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Digital Sum Problems and Substitutions on a Finite Alphabet

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Let $(u_i)_{i\geqslant 1}$ be a fixed point for a substitution σ on a finite alphabet A and for $a \in A$, $f(a)$ a real number. We establish an asymptotic formula for $S(N) = \sum_{n<N} \sum_{i\leqslant n} f(u_i)$ in the case where the second largest eigenvalue of the substitution matrix equals one and under some additional hypothesis. More precisely $S(N) = \alpha N \log_\theta N + NF(N) + o(N)$, where the real number α depending on σ and f is explicitly determined and $\theta > 1$ is the largest eigenvalue of the substitution matrix; F is a continuous, nowhere differentiable (if $\alpha \neq 0$), real function such that $F(\theta x) = F(x)$ for all $x > 0$. Using the same method we prove a similar formula for $\sum_{n<N} s(n)$, $s(n)$ the sum of digits function with respect to the system of numeration associated with σ. These formulae generalize some recent work concerning digital sum problems.

1. INTRODUCTION

In recent years, various summation formulae related to digit expansions have been proved in various ways. For instance

(1) $\sum_{n<N} (-1)^{s(3n)} = N^\beta F(N) + O(1)$, where $s(n)$ = sum of digits in the binary expansion of n and $\beta = \log_4(3)$ (see Coquet [8]).

(2) $\sum_{n<N} (-1)^{r(n)} = N^{1/2} G(N)$, where $r(n)$ = number of blocks 11 in the binary expansion of n (see Brillhart, Erdos, and Morton [5] and, for a generalization to other blocks, [3]).

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(3) $\sum_{n<N} \Delta(n) = \frac{3}{8} N \log_4(N) + NH(N) + O(1)$, where $Q(n)$ counts those $k \leqslant n$ which are representable as sums of three squares, $\Delta(n) = Q(n) - \frac{5}{6}n$ (see Osbaldstin and Shiu [17]). The functions F, G, H are defined for any real $x > 0$, continuous and nowhere differentiable, and satisfy the equation $\Phi(4x) = \Phi(x)$ for all $x > 0$.

(4) $\sum_{n<N} s_q(n) = ((q-1)/2)\, N \log_q(N) + NF_q(N)$, where $s_q(n) =$ sum of the q-ary digits of n (q is an integer $\geqslant 2$) and F_q is continuous, satisfying $F_q(qx) = F_q(x)$ for all $x > 0$ (see [10]). This is the famous Delange formula and was generalized recently by P. J. Grabner and R. F. Tichy to digit expansions with respect to linear recurrences sequence G in the form

(5) $\sum_{n<N} s_G(n) = c_G N \log_\alpha(N) + NF_G(N) + O(\log N)$, where c_G is a constant, α the dominating characteristic root of G, and F_G a continuous function satisfying $F(\alpha x) = F(x)$ for all $x > 0$ (see [16, 18]).

In an earlier paper [13], we gave a generalization of (1) and (2) in the following way. Let A be a finite "alphabet" $A = \{1, 2, ..., d\}$; σ a substitution over A; $(u_n)_{n \geqslant 1}$ a sequence of elements of A which is invariant under σ; for each $a \in A$, $f(a)$ a real number; and

$$s^f(n) = \sum_{i \leqslant n} f(u_i).$$

Define $L_i(m) =$ number of occurrences of the letter i in the word m, and M (the "matrix of the substitution") $= (L_i(\sigma(j)))_{(i,j) \in A^2}$. Let $\{\theta_i / 1 \leqslant i \leqslant \delta\}$ be the set of the distinct eigenvalues of M such that $i \leqslant j \Rightarrow |\theta_i| \geqslant |\theta_j|$. Assume that

(H_1) M is primitive and $\theta = \theta_1 > 1$. As a consequence $i \geqslant 2 \Rightarrow |\theta_i| < \theta$ and there is a unique vector $\Lambda = (\lambda_i)_{i \in A}$, $M\Lambda = \theta\Lambda$, and $\sum_{i \in A} \lambda_i = 1$.

(H_2) $\theta_2 \in \mathbb{R}_+$, $\theta_2 > 1$, $i \geqslant 3 \Rightarrow |\theta_i| < \theta_2$.

