h_1 \otimes x^3 h_2$, $\forall n \in N$, $a, h, b \in H$. The inverse of ν is given by

$$\nu^{-1}(m) = \sum E(m_{(0)}) \otimes m_{(1)}, \quad \forall m \in M. \tag{3.6}$$

Now, let H be a finite-dimensional quasi-Hopf algebra; recall that the antipode is then automatically bijective. Let $\{e_i\}_{i=\overline{1,n}}$ be a basis in H with dual basis $\{e^i\}_{i=\overline{1,n}}$ in H^* and consider

$$U = \sum g^1 S\big(q^2\big) \otimes g^2 S\big(q^1\big), \qquad V = \sum S^{-1}\big(f^2 p^2\big) \otimes S^{-1}\big(f^1 p^1\big), \tag{3.7}$$

where $f = \sum f^1 \otimes f^2$ is the element defined by (1.15), $f^{-1} = \sum g^1 \otimes g^2$, and $q_R = \sum q^1 \otimes q^2$ and $p_R = \sum p^1 \otimes p^2$ are defined as in (1.19). Following [11], H^* is right quasi-Hopf H-bimodule. The structure is the following:

$$h \cdot h^* \cdot h' = S\big(h'\big) \rightharpoonup h^* \leftharpoonup S^{-1}(h), \quad \forall h, h' \in H, h^* \in H^*, \tag{3.8}$$

$$\rho\big(h^*\big) = \sum_{i=1}^{n} e^i * h^* \otimes e_i, \quad \forall h^* \in H^*, \tag{3.9}$$

where the (non-associative) multiplication $* : H^* \otimes H^* \to H^*$ is given by

$$\big\langle h^* * g^*, h\big\rangle = \sum h^*\big(V^1 h_1 U^1\big) g^*\big(V^2 h_2 U^2\big), \quad \forall h^*, g^* \in H^*,\ h \in H. \tag{3.10}$$

The coinvariants $\lambda \in H^{*\mathrm{co}H}$ are called left cointegrals on H and the space of left cointegrals is denoted by $\mathcal{L}$. Thus $\dim_k \mathcal{L} = 1$ and the projection $E : H^* \to \mathcal{L}$ is given by

$$\langle E(h^*), h\rangle = \sum_{i=1}^{n} \langle e^i \otimes h^*, \overline{\Delta}\big(S^{-1}(q^1) h S^2(q^2 e_i) S(\beta)\big)\rangle, \tag{3.11}$$

for all $h^* \in H^*$ and $h \in H$. Here $\overline{\Delta}(h) = \sum V^1 h_1 U^1 \otimes V^2 h_2 U^2$, for any $h \in H$. The transpose $E^{\mathrm{T}} : H \to H$ is given by

$$E^{\mathrm{T}}(h) = \sum_{i=1}^{n} \langle e^i \otimes \mathrm{id}, \overline{\Delta}\big(S^{-1}(q^1) h S^2(q^2 e_i) S(\beta)\big)\rangle, \quad \forall h \in H \tag{3.12}$$

and provides a projection onto the space of right integrals in H. Moreover, the dual pairing $\mathcal{L} \otimes \int_r^H \to k$, $\lambda \otimes r \mapsto \langle \lambda, r\rangle$ is nondegenerate [11, Lemma 4.4].

When H is an ordinary Hopf algebra the left cointegrals on H are precisely the left integrals in H^* (i.e., an element $\lambda \in H^*$ such that $\sum \lambda(h_2)h_1 = \lambda(h)1$, $\forall h \in H$). For quasi-Hopf algebras H we will give some characterizations for left cointegrals. First, we need another formula for the projection (3.11), giving the connection between the projection P onto the space of left integrals defined by (2.1) and the projection E^{T}, defined by (3.12). This provides an answer to a question raised in [11].

Lemma 3.2. *Let H be a finite-dimensional quasi-Hopf algebra and E the map defined by* (3.11). *Then, for all $h^* \in H^*$ and $h \in H$, we have*:

$$\langle E(h^*), h\rangle = \langle h^*, S^{-1}\big(P(S(h))\big)\rangle. \tag{3.13}$$

In particular, if E^{T} is the transpose map (3.12) *and $P : H \to \int_l^H$ is the projection* (2.1) *then $E^{\mathrm{T}}(h) = S^{-1}(P(S(h)))$, for all $h \in H$.*

Proof. Take $f = \sum f^1 \otimes f^2 = \sum F^1 \otimes F^2$ and $f^{-1} = \sum g^1 \otimes g^2 = \sum G^1 \otimes G^2$ as in (1.15) and (1.16). It is easy to see that

$$\sum g^1 S(g^2\alpha) = \beta, \qquad \sum S(\beta f^1) f^2 = \alpha,$$
$$\sum f^1 \beta S(f^2) = S(\alpha), \tag{3.14}$$

and we compute, for $h^* \in H^*$ and $h \in H$:

$$
\begin{aligned}
\langle E(h^*), h\rangle &= \sum_{i=1}^{n} \langle e^i \otimes h^*, \overline{\Delta}(S^{-1}(q^1)hS^2(q^2e_i)S(\beta))\rangle \\
\text{(3.7), (1.11)} \quad &= \sum_{j=1}^{n} \langle e^j, hS^2(q^2S^{-1}(f^2q_2^1p^2)(e_j)_1U^1)S(\beta)\rangle \\
&\quad \langle h^*, S^{-1}(f^1q_1^1p^1)(e_j)_2U^2\rangle \\
\text{(1.21), (3.7), (1.19)} \quad &= \sum_{j=1}^{n} \langle e^j, hS^2(S^{-1}(f^2)(e_j)_1g^1S(X^2)\alpha X^3S^{-1}(\beta))\rangle \\
&\quad \langle h^*, S^{-1}(f^1)(e_j)_2g^2S(X^1)\rangle \\
\text{(1.9), (1.18)} \quad &= \sum_{j=1}^{n} \langle e^j, hS^2(S^{-1}(f^2)(e_j)_1F_1^2X^2g_2^1G^2\alpha S^{-1}(\beta F^1X^1g_1^1G^1))\rangle \\
&\quad \langle h^*, S^{-1}(f^1)(e_j)_2F_2^2X^3g^2\rangle \\
\text{(3.14), (1.5)} \quad &= \sum_{i=1}^{n} \langle e^i, hS^2(S^{-1}(f^2)(e_i)_1X^2)S(\beta F^1X^1\beta)F^2\rangle \\
&\quad \langle h^*, S^{-1}(f^1(e_i)_2X^3\rangle \\
\text{(3.14), (1.11)} \quad &= \sum_{i=1}^{n} \langle e^i, hS^2(S^{-1}(S(e_i)_2)S^{-1}(f^2)X^2S^{-1}(S^{-1}(\alpha)X^1\beta))\rangle \\
&\quad \langle h^*, S^{-1}(S(e_i)_1)S^{-1}(f^1)X^3\rangle \\
\text{(1.9), (1.18)} \quad &= \sum_{i=1}^{n} \langle e^i, hS(S(e_i)_2)S(\beta F^1f_1^2X^2g_2^1)F^2f_2^2X^3g^2\alpha\rangle \\
&\quad \langle h^*, S^{-1}(S(e_i)_1)S^{-1}(f^1X^1g_1^1)\rangle \\
\text{(3.14), (1.5)} \quad &= \sum_{j=1}^{n} \langle e^j, g^1S(hS((e_j)_2)S(X^2)\alpha X^3g^2\alpha)\rangle\langle h^*, S^{-1}(X^1(e_j)_1)\rangle \\
\text{(3.14), (1.19), (2.1)} \quad &= \langle h^*, S^{-1}(P(S(h)))\rangle.
\end{aligned}
$$

The last assertion follows easily by (3.13) and (3.12). □

Lemma 3.3. *Let H be a finite-dimensional quasi-Hopf algebra and $E : H^* \to \mathcal{L}$ the projection* (3.11). *Then*

$$E(h^* \leftharpoonup S^{-1}(h)) = \mu(h)E(h^*), \quad \forall h^* \in H^* \text{ and } h \in H, \tag{3.15}$$

$$\lambda(S^{-1}(h)h') = \sum \mu(h_1)\lambda(h'S(h_2)), \quad \forall h, h' \in H, \tag{3.16}$$

where μ is the distinguished group-like element of H^*.

