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Note

Triangle-free graphs with large chromatic numbers

A. Nilli

Department of Mathematics, Tel Aviv University, 69978 Ramat Aviv, Tel Aviv, Israel

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Abstract

It is shown that there are two positive constants c_1, c_2 such that the maximum possible chromatic number of a triangle-free graph with $m > 1$ edges is at most $c_1 m^{1/3}/(\log m)^{2/3}$ and at least $c_2 m^{1/3}/(\log m)^{2/3}$. This is deduced from results of Ajtai, Komlós, Szemerédi, Kim and Johansson, and settles a problem of Erdős. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Chromatic number; Triangle-free graphs

In his open problems appendix [2, pp. 241–243], Paul Erdős raised the problem of determining or estimating the two functions $f(n)$ and $g(m)$, where $f(n)$ is the maximum possible chromatic number of a triangle-free graph with n vertices, and $g(m)$ is the maximum possible chromatic number of a triangle-free graph with m edges.

He mentioned that $n^{1/2}/(\log n)^{c_1} \leqslant f(n) \leqslant n^{1/2}/(\log n)^{c_2}$, where here and throughout this note $c_1, c_2, c_3, c_4, \ldots$ always denote absolute positive constants. Similarly, he remarked that $m^{1/3}/(\log m)^{c_3} \leqslant g(m) \leqslant m^{1/3}/(\log m)^{c_4}$, and added that the best possible values of the constants c_i in both estimates above are not known.

The asymptotic behaviour of the first function, $f(n)$, has been determined, up to a constant factor, by the main theorem of Kim [4], together with the earlier results of Ajtai et al., [1]. In [1] it is shown that any triangle-free graph on n vertices contains an independent set of size at least $\Omega(\sqrt{n}\sqrt{\log n})$. By repeatedly omitting such independent sets from such a graph we conclude that its chromatic number is at most $O(\sqrt{n}/\sqrt{\log n})$, showing that

$$f(n) \leqslant c_5 \frac{\sqrt{n}}{\sqrt{\log n}}. \tag{1}$$

Kim proved that this is tight, up to a constant factor, by showing that $f(n) \geqslant c_6 \sqrt{n}/\sqrt{\log n}$. To do so, he proved the existence of a triangle-free graph on n vertices with

E-mail address: nilli@math.tau.ac.il (A. Nilli)

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no independent set of size $c_7\sqrt{n}\sqrt{\log n}$. The total number of edges in his graph is $m = \Theta(n^{3/2}(\log n)^{1/2})$ and hence his construction also shows that $g(m) \geqslant \Omega(m^{1/3}/(\log m)^{2/3})$. In this brief note we prove that this is tight, up to a constant factor, namely, $g(m) \leqslant \mathrm{O}(m^{1/3}/(\log m)^{2/3})$. In order to do so, it suffices to prove the following.

Theorem 1. There exists an absolute positive constant c_7 so that the chromatic number of any triangle-free graph with at most m edges does not exceed $c_7 m^{1/3}/(\log m)^{2/3}$. □

To prove this theorem, we need the following result of Johansson.

Theorem 2 (Johansson [3]). There exists an absolute positive constant c_8 such that for any triangle-free graph G with maximum degree at most d, $\chi(G) \leqslant c_8(d/\log d)$. □

Proof of Theorem 1. Let $G=(V,E)$ be a triangle-free graph with at most m edges. Define $n = m^{2/3}/(\log m)^{1/3}$. If $|V| \leqslant n$ then, by (1)

$$\chi(G) \leqslant \mathrm{O}\left(\frac{\sqrt{n}}{\sqrt{\log n}}\right) = \mathrm{O}\left(\frac{m^{1/3}}{(\log m)^{2/3}}\right),$$

as needed. Otherwise, let V_1 be the n vertices of largest degree in G, and let G_1 be the induced subgraph of G on V_1. Then, by (1),

$$\chi(G_1) \leqslant \mathrm{O}\left(\frac{\sqrt{n}}{\sqrt{\log n}}\right) = \mathrm{O}\left(\frac{m^{1/3}}{(\log m)^{2/3}}\right).$$

Let G_2 be the induced subgraph of G on the rest of the vertices. Clearly, the maximum degree in G_2 is at most $d = 2m/(n+1) < 2m^{1/3}(\log m)^{1/3}$. By the result of Johansson (Theorem 2 above) it follows that the chromatic number of G_2 satisfies

$$\chi(G_2) \leqslant \mathrm{O}\left(\frac{d}{\log d}\right) = \mathrm{O}\left(\frac{m^{1/3}}{(\log m)^{2/3}}\right).$$

Since $\chi(G) \leqslant \chi(G_1) + \chi(G_2)$, this completes the proof. □

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