Let m be the integer such that $m+1$ is the order of θ_2 in the minimal polynomial of M, and $\beta = \log_\theta(\theta_2)$. Then there exists a continuous function F, defined for $x > 0$ such that

(i) $s^f(N) = (\Lambda \cdot f)\, N + (\log_\theta N)^m N^\beta F(N) + o((\log N)^m N^\beta)$

(ii) $x > 0 \Rightarrow F(\theta x) = F(x)$.

Moreover F is Hölder continuous with exponent β, and except for the case $F \equiv 0$, F is nowhere differentiable. The fact that the sums in (1) and (2) are of the form $s^f(N)$ can be found respectively in [7] and [6].

In the present paper, we want to study the "double sum" $\sum_{n<N} s^f(n)$ in the case where (H_1) is true and (H_2) is replaced by

(H_2') $\theta_2 = 1$, $i \geqslant 3 \Rightarrow |\theta_i| < 1$.

Without loss of generality, we may suppose that $\Delta \cdot f = 0$ (if this is not the case we take $f'(i) = f(i) - \Delta \cdot f$ in place of f). For technical reasons we assume that there exists a base of $\mathbb{C}^d$ with eigenvectors of the matrix ${}^tM$. Under the above hypothesis, the main result is the following:

THEOREM. *There exist a constant α and a continuous function F nowhere differentiable if $\alpha \neq 0$ defined for $x > 0$ such that*

(i) $\sum_{n<N} s^f(n) = \alpha N \log_\theta N + NF(N) + o(N)$

(ii) $x > 0 \Rightarrow F(\theta x) = F(x)$.

More precisely

$$\alpha = \theta^{-1} \sum_{\substack{a \in A \\ mc \leqslant \sigma(a)}} \lambda_a f_2(m)\, \varepsilon(c),$$

where $f = \sum_{i=1}^{\delta} f_i$ with f_i an eigenvector for ${}^tM$ and $\theta_i, \varepsilon(a) = \lim_{n \to \infty} \theta^{-n} |\sigma^n(a)|$ ($|m| =$ length of the word m), and the relation $mc \leqslant \sigma(a)$ means that the summation is extended to all the $(m, c) \in A^* \times A$ such that the word mc is a prefix of the word $\sigma(a)$.

For instance the sequence $Q(n)$ in (3) is of the form $s^f(n)$ with $A = \{1, 2, ..., 6\}$; $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 14$, $\sigma(4) = 54$, $\sigma(5) = 62$, $\sigma(6) = 52$; $f(1) = f(2) = f(3) = f(5) = 1$, $f(4) = f(6) = 0$; and $(u_n)_{n \geqslant 1} = \lim_{k \to \infty} \sigma^k(1)$ (cf. [7, p. 172–173).

The matrix M of σ has $\{2, \pm 1, 0\}$ as eigenvalues and hence does not satisfy (H'_2). But if we consider σ^2 instead of σ, with the same f, $s^f(n)$ remains the same and H'_2 is true (in H_1, θ is now 4). Moreover $\Delta \cdot f = \frac{5}{6}$ and if $f' = \frac{1}{6}(1, 1, 1, -5, 1, -5)$ we have $\Delta(n) = s^{f'}(n)$. One has (for σ^2 and f') $f_2 = \frac{1}{3}(2, -1, -1, -4, 2, -1)$, $\forall a, a \in A, \varepsilon(a) = 1$, and $\alpha = \frac{3}{8}$ in accordance with [17].

As another class of application of our theorem, we mention that if α is a quadratic algebraic number, the sequence

$$\chi(n, \alpha) = \begin{cases} 0 & \text{if } \ 0 \leqslant \operatorname{frac}(n\alpha) < \frac{1}{2} \\ 1 & \text{if } \ \frac{1}{2} \leqslant \operatorname{frac}(n\alpha) < 1 \end{cases}$$

can be described by a sequence $\varphi(u_1)\, \varphi(u_2) \cdots$, where (u_n) is a fixed point for an appropriate substitution on a finite alphabet A and $\varphi: A \to \{0, 1\}^*$ (see [1, 15, 19]).

For instance, if $\alpha = (\sqrt{3} - 1)/2$ one has $A = \{1, 2, 3\}$, $\sigma(1) = 13$, $\sigma(2) = 13223$, $\sigma(3) = 1323$, $\varphi(1) = 0$, $\varphi(2) = 011$, $\varphi(3) = 01$, $(u_n)_{n \geqslant 1} =$ 1313231313223..., and $(\chi(n, \alpha)) =$ 001001011... (see [19]).