Proof. The relations (3.15) follow easily from (3.13) and the fact that $P(h) \in \int_l^H$, $\forall h \in H$. To prove (3.16), let $\{e_i\}_{i=\overline{1,n}}$ be a basis in H with dual basis $\{e^i\}_{i=\overline{1,n}}$ in H^* and q_R the element defined by (1.19). For all $h, h' \in H$ and $\lambda \in \mathcal{L}$ we have:

$$\begin{aligned}
\lambda\big(S^{-1}(h)h'\big) &= \big\langle E(\lambda), S^{-1}(h)h'\big\rangle \\
\overset{(3.13),\,(2.1)}{} &= \sum_{i=1}^{n}\big\langle e^i, \beta S^2\big(q^2(e_i)_2\big)S\big(h'\big)h\big\rangle\big\langle\lambda, S^{-1}\big(q^1(e_i)_1\big)\big\rangle \\
&= \sum_{j=1}^{n}\big\langle e^j, \beta S^2\big(q^2(e_j)_2h_2\big)S\big(h'\big)\big\rangle\big\langle\lambda, S^{-1}\big(q^1(e_j)_1h_1\big)\big\rangle \\
\overset{(2.1),\ (3.13)}{} &= \sum\big\langle E\big(\lambda \leftharpoonup S^{-1}(h_1)\big), h'S(h_2)\big\rangle \\
\overset{(3.15)}{} &= \sum \mu(h_1)\lambda\big(h'S(h_2)\big). \qquad \square
\end{aligned}$$

Proposition 3.4. *Let H be a finite-dimensional quasi-Hopf algebra and μ the distinguished group-like element of H^*. For $\lambda \in H^*$, the following statements are equivalent*:

(a) *λ is a left cointegral on H*;
(b) *for all $h \in H$, we have*

$$\begin{aligned}
&\sum \lambda\big(S^{-1}\big(f^1\big)h_2S^{-1}\big(q^1g^1\big)\big)S^{-1}\big(f^2\big)h_1S^{-1}\big(q^2g^2\big) \\
&\quad= \sum \mu\big(q_1^1x^1\big)\big\langle\lambda, hS^{-1}\big(f^1\big)g^2S\big(q_2^1x^2\big)\big\rangle q^2x^3S^{-1}\big(S^{-1}\big(f^2\big)g^1\big); \qquad (3.17)
\end{aligned}$$

(c) *for all $h \in H$, we have*

$$\sum \lambda\big(S^{-1}\big(f^1\big)h_2U^2\big)S^{-1}\big(f^2\big)h_1U^1 = \sum \mu\big(q_1^1x^1\big)\big\langle\lambda, hS\big(q_2^1x^2\big)\big\rangle q^2x^3.$$

Here $f = \sum f^1 \otimes f^2$, $q_R = \sum q^1 \otimes q^2$, and $U = \sum U^1 \otimes U^2$ are defined respectively by (1.15), (1.19), *and* (3.7), *and $f^{-1} = \sum g^1 \otimes g^2$.*

Proof. (a) $\Rightarrow$ (b). Suppose that λ is a left cointegral. As before, we write $f = \sum f^1 \otimes f^2 = \sum F^1 \otimes F^2 = \sum \boldsymbol{F}^1 \otimes \boldsymbol{F}^2$, $f^{-1} = \sum g^1 \otimes g^2 = \sum G^1 \otimes G^2 = \sum \boldsymbol{G}^1 \otimes \boldsymbol{G}^2$, $q_R = \sum q^1 \otimes q^2$, and $p_R = \sum p^1 \otimes p^2$. Using (3.4), (3.8), (3.9), and (3.15), we find that

$$\sum \lambda\big(V^2h_2U^2\big)V^1h_1U^1 = \sum \mu\big(x^1\big)\lambda\big(hS\big(x^2\big)\big)x^3, \qquad (3.18)$$

for all $h \in H$, and we compute that

$$
\begin{aligned}
&\sum \mu\big(q_1^1x^1\big)\big\langle\lambda, hS^{-1}\big(f^1\big)g^2S\big(q_2^1x^2\big)\big\rangle q^2x^3S^{-1}\big(S^{-1}\big(f^2\big)g^1\big)\\
(3.15),\ (3.18)\quad &= \sum \big\langle\lambda, V^2\big[S^{-1}\big(q^1\big)hS^{-1}\big(f^1\big)g^2\big]_2U^2\big\rangle q^2V^1\big[S^{-1}\big(q^1\big)hS^{-1}\big(f^1\big)g^2\big]_1\\
&\qquad U^1S^{-1}\big(S^{-1}\big(f^2\big)g^1\big)\\
(3.7),\ (1.11)\quad &= \sum \big\langle\lambda, S^{-1}\big(\boldsymbol{F}^1q_1^1p^1\big)h_2S^{-1}\big(F^1f_1^1\boldsymbol{G}^1\big)g_2^2U^2\big\rangle q^2S^{-1}\big(\boldsymbol{F}^2q_2^1p^2\big)h_1S^{-1}\\
&\qquad \big(F^2f_2^1\boldsymbol{G}^2\big)g_1^2U^1S^{-1}\big(S^{-1}\big(f^2\big)g^1\big)\\
(1.21),\ (1.11)\quad &= \sum \big\langle\lambda, S^{-1}\big(S(h)_1\big)S^{-1}\big(F^1f_1^1\big)g_2^2G^2S\big(X^1\big)\big\rangle S^{-1}\big(S(h)_2\big)S^{-1}\big(F^2f_2^1\big)\\
&\qquad g_1^2G^1S\big(X^2\big)\alpha S^{-1}\big(S^{-1}\big(f^2\big)g^1S\big(X^3\big)\big)\\
(1.9),\ (1.18)\quad &= \sum \big\langle\lambda, S^{-1}\big(S(h)_1\big)S^{-1}\big(S\big(X^3\big)F^1f_1^1\big)g^2\big\rangle S^{-1}\big(S(h)_2\big)S^{-1}\big(S\big(X^2\big)F^2f_2^1\big)\\
&\qquad g_2^1G^2\alpha S^{-1}\big(S^{-1}\big(S\big(X^1\big)f^2\big)g_1^1G^1\big)\\
(3.14),\ (1.5)\quad &= \sum \big\langle\lambda, S^{-1}\big(S(h)_1\big)S^{-1}\big(S\big(X^3\big)F^1f_1^1\big)\big\rangle S^{-1}\big(S(h)_2\big)\\
&\qquad S^{-1}\big(S^{-1}\big(S\big(X^1\big)f^2\big)\beta S\big(X^2\big)F^2f_2^1\big)\\
(1.9),\ (1.18)\quad &= \sum \big\langle\lambda, S^{-1}\big(S(h)_1\big)S^{-1}\big(f^1X^1\big)\big\rangle S^{-1}\big(S(h)_2\big)\\
&\qquad S^{-1}\big(S^{-1}\big(F^2f_2^2X^3\big)\beta F^1f_1^2X^2\big)\\
(3.14),\ (1.5)\quad &= \sum \big\langle\lambda, S^{-1}\big(X^1S(h)_1\big)\big\rangle S^{-1}\big(S^{-1}\big(\alpha X^3\big)X^2S(h)_2\big)\\
(1.19),\ (1.11)\quad &= \sum \big\langle\lambda, S^{-1}\big(f^1\big)h_2S^{-1}\big(q^1g^1\big)\big\rangle S^{-1}\big(f^2\big)h_1S^{-1}\big(q^2g^2\big).
\end{aligned}
$$

(b) $\Rightarrow$ (a). Assume that $\lambda \in H^*$ satisfies (3.17). It follows from (1.11) that

$$
\sum \lambda\big(S^{-1}\big(q^1h_1\big)\big)q^2h_2 = \sum \mu\big(q_1^1x^1\big)\big\langle\lambda, S^{-1}\big(f^1h\big)g^2S\big(q_2^1x^2\big)\big\rangle S^{-1}\big(f^2\big)g^1S\big(q^2x^3\big), \tag{3.19}
$$

for all $h \in H$ and

$$
\begin{aligned}
&\big\langle\lambda, S^{-1}\big(P\big(S(h)\big)\big)\big\rangle\\
(2.1),\ (3.19)\quad &= \sum \mu\big(q_1^1x^1\big)\ \big\langle\lambda, S^{-1}\big(f^1\beta S\big(f^2\big)S^2\big(g^1\big)S^3\big(q^2x^3\big)S(h)\big)\\
&\qquad\qquad g^2S\big(q_2^1x^2\big)\big\rangle\\
(3.14)\quad &= \sum \mu\big(q_1^1x^1\big)\ \big\langle\lambda, hS^2\big(q^2x^3\big)S\big(g^1\big)\alpha g^2S\big(q_2^1x^2\big)\big\rangle\\
(3.14),\ (1.19),\ (1.20)\quad &= \lambda(h),
\end{aligned}
$$

and it follows from Lemma 3.2 that $\lambda \in \mathcal{L}$.

(c) ⇒ (a). Repeating the computations of the first part of the proof of Lemma 3.2, we find that the projection E (cf. (3.11)) is given by

$$\big\langle E\big(h^*\big), h\big\rangle = \sum_{i=1}^{n} \big\langle e^i, hS\big(f^2\big)S^2\big((e_i)_1U^1\big)S(\beta)\big\rangle\big\langle h^*, S^{-1}\big(f^1\big)(e_i)_2U^2\big\rangle, \tag{3.20}$$

for all $h^* \in H$, $h \in H$. Using (3.20), we can compute that $E(\lambda) = \lambda$, so $\lambda \in \mathcal{L}$.

(a) ⇒ (c). Assume that $\lambda \in \mathcal{L}$. We calculate

$$\begin{aligned} &\sum \mu\big(q_1^1x^1\big)\big\langle\lambda, hS\big(q_2^1x^2\big)\big\rangle q^2x^3 \\ \overset{(3.16),\ (3.18)}{=} &\sum \big\langle\lambda, V^2\big[S^{-1}\big(q^1\big)h\big]_2U^2\big\rangle q^2V^1\big[S^{-1}\big(q^1\big)h\big]_1U^1 \\ \overset{(3.7),\ (1.11),\ (1.21)}{=} &\sum \big\langle\lambda, S^{-1}\big(f^1\big)h_2U^2\big\rangle S^{-1}\big(f^2\big)h_1U^1 \end{aligned}$$

and the proof is complete. □

Remark 3.5. Formula (3.19) is equivalent to (3.17), and can be viewed as another characterization of left cointegrals. In the case of a Hopf algebra, (3.19) takes the form

$$\sum \lambda\big(S^{-1}(h_1)\big)h_2 = \lambda\big(S^{-1}(h)\big)1$$

which is the well known statement that λ is a left integral if and only if $\lambda \circ S^{-1}$ is a right integral.

Observe that our definition of cointegral only makes sense in the case where H is finite-dimensional: indeed, we need a dual basis of H in order to make H^* into a right quasi-Hopf bimodule (see 3.9). Also the equivalent characterizations from Proposition 3.4 make no sense in the infinite-dimensional case, as they involve the distinguished element, which can only be defined in the finite-dimensional case. Nevertheless, the cointegral has a lot of applications in the finite-dimensional case (see [11]).

Following [5, Lemma 4.1], we now give an alternative definition for the space of coinvariants of a right quasi-Hopf H-bimodule M:

$$M^{\overline{\mathrm{co}H}} = \Big\{n \in M \;\Big|\; \rho(n) = \sum x^1 \cdot n \cdot S\big(x_2^3X^3\big)f^1 \otimes x^2X^1\beta S\big(x_1^3X^2\big)f^2\Big\}. \tag{3.21}$$

Using this definition, we will prove a second Structure Theorem for right quasi-Hopf bimodules. For $M \in {}_H\mathcal{M}_H^H$, we define

$$\overline{E} : M \to M, \quad \overline{E}(m) := \sum m_{(0)} \cdot \beta S(m_{(1)}), \quad \forall m \in M. \tag{3.22}$$

It follows that

$$\overline{E}(m) = \sum E\big(p^1 \cdot m\big) \cdot p^2, \quad E(m) = \sum X^1 \cdot \overline{E}(m) \cdot S\big(X^2\big)\alpha X^3, \quad \forall m \in M. \tag{3.23}$$

By [11, Proposition 3.4], we have

$$E(m\cdot h)=\varepsilon(h)E(m),\qquad E\big(h\cdot E(m)\big)=E(h\cdot m),\qquad h\cdot E(m)=\sum E(h_1\cdot m)\cdot h_2 \tag{3.24}$$

for all $m\in M$ and $h\in H$ and therefore the maps

$$\overline{E}:M^{\mathrm{co}H}\to M^{\overline{\mathrm{co}H}}\quad\text{and}\quad E:M^{\overline{\mathrm{co}H}}\to M^{\mathrm{co}H} \tag{3.25}$$

are each others inverses. In the case of a classical Hopf algebra, the maps E and $\overline{E}$ are equal to the identity on $M^{\mathrm{co}H}=M^{\overline{\mathrm{co}H}}$. Moreover, in this case $M^{\mathrm{co}H}$ is invariant under the left adjoint H-action $h\triangleright m:=\sum h_1\cdot m\cdot S(h_2)$, in the sense that $E(h\triangleright m)=h\triangleright E(m)$, $h\in H$, $m\in M$. In the quasi-Hopf case the projection E generalizes this property. More precisely, if we define $h\neg m=E(h\cdot m)$, then by [11, Proposition 3.4], we have that $h\cdot E(m)=\sum[h_1\neg E(m)]\cdot h_2$. Now, as in the Hopf algebra case, we will prove that $M^{\overline{\mathrm{co}H}}$ is invariant under the left adjoint H-action.