The second largest eigenvalues of the substitutions matrices are ± 1. These sequences and some more general ones are important in recent works in mathematical physics (see [1, 2, 5]).

In Section 2, we give some results concerning systems of numeration associated with a substitution; some representations of the integers with respect to linear recurrence sequences appear as particular cases of our systems of numeration.

In Section 3, we give an expression of $S(N)$ related to the "digits" of N in the system of numeration described in Section 2, using a summation by column method.

In Section 4, we establish the continuity and the nowhere differentiability of the function F. For this we investigate for the first time the properties of "self-similarity" of F; this method is very distinct from that used, for instance, in [16, 18] and allows more generality.

In Section 5, we show how the computation developed in the preceding sections gives some results concerning the "sum of digits functions."

To conclude, let us note that, using another method, some results concerning the sums $\sum_{n<N} (s^J(n) - \alpha \log_\theta n)^k$ ($k \in \mathbb{N}$) were proved by the first author of this paper [12].

2. Systems of Numeration Associated with a Substitution

Here we recall some results proved in [13] and prove some useful new results. A^* is the set of words on A and ω the empty word; for $m \in A^*$, $|m| =$ length of m. If m, m' are in A^* the relation "m is a prefix of m'" means that there exists a word u such that $m' = mu$ and is written $m \leqslant m'$; $m < m' \Leftrightarrow m \leqslant m'$ and $m \neq m'$. σ is a morphism from A^* into itself ($\sigma(mm') = \sigma(m)\,\sigma(m')$) such that $1 < \sigma(1)$, and $u = (u_i)_{i \geqslant 1}$ is $\lim_{n \to \infty} \sigma^n(1)$, i.e., the fixed point for σ such that $u_1 = 1$.

2.1. *Representations of Integers*

Definition. A sequence $(m_i, a_i)_{i=0,1,\ldots,n}$ in $A^* \times A$ is a-admissible ($a \in A$) iff

(i) $m_n a_n \leqslant \sigma(a)$

(ii) $1 \leqslant i \leqslant n \Rightarrow m_{i-1} a_{i-1} \leqslant \sigma(a_i)$.

Theorem 2.1.1. *Let N be an integer, $N \geqslant 1$. Then there exist a unique integer $n = n(N)$ and a unique* 1*-admissible sequence $(m_i, a_i)_{i=0,\ldots,n}$ such that $m_n \neq \omega$ and*

$$u_1, u_2, \ldots, u_N = \sigma^n(m_n) \cdots \sigma^0(m_0). \tag{2.1}$$

For the proof see [13].

For instance if $A = \{1, 2, 3\}$, $\sigma(1) = 123$, $\sigma(2) = 31$, $\sigma(3) = 22$, and $N = 13$ (in decimal system!), $n = 2$, $(m_2, a_2) = (12, 3)$, $(m_1, a_1) = (\omega, 2)$, $(m_0, a_0) =$

(3, 1). Note that the digits m_i belong to a finite set, the set of the proper prefixes of the words $\sigma(a)$, $a \in A$, but are in general not independent of each other. For instance, if $A = \{1, 2\}$, $\sigma(1) = 12$, $\sigma(2) = 1$, one has $m_i \in \{\omega, 1\}$ and in an "admissible writing" (2.1), $(m_{i+1}, m_i) \neq (1, 1)$ for all $i \leqslant n-1$. Clearly in this example the representation

$$N = |u_1 \cdots u_N| = \sum_{j=0}^{n} |\sigma^j(m_j)|$$

is the "normal" representation of N in the Fibonacci base. We now give a more general case in which the representation (2.1) leads to the ordinary reresentation (in the sense of [14]).