Lemma 3.6. *Let H be a quasi-Hopf algebra, M a right quasi-Hopf H-bimodule.*

(a) $\mathrm{Im}(\overline{E})\subseteq M^{\overline{\mathrm{co}H}}$. *If $n\in M$ then $n\in M^{\overline{\mathrm{co}H}}\Longleftrightarrow \overline{E}(n)=n$.*
(b) $M^{\overline{\mathrm{co}H}}$ *is a left H-submodule of M, where M is considered a left H-module via the left adjoint action, that is $h\triangleright m=\sum h_1\cdot m\cdot S(h_2)$, for all $h\in H$ and $m\in M$.*

Proof. (a) Let $m\in M$ and $\delta=\sum\delta^1\otimes\delta^2$ be given by (1.14). Then

$$\begin{aligned}
\rho\big(\overline{E}(m)\big) &= \sum m_{(0,0)}\cdot\beta_1 S(m_{(1)})_1\otimes m_{(0,1)}\beta_2 S(m_{(1)})_2\\
(1.17),\ (1.11)\quad &= \sum m_{(0,0)}\cdot\delta^1 S(m_{(1)_2})f^1\otimes m_{(0,1)}\delta^2 S(m_{(1)_1})f^2\\
(1.14),\ (1.3),\ (1.5)\quad &= \sum m_{(0,0)}\cdot x^1\beta S\big((m_{(1)}x^3)_2X^3\big)f^1\\
&\quad \otimes m_{(0,1)}x^2X^1\beta S\big((m_{(1)}x^3)_1X^2\big)f^2\\
(3.2),\ (1.1),\ (1.5)\quad &= \sum x^1\cdot m_{(0)}\cdot\beta S\big(x^3_2X^3m_{(1)}\big)f^1\otimes x^2X^1\beta S\big(x^3_1X^2\big)f^2\\
&= \sum x^1\cdot\overline{E}(m)\cdot S\big(x^3_2X^3\big)f^1\otimes x^2X^1\beta S\big(x^3_1X^2\big)f^2.
\end{aligned}$$

Therefore $\mathrm{Im}(\overline{E})\subseteq M^{\overline{\mathrm{co}H}}$, implying immediately one implication of the second part. Conversely, if $n\in M^{\overline{\mathrm{co}H}}$ then

$$\begin{aligned}
\overline{E}(n) &= \sum n_{(0)}\cdot\beta S(n_{(1)})\\
(3.21),\ (3.14)\quad &= \sum x^1\cdot n\cdot S\big(x^2X^1\beta S\big(x^3_1X^2\big)\alpha x^3_2X^3\big)\\
(1.5),\ (1.6)\quad &= n.
\end{aligned}$$

(b) If M is an H-bimodule then is not hard to see that M is a left H-module via the H-action $h \triangleright m = \sum h_1 \cdot m \cdot S(h_2)$, $h \in H$, $m \in M$. Using (3.23), (3.24) and (1.20) we have that $\overline{E}(h \triangleright m) = h \triangleright \overline{E}(m)$ for all $h \in H$, $m \in M$. By part (a), we see that $M^{\overline{\mathrm{co}H}}$ is a left H-module under the left adjoint action. □

Let H be a quasi-Hopf algebra and M a right quasi-Hopf H-bimodule. We already have seen that the map $\overline{E} : M^{\mathrm{co}H} \to M^{\overline{\mathrm{co}H}}$ is an isomorphism. It is also H-linear since

$$\begin{aligned} \overline{E}(h \neg n) &= \overline{E}\big(E(h \cdot n)\big) \\ \text{(3.23), (3.24)} \quad &= \sum E\big(p^1 h \cdot n\big) \cdot p^2 \\ \text{(1.24) (3.24)} \quad &= \sum h_1 \cdot E\big(p^1 \cdot n\big) \cdot p^2 S(h_2) \\ \text{(3.23)} \quad &= h \triangleright \overline{E}(n) \end{aligned}$$

for all $h \in H$ and $n \in M^{\mathrm{co}H}$.

$M^{\overline{\mathrm{co}H}}$ is a left H-module, so, by [11, Lemma 3.2], $M^{\overline{\mathrm{co}H}} \otimes H$ becomes a right quasi-Hopf H-bimodule with the following structure:

$$a \cdot (n \otimes h) \cdot b = \sum a_1 \triangleright n \otimes a_2 h b, \qquad \rho'(n \otimes h) = \sum x^1 \triangleright n \otimes x^2 h_1 \otimes x^3 h_2 \tag{3.26}$$

for all $a, b, h \in H$ and $n \in M^{\overline{\mathrm{co}H}}$. We can now state the second Structure Theorem for right quasi-Hopf bimodules.

Theorem 3.7. *Let H be a quasi-Hopf algebra and M a right quasi-Hopf H-bimodule. Consider $M^{\overline{\mathrm{co}H}} \otimes H$ as a right quasi-Hopf H-bimodule as in* (3.26). *The map*

$$\overline{\nu} : M^{\overline{\mathrm{co}H}} \otimes H \to M, \quad \overline{\nu}(n \otimes h) = \sum X^1 \cdot n \cdot S\big(X^2\big)\alpha X^3 h \tag{3.27}$$

is an isomorphism of quasi-Hopf H-bimodules. The inverse of $\overline{\nu}$ is given by the formula

$$\overline{\nu}^{-1}(m) = \sum \overline{E}(m_{(0)}) \otimes m_{(1)}. \tag{3.28}$$

Proof. We have seen that $M^{\overline{\mathrm{co}H}} \cong M^{\mathrm{co}H}$ are isomorphic as left H-modules, and therefore $M^{\overline{\mathrm{co}H}} \otimes H \cong M^{\mathrm{co}H} \otimes H$ as quasi-Hopf H-bimodules (in both cases, the structure is determined by [11, Lemma 3.2]). From the Hausser–Nill Structure Theorem [11, Theorem 3.8], it follows that $M \cong M^{\mathrm{co}H} \otimes H$ as quasi-Hopf H-bimodules. Thus we find that $M^{\overline{\mathrm{co}H}} \otimes H \cong M$ as quasi-Hopf H-bimodules, and it is straightforward to verify that the connecting isomorphism is exactly $\overline{\nu}$. □

Remark 3.8. Theorem 3.7 can be proved also in a direct way, without using the Hausser–Nill Structure Theorem. This proof is straightforward, but long and technical. However, it has the advantage that it is independent of the bijectivity of the antipode, which makes it more general in the infinite-dimensional case. At first sight, the Hausser–Nill definition of coinvariants makes use of the bijectivity of the antipode; however, a careful inspection of their definition and their proof of the Structure Theorem shows that we do not really need the bijectivity.

The antipode of a finite-dimensional quasi-Hopf algebra is bijective, so its dual H^* is a right quasi-Hopf H-bimodule with structures defined in (3.8), (3.9). The coinvariants $\bar{\lambda} \in H^{*\overline{\mathrm{co}H}}$ are called left alternative cointegrals on H, and the space of left alternative cointegrals is denoted by $\overline{\mathcal{L}} = H^{*\overline{\mathrm{co}H}}$. From Theorem 3.7, we obtain immediately the following result.

Theorem 3.9. *Let H be a finite-dimensional quasi-Hopf algebra. Then* $\dim_k \overline{\mathcal{L}} = 1$.