PROPOSITION 2.1.2. *Let d be an an integer, $d \geqslant 2$, $a_1 \cdots a_d$ be integers with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_d > 0$, and:*

$$\sigma(i) = 1^{a_i}(i+1) \qquad (i \neq d)$$

$$\sigma(d) = 1^{a_d}.$$

Let $N = \sum_{j=0}^{n} |\sigma^j(m_j)|$ be the admissible representation and for $j \in \mathbb{N}$ define $G_j = |\sigma^j(1)|$, $\varepsilon_j = |m_j|$. Then

(i) $G_0 = 1$, $1 \leqslant k \leqslant d-1 \Rightarrow G_k = a_1 G_{k-1} + \cdots + a_k G_0 + 1$ *and* $k \geqslant 0 \Rightarrow G_{k+d} = a_1 G_{k+d-1} + a_2 G_{k+d-2} + \cdots + a_d G_k$;

(ii) $N = \sum_{j=0}^{n} \varepsilon_j G_j$, *where* $G_n \leqslant N < G_{n+1}$ *and* $\varepsilon_j = [N_j/G_j]$, $N_n = N$, $1 \leqslant j \leqslant n \Rightarrow N_{j-1} = N_j - \varepsilon_j G_j$.

In other words, we obtain the G-ary *representation of N with digits ε_j and initial canonical values (cf. [16, 18]).*

Proof. (i) Immediate using $\sigma^k(i) = \sigma^{k-1}(\sigma(i))$ and the definition of $\sigma(i)$ for $i \in A$.

(ii) We remember that for an admissible sequence $(m_j, a_j)_{j=0,\dots,n}$ we have for k, $0 \leqslant k \leqslant n$, $\sum_{j=0}^{k} |\sigma^j(m_j)| < |\sigma^k(m_k a_k)|$ (see [13, Lemma 1.1]). Moreover, if $1 \leqslant j \leqslant d-1$, $|\sigma^n(j)| = a_j |\sigma^{n-1}(1)| + |\sigma^{n-1}(j+1)|$, and hence, using $a_j \geqslant a_{j+1}$, $|\sigma^n(j)| \geqslant |\sigma^n(j+1)|$. Furthermore, $|\sigma^n(d)| = a_d |\sigma^{n-1}(1)| < |\sigma^n(d-1)|$. Thence $1 \leqslant j \leqslant d \Rightarrow |\sigma^n(j)| \leqslant G_n$, and we have for k, $0 \leqslant k \leqslant n$,

$$\sum_{j=0}^{k} \varepsilon_j G_j = \sum_{j=0}^{k} |\sigma^j(m_j)| < |\sigma^{k+1}(a_{k+1})| \leqslant G_{k+1}.$$

The relations (ii) are then very easy to prove.

Remark. Part (ii) of Proposition 2.1.2 can be false if $a_1 < a_2$. For instance $d = 2$, $a_1 = 1$, $a_2 = 3$, $N = 4 = |\sigma^1(1)| + |\sigma^0(11)|$, and $G_1 = 2$, $G_2 = 15$.

Now, we give a "technical lemma" useful in the next section. This lemma generalizes for the substitution of an elementary well-known fact about numeration systems in an integer base g. Namely if c_i are digits and

$$\sum_{h=k}^{m} c_h g^h \leqslant n < g^k + \sum_{h=k}^{m} c_h g^h, \qquad \text{then the } k\text{th digit of } n \text{ is } c_k.$$

LEMMA 2.1.3. (i) *Let* $u_1 u_2 \cdots u_N = \sigma^n(m_n) \cdots \sigma^0(m_0)$ *be the admissible representation of* N. *Then* $a_0 = u_{N+1}$.

(ii) *Let* $(m_i, a_i)_{0 \leqslant i \leqslant v}$ *be a* 1*-admissible sequence*; k *be an integer,* $0 \leqslant k \leqslant v$; m *and* $m' \in A^*$; $b, b' \in A$ *such that* $mb \leqslant \sigma^{v-k}(1)$, $m'b' \leqslant \sigma(b)$; *and* $v \in A^*$ *such that* $\sigma^{k+1}(m)\,\sigma^k(m') \leqslant v < \sigma^{k+1}(m)\,\sigma^k(m'b')$, $v = \sigma^n(m_n(v)) \cdots \sigma^0(m_0(v))$ *the admissible representation of* v. *Then* $m_k(v) = m'$.

Proof. (i) The definition of a 1-admissible sequence implies that $\sigma^n(m_n) \cdots \sigma^0(m_0)\, a_0 \leqslant \sigma^{n+1}(1)$. Thus $a_0 = u_{N+1}$.

(ii) First, it is easy to prove that if $a \in A$, $t \in A^*$, $t < \sigma^k(a)$, then there exists and a-admissible sequence $(m'_i, a'_i)_{0 \leqslant i < k}$ such that $t = \sigma^{k-1}(m'_{k-1}) \cdots \sigma^