Applying (1.3) and (1.5), we find that $\bar{\lambda} \in H^*$ is an alternative left cointegral if and only if

$$\sum \bar{\lambda}\big(V^2 h_2 U^2\big) V^1 h_1 U^1 = \sum \bar{\lambda}\big(S^{-1}\big(X_1^1 p^1\big) h S\big(S\big(X^3\big) f^1\big)\big) X_2^1 p^2 S\big(X^2\big) f^2 \quad (3.29)$$

for all $h \in H$. (3.29) can be used to extend the definition of left alternative cointegral to infinite-dimensional quasi-Hopf algebras.

4. Integrals for dual quasi-Hopf algebras

Sullivan's Theorem [20] asserts that the space of left integrals on a Hopf algebra has dimension at most one. Various new proofs of this result have been given in recent years, see [3,6,18,22]. The aim of this section is to give a proof of the uniqueness of integrals on a dual quasi-Hopf algebra. Our approach is based on the methods developed in [3].

Throughout, A will be a dual quasi-bialgebra or a dual quasi-Hopf algebra. Following [14], a dual quasi-bialgebra A is a coassociative coalgebra A with comultiplication Δ and counit ε together with coalgebra morphisms $M : A \otimes A \to A$ (the multiplication; we write $M(a \otimes b) = ab$) and $u : k \to A$ (the unit; we write $u(1) = 1$), and an invertible element $\varphi \in (A \otimes A \otimes A)^*$ (the reassociator), such that for all $a, b, c, d \in A$ the following relations hold:

$$\sum a_1(b_1 c_1)\varphi(a_2, b_2, c_2) = \sum \varphi(a_1, b_1, c_1)(a_2 b_2) c_2, \quad (4.1)$$

$$1a = a1 = a, \quad (4.2)$$

$$\sum \varphi(a_1, b_1, c_1 d_1)\varphi(a_2 b_2, c_2, d_2) = \sum \varphi(b_1, c_1, d_1)\varphi(a_1, b_2 c_2, d_2)\varphi(a_2, b_3, c_3), \quad (4.3)$$

$$\varphi(a, 1, b) = \varepsilon(a)\varepsilon(b). \quad (4.4)$$

A is called a dual quasi-Hopf algebra if, moreover, there exist an antimorphism S of the coalgebra A and elements $\alpha, \beta \in H^*$ such that, for all $a \in A$:

$$\sum S(a_1)\alpha(a_2)a_3 = \alpha(a)1, \qquad \sum a_1\beta(a_2)S(a_3) = \beta(a)1, \tag{4.5}$$

$$\sum \varphi\big(a_1\beta(a_2), S(a_3), \alpha(a_4)a_5\big) = \sum \varphi^{-1}\big(S(a_1), \alpha(a_2)a_3, \beta(a_4)S(a_5)\big) = \varepsilon(a). \tag{4.6}$$

It follows from the axioms that $\alpha(1)\beta(1) = 1$, so we can assume that $\alpha(1) = \beta(1) = 1$ and, $S(1) = 1$. Moreover (4.3) and (4.4) imply

$$\varphi(1, a, b) = \varphi(a, b, 1) = \varepsilon(a)\varepsilon(b), \quad \forall a, b \in A. \tag{4.7}$$

If $A = (A, M, u, \varphi, S, \alpha, \beta)$ is a dual quasi-Hopf algebra, then $A^{\mathrm{op,cop}}$ is also a dual quasi-Hopf algebra. The structure maps are

$$\varphi_{\mathrm{op,cop}}(a, b, c) = \varphi(c, b, a), \qquad S_{\mathrm{op,cop}} = S, \qquad \alpha_{\mathrm{op,cop}} = \beta, \quad \text{and} \quad \beta_{\mathrm{op,cop}} = \alpha.$$

If A is a dual quasi-bialgebra, then A^* is an algebra, with multiplication given by convolution and unit ε.

Definition 4.1. A map $T \in A^*$ is called a left integral on the dual quasi-bialgebra A if $a^*T = a^*(1)T$ for any $a^* \in A^*$. Left integrals on $A^{\mathrm{op,cop}}$ are called right integrals on A.

The set of left (right) integrals on A is denoted by $\int_l$ ($\int_r$). We keep the same notation as in Section 3 but we will specify every time which kind of integral we are using. It is clear that $T \in A^*$ is a left integral if and only if $\sum T(a_2)a_1 = T(a)1$ for all $a \in A$. $\int_l$ is a subspace of A^*. Moreover, as in the Hopf algebra case, $\int_l$ is a two-sided ideal in the algebra A^*.

Let $A^{*\mathrm{rat}}$ be the left rational part of A^*. $A^{*\mathrm{rat}}$ is the sum of rational left ideals of the algebra A^*, see [21, Chapter II]. Note that $A^{*\mathrm{rat}} \subseteq A^*$ and

$$\begin{aligned} a^* \in A^{*\mathrm{rat}} \quad &\Longleftrightarrow \quad \exists\big(a_i^*\big)_{i=\overline{1,n}} \subseteq A^* \quad \text{and} \quad (a_i)_{i=\overline{1,n}} \subseteq A \quad \text{such that} \\ &\qquad b^*a^* = \sum_{i=1}^{n} b^*(a_i)a_i^*, \quad \forall b^* \in A^*. \end{aligned} \tag{4.8}$$

It follows that $\int_l \subseteq A^{*\mathrm{rat}}$. In particular, if $A^{*\mathrm{rat}} = 0$, then $\int_l = 0$.

Later in this section we will show that the left and right rational parts of A^* are equal, justifying our notation. As in the Hopf algebra case, we first describe the connection between $A^{*\mathrm{rat}}$ and $\int_l$. It is well known that $A^{*\mathrm{rat}}$ is a rational left A^*-module, and this induces a right A-comodule structure on $A^{*\mathrm{rat}}$ defined by

$$\begin{aligned} &\rho : A^{*\mathrm{rat}} \to A^{*\mathrm{rat}} \otimes A, \\ &\rho(a^*) = \sum a^*_{(0)} \otimes a^*_{(1)} \quad \Longleftrightarrow \quad b^*a^* = \sum b^*\big(a^*_{(1)}\big)a^*_{(0)}, \end{aligned} \tag{4.9}$$

for all $b^* \in A^*$. This can be rewritten as follows:

$$\rho(a^*) = \sum a^*_{(0)} \otimes a^*_{(1)} \quad \Longleftrightarrow \quad \sum a^*(a_2)a_1 = \sum a^*_{(0)}(a)a^*_{(1)}, \quad \forall a \in A. \tag{4.10}$$

Now, define the map $\sigma : A \otimes A \to A^*$ by

$$\sigma(a \otimes b)(c) = \varphi(c, a, b), \quad \forall a, b, c \in A. \tag{4.11}$$

σ is convolution invertible, the inverse σ^{-1} is given by $\sigma^{-1}(a \otimes b)(c) = \varphi^{-1}(c, a, b)$, for all $a, b, c \in A$. We introduce the following notation, for $a^* \in A^*$ and $a \in A$:

$$a^* \leftharpoondown a = S(a) \rightharpoonup a^* \in A^*.$$

For all $a^* \in A^{*\mathrm{rat}}$, we define $P^*(a^*)$ by

$$P^*(a^*) = \sum \beta\big(a^*_{(1)}\big)\alpha\big(S\big(a^*_{(3)}\big)\big)\sigma\big(S^2\big(a^*_{(4)}\big) \otimes S\big(a^*_{(2)}\big)\big)\big(a^*_{(0)} \leftharpoondown S\big(a^*_{(5)}\big)\big). \tag{4.12}$$

We now claim that $P^*(a^*) \in \int_l$, for all $a^* \in A^{*\mathrm{rat}}$. Indeed, for all $b^* \in A^*$, we calculate:

$$\begin{aligned}
&\big\langle b^*(1)P^*(a^*), a\big\rangle \\
\text{(4.12), (4.5)}\quad &= \sum \big\langle b^*, a^*_{(1)}\beta\big(a^*_{(2)}\big)S\big(a^*_{(3)}\big)\big\rangle\alpha\big(S\big(a^*_{(5)}\big)\big)\varphi\big(a_1, S^2\big(a^*_{(6)}\big), S\big(a^*_{(4)}\big)\big) \\
&\quad \big\langle a^*_{(0)}, a_2 S^2\big(a^*_{(7)}\big)\big\rangle \\
\text{(4.10), (4.1)}\quad &= \sum \beta\big(a^*_{(1)}\big)\big\langle b^*, a_1\big[S^2\big(a^*_{(5)}\big)\alpha\big(S\big(a^*_{(4)}\big)\big)S\big(a^*_{(3)}\big)\big]\big\rangle\varphi\big(a_2, S^2\big(a^*_{(6)}\big), S\big(a^*_{(2)}\big)\big) \\
&\quad \big\langle a^*_{(0)}, a_3 S^2\big(a^*_{(7)}\big)\big\rangle \\
\text{(4.5), (4.12)}\quad &= \sum b^*(a_1)\big\langle P^*(a^*), a_2\big\rangle = \big\langle b^* P^*(a^*), a\big\rangle.
\end{aligned}$$

Proposition 4.2. *Let A be a dual quasi-Hopf algebra and $\sigma : A \otimes A \to A^*$ the map defined in* (4.11). *Then $\theta^* : \int_l \otimes A \to A^{*\mathrm{rat}}$, given by*

$$\theta^*(T \otimes a) = \sum \sigma\big(S(a_5), \alpha(a_6)a_7\big)(T \leftharpoondown a_4)\sigma^{-1}\big(S(a_3) \otimes \beta\big(S(a_2)\big)S^2(a_1)\big), \tag{4.13}$$

is an isomorphism of right A-comodules.

Proof. In the situation where A is finite-dimensional, the proof follows from Theorem 2.2 by duality. This is why we restrict to proving that θ^* is well-defined and has an inverse, leaving other details to the reader. For $a^* \in A^*$, $T \in \int_l$ and $a \in A$, we compute that

$$\begin{aligned}
&\big\langle a^*\theta^*(T \otimes a), b\big\rangle \\
&\quad = \sum a^*(b_1)\varphi\big(b_2, S(a_5), \alpha(a_6)a_7\big)T\big(b_3 S(a_4)\big)\varphi^{-1}\big(b_4, S(a_3), \beta\big(S(a_2)\big)S^2(a_1)\big)
\end{aligned}$$

$$\begin{aligned}
&\overset{(4.5)}{=} \sum \big\langle a^*, b_1\big(S(a_6)\alpha(a_7)a_8\big)\big\rangle \varphi\big(b_2, S(a_5), a_9\big) T\big(b_3 S(a_4)\big)\\
&\qquad \varphi^{-1}\big(b_4, S(a_3), \beta\big(S(a_2)\big)S^2(a_1)\big)\\
&\overset{(4.1)}{=} \sum \big\langle (a_8 \rightharpoonup a^*)T, b_2 S(a_4)\big\rangle \varphi\big(b_1, S(a_5), \alpha(a_6)a_7\big)\\
&\qquad \varphi^{-1}\big(b_3, S(a_3), \beta\big(S(a_2)\big)S^2(a_1)\big)\\
&= \sum a^*(a_2)\big\langle \theta^*(T \otimes a_1), b\big\rangle,
\end{aligned}$$

hence

$$a^*\theta^*(T \otimes a) = \sum a^*(a_2)\theta^*(T \otimes a_1). \tag{4.14}$$

From (4.8) it follows that $\theta^*(T \otimes a) \in A^{*\mathrm{rat}}$, as needed.

$\int_l \otimes A$ is a right A-comodule with structure induced by the comultiplication on A, and it follows then from (4.9) that θ^* is right A-colinear.

We claim that the inverse of θ^* is given by

$$\theta^{*-1}: A^{*\mathrm{rat}} \to \int_l \otimes A, \quad \theta^{*-1}(a^*) = \sum P^*\big(a^*_{(0)}\big) \otimes a^*_{(1)}, \tag{4.15}$$

where $\rho(a^*) = \sum a^*_{(0)} \otimes a^*_{(1)} \in A^{*\mathrm{rat}} \otimes A$ is defined as in (4.9). It is clear that θ^{*-1} is well defined. To show that θ^* and θ^{*-1} are each others inverses, we need the following equalities, for any $T \in \int_l$ and $a, b \in A$:

$$\begin{aligned}
&\sum \varphi\big(a_2, S(b_2), \alpha(b_3)b_4\big) T\big(a_3 S(b_1)\big) a_1\\
&\quad = \sum \varphi\big(a_1, S(b_2), \alpha(b_3)b_4\big) T\big(a_2 S(b_1)\big) b_5,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
T\big(aS(b)\big) &= \sum \varphi\big(a_1, S(b_2)\alpha(b_3), b_4\beta(b_5)\big) T\big(a_2 S(b_1)\big)\\
&= \sum \varphi\big(\beta(a_1)a_2, S(b_2), \alpha(b_3)b_4\big) T\big(a_3 S(b_1)\big).
\end{aligned} \tag{4.17}$$

Note that these formulas are the formal duals of (2.2) and (2.3). □

Corollary 4.3. *Let A be a dual quasi-Hopf algebra. Then $A^{*\mathrm{rat}} = 0$ if and only if $\int_l = 0$.*

Corollary 4.4. *Assume that a dual quasi-Hopf algebra A has a non-zero left integral. Then the antipode S is injective. If A is finite-dimensional, then the left integral space has dimension 1, and the antipode S is bijective.*

Proof. Let T be a non-zero left integral, and assume that $S(a) = 0$. Let $\overline{S}$ be the algebra antimorphism dual to S. If θ^* is the map defined by (4.13) then for all $a^* \in A^*$ and $b \in A$ we have:

$$\begin{aligned}
&\big\langle\theta^*\big(T\otimes\beta\overline{S}(a^*)\rightharpoonup a\big),b\big\rangle\\
&\quad=\sum\beta(a_2)a^*\big(S(a_3)\big)\big\langle\theta^*(T\otimes a_1),b\big\rangle\\
&\quad=\sum\beta(a_8)a^*\big(S(a_9)\big)\varphi\big(b_1,S(a_5),\alpha(a_6)a_7\big)T\big(b_2S(a_4)\big)\\
&\qquad\quad\varphi^{-1}\big(b_3,S(a_3),\beta\big(S(a_2)\big)S^2(a_1)\big)\\
&\quad=\sum a^*\big(S(a_5)\big)T\big(b_1S(a_4)\big)\varphi^{-1}\big(b_2,S(a_3),\beta\big(S(a_2)\big)S^2(a_1)\big)=0. \qquad (4.17)
\end{aligned}$$

Since θ^* is bijective and $T\neq 0$, it is follows that $\beta\overline{S}(a^*)\rightharpoonup a=0$, for any $a^*\in A^*$. Thus $\sum\beta(a_2)a_1\otimes S(a_3)=0$ and therefore

$$\sum\beta(a_5)a_1\otimes a_2\otimes a_3\otimes a_4\otimes S(a_6)=0.$$

By (4.6) we obtain that $a=\sum\varphi^{-1}(S(a_2),\alpha(a_3)a_4,\beta(a_5)S(a_6))a_1=0$, and S is injective.

If A is finite-dimensional then $A^{*\mathrm{rat}}=A^*$. We obtain that $\theta^*:\int_l\otimes A\to A^*$ is an isomorphism of right A-comodules. The final assertion is then obvious. □

Let C be a coalgebra and C^* the dual algebra. Then C is a left (right) C^*-module under the left (right) action $\rightharpoonup$ ($\leftharpoonup$) of C^* on C given by $c^*\rightharpoonup c=\sum c^*(c_2)c_1$ ($c\leftharpoonup c^*=\sum c^*(c_1)c_2$), for all $c^*\in C^*$ and $c\in C$. Recall that C is called a left (right) quasi-co-Frobenius coalgebra (shortly QcF coalgebra) if there exists an injective morphism of left (right) C^*-modules from C to a free left (right) C^*-module. The coalgebra C is called left (right) co-Frobenius if there exists a monomorphism of (left) right C^*-modules from C to C^* or, equivalently, if there exists a bilinear form $\boldsymbol{b}:C\otimes C\to k$ which is left (right) non-degenerated and C^*-balanced, i.e., if $\boldsymbol{b}(c,x)=0$ for any $c\in C$ (respectively $\boldsymbol{b}(x,c)=0$ for any $c\in C$) then $x=0$ and $\boldsymbol{b}(x\leftharpoonup c^*,y)=\boldsymbol{b}(x,c^*\rightharpoonup y)$ for any $x,y\in C$, $c^*\in C^*$.

Finally, C^* is a left (right) C^*-module, so we can consider the left (right) rational part of C^*. We will denote this C^*-submodule of C^* by $C_l^{*\mathrm{rat}}$ (respectively $C_r^{*\mathrm{rat}}$). C is called a left (right) semiperfect coalgebra if the category ${}^C\mathcal{M}$ ($\mathcal{M}^C$) of left (right) C-comodules has enough projectives or, equivalently, $C_r^{*\mathrm{rat}}$ ($C_l^{*\mathrm{rat}}$) is a dense subset of C^* (in the finite topology, see [1,7] for the definition). Following [3], if C is a left and right semiperfect coalgebra then $C_l^{*\mathrm{rat}}=C_r^{*\mathrm{rat}}:=C^{*\mathrm{rat}}$, $C^{*\mathrm{rat}}$ is dense in C^* and C is projective generator in the categories ${}^C\mathcal{M}$ and $\mathcal{M}^C$.

It is known that a left (right) co-Frobenius coalgebra is a left (right) QcF coalgebra and a left (right) QcF coalgebra is a left (right) semiperfect coalgebra, but the converse implications are not true. There are examples showing that none of the three concepts is left–right symmetric. However, if H is a Hopf algebra all these concepts are equivalent to H having non-zero left (or right) integrals. We prove now that a similar result holds for dual quasi-Hopf algebras. Except for the implication (i) ⇒ (ii), the proof is identical to the proof of [7, Theorem 5.3.2], so we omit it here.

Theorem 4.5. *Let A be a dual quasi-Hopf algebra. Then the following assertions are equivalent*:

(i) *A has a non-zero integral.*
(ii) *A is a left co-Frobenius coalgebra.*
(iii) *A is a left QcF coalgebra.*
(iv) *A is a left semiperfect coalgebra.*
(v) *A has a non-zero right integral.*
(vi) *A is a right co-Frobenius coalgebra.*
(vii) *A is a right QcF coalgebra.*
(viii) *A is a right semiperfect coalgebra.*
(ix) *A is a generator in the category ${}^A\mathcal{M}$ (or in $\mathcal{M}^A$).*
(x) *A is a projective object in the category ${}^A\mathcal{M}$ (or in $\mathcal{M}^A$).*

Proof. (i) $\Rightarrow$ (ii). Let $T \in A^*$ be a non-zero left integral. We define a bilinear form $\boldsymbol{b} : A \otimes A \to k$ as follows:

$$\boldsymbol{b}(a,b) = \theta^*(T \otimes b)(a), \quad \forall a, b \in A, \tag{4.18}$$

where θ^* is the map defined as in (4.13). Then, for all $a, b \in A$ and $a^* \in A^*$ we compute:

$$\begin{aligned} \boldsymbol{b}(a \leftharpoonup a^*, b) &= \sum a^*(a_1)\boldsymbol{b}(a_2, b) \\ \text{(4.18)} \quad &= \langle a^*\theta^*(T \otimes b), a\rangle \\ \text{(4.14), (4.18)} \quad &= \sum a^*(b_2)\boldsymbol{b}(a, b_1) = \boldsymbol{b}(a, a^* \rightharpoonup b) \end{aligned}$$

proving that $\boldsymbol{b}$ is C^*-balanced. Now we prove that $\boldsymbol{b}$ is left non-degenerate. If $x \in A$ such that $\boldsymbol{b}(a, x) = 0$ for any $a \in A$ then $\theta^*(T \otimes x) = 0$, and it follows from Proposition 4.2 that $x = 0$. Thus $\boldsymbol{b}$ is C^*-balanced and left non-degenerate, so A is a left co-Frobenius coalgebra. □

By [6, Proposition 2.2], any subcoalgebra D of a left semiperfect coalgebra C is itself left semiperfect. As a consequence of Theorem 4.5, we therefore have

Corollary 4.6. *Let A be a dual quasi-Hopf algebra with non-zero integrals. Then any dual quasi-Hopf subalgebra B of A (i.e., a subcoalgebra B of A which is closed under multiplication of A, $1 \in B$ and $S(B) \subseteq B$) has non-zero integrals.*

We proceed with a proof of the uniqueness of integrals for dual quasi-Hopf algebras. First, by [7, Remark 5.4.3], if C is a left and right co-Frobenius coalgebra, and M is a finite-dimensional right C-comodule, then $\dim_k \mathrm{Hom}_{C^*}(C, M) \leqslant \dim_k M$ (recall that M and N are in a natural way left C^*-modules).

Proposition 4.7. *Let A be a dual quasi-Hopf algebra with non-zero integral. Then $\dim_k \int_l = \dim_k \int_r = 1$.*

Proof. A is a left and right co-Frobenius dual quasi-Hopf algebra by Theorem 4.5. k is a right A-comodule, and $\int_r = \mathrm{Hom}_{A^*}(A, k)$. Thus, $\int_r$ has dimension at most 1, so this has dimension precisely 1 since $\int_r \neq 0$. Now, if we replace the dual quasi-Hopf algebra A with $A^{\mathrm{op,cop}}$ then we obtain that $\dim_k \int_l = 1$. □

A result of Radford [17] asserts that a co-Frobenius Hopf algebra has bijective antipode. It is not known if a similar result holds for a co-Frobenius dual quasi-Hopf algebra. To this end, we would need the dual version of Lemma 2.4, without the assumption that the antipode is bijective. However, we know that the antipode of a co-Frobenius dual quasi-Hopf algebra is injective (Corollary 4.4). We apply this to prove the following result.

Lemma 4.8. *Let A be a dual quasi-Hopf algebra with a non-zero left integral T and antipode S. Then $T \circ S$ is a non-zero right integral on A. In particular, $\overline{S}(\int_l) = \int_r$.*

Proof. We omit the proof, since it is identical to proof of [7, Lemma 5.4.4]. □

The right integral $T \circ S$ can be described more explicitly. Indeed, let A be a dual quasi-Hopf algebra with non-zero left integral T. Then, for all $a^* \in A^*$, Ta^* is also a left integral and $Ta^* = \chi(a^*)T$ for some map χ from A^* to k. Since $Ta^*b^* = \chi(a^*b^*)T = \chi(a^*)\chi(b^*)T$ and $T = \chi(\varepsilon)T$, χ is an algebra map.

Let $G(C)$ be the set of group-like elements of a coalgebra C. From (4.6) and (4.5), it follows that a group-like element g in a dual quasi-Hopf algebra is invertible and $g^{-1} = S(g)$.

The following result generalizes [2, Proposition 1.3].

Proposition 4.9. *Let A be a dual quasi-Hopf algebra with a non-zero left integral T. Then there exists a group-like element g in A such that*

(i) $Ta^* = a^*(g)T$, *for any* $a^* \in A^*$;
(ii) $\overline{S}(T) = \Lambda(g \rightharpoonup T)$, *where* $\Lambda \in A^*$ *is given by* $\Lambda(b) = \sum \alpha(b_2)\varphi^{-1}(S(b_1), b_3, g)$, *for all* $b \in A$.

Proof. Since $T \neq 0$, there exists $a \in A$ such that $T(a) = 1$. We will show that

$$g = a \leftharpoonup T = \sum T(a_1)a_2$$

has the required properties. As in Section 3, we denote by $\rightharpoonup$ and $\leftharpoonup$ the usual left and right action respectively of A on A^*, that is

$$\langle b \rightharpoonup a^*, c\rangle = \langle a^*, cb\rangle \quad \text{and} \quad \langle a^* \leftharpoonup b, c\rangle = \langle a^*, bc\rangle.$$

It is easy to show that

$$(a^*b^*) \leftharpoonup b = \sum (a^* \leftharpoonup b_1)(b^* \leftharpoonup b_2); \tag{4.19}$$

$$b \rightharpoonup (a^*b^*) = \sum (b_1 \rightharpoonup a^*)(b_2 \rightharpoonup b^*). \tag{4.20}$$

Let $\chi \in A^{**}$ be defined as above. For all $a^* \in A^*$, we have that

$$\chi(a^*) = \chi(a^*)T(a) = \langle Ta^*, a\rangle = \sum T(a_1)a^*(a_2) = \Big\langle a^*, \sum T(a_1)a_2\Big\rangle = a^*(g),$$

and therefore $Ta^* = \chi(a^*)T = a^*(g)T$, for any $a^* \in A^*$. g is a group-like element, because χ is an algebra map.

(ii) Define $\overline{\Lambda} : A \to A^*$ by $\overline{\Lambda}(b)(c) = \sum \alpha(b_2)\varphi^{-1}(S(b_1), b_3, c)$ for any $b, c \in A$. It follows that $\Lambda(b) = \overline{\Lambda}(b)(g)$, for any $b \in A$, and $\overline{\Lambda}(1) = \varepsilon$. Moreover, by (4.1), (4.5) and (i) we have that

$$\sum \overline{\Lambda}(b_2)\big[\big(T \leftharpoonup S(b_1)\big) \leftharpoonup b_3\big] = T\overline{\Lambda}(b) = \Lambda(b)T, \quad \forall b \in B. \tag{4.21}$$

Now, the relation in (ii) follows from the following computation, for all $b \in A$:

$$\begin{aligned}
\big\langle \Lambda(g \rightharpoonup T), b\big\rangle &= \sum \Lambda(b_1)\big\langle T \leftharpoonup b_2, g\big\rangle T(a) \\
\text{(i)} \quad &= \sum \Lambda(b_1)\big\langle T(T \leftharpoonup b_2), a\big\rangle \\
(4.21),\ (4.19) \quad &= \sum \big\langle \overline{\Lambda}(b_2)\big\{\big[\big(T \leftharpoonup S(b_1)\big)T\big] \leftharpoonup b_3\big\}, a\big\rangle \\
&= \sum T\big(S(b_1)\big)\big\langle \overline{\Lambda}(b_2)(T \leftharpoonup b_3), a\big\rangle \quad \text{since } T \in \textstyle\int_l \\
&= \big\langle \overline{\Lambda}(1)T, a\big\rangle\big\langle \overline{S}(T), b\big\rangle \quad \text{since } \overline{S}(T) \in \textstyle\int_r, \text{ see Lemma 4.8} \\
&= \big\langle \overline{S}(T), b\big\rangle. \qquad \square
\end{aligned}$$

We conclude this paper with a Maschke type theorem for dual quasi-Hopf algebras. Let C be a coalgebra and M a right C-comodule. Recall that M is called cosemisimple if for any subcomodule N of M there exists a C-colinear map $\pi : M \to N$ such that $\pi \circ i = \mathrm{id}$, where $i : N \to M$ is the inclusion map. C is called cosemisimple if any right C-comodule is cosemisimple.

Theorem 4.10. *For a dual quasi-Hopf algebra A, the following statements are equivalent*:

(i) *A is a cosemisimple coalgebra*;
(ii) *A, viewed as a right A-comodule via comultiplication, is cosemisimple as an A-comodule*;
(iii) *there exists a right integral $T \in A^*$ such that $T(1) = 1$*;
(iv) *there exists a left integral $T \in A^*$ such that $T(1) = 1$.*

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Let $u : k \to A$ be the unit map of A. k is trivially a right A-comodule. Since u is injective, there exists an A-colinear map $T : A \to k$ such that $T(1) = 1$. This means that T is a right integral for A and $T(1) = 1$.

(iii) $\Rightarrow$ (iv). Let T be a right integral on A such that $T(1) = 1$. Then T is a left integral on $A^{\mathrm{op,cop}}$ and $T \circ S$ is a right integral on $A^{\mathrm{op,cop}}$, by Lemma 4.8. Therefore, $T \circ S$ is a left integral for A, and since $T(1) = 1$ it follows that $(T \circ S)(1) = 1$.

(iv) $\Rightarrow$ (i). Let $T \in \int_l$ be such that $T(1) = 1$ and M a right A-comodule. Then $M \otimes A$ is also a right A-comodule right A-action $\mathrm{id} \otimes \Delta$. $\rho_M : M \to M \otimes A$ is an injective right A-colinear map. Moreover, the map $\omega_M : M \otimes A \to M$ given by

$$\omega_M(m \otimes a) = \sum \alpha(a_3)\varphi\big(m_{(1)}, S(a_2), a_4\big)\beta(m_{(3)})T\big(m_{(2)}S(a_1)\big)m_{(0)}, \tag{4.22}$$

is right A-colinear (by (4.16)), and $\omega_M \circ \rho_M = \mathrm{id}$ (by (4.5) and (4.6)).

Now, if N is an A-subcomodule of M, then there is a k-linear map $\tilde{\pi} : M \to N$ such that $\tilde{\pi} \circ i = \mathrm{id}$, where i is the inclusion map $i : N \to M$. We define $\pi : M \to N$, $\pi = \omega_N \circ (\tilde{\pi} \otimes \mathrm{id}) \circ \rho_M$, where ω_N is the corresponding map (4.22) for the right A-comodule N. Is not hard to see that $\tilde{\pi} \otimes \mathrm{id}$ is A-colinear, hence π is A-colinear. It also follows from the above considerations that $\pi \circ i = \mathrm{id}$, proving that A is a cosemisimple coalgebra. $\square$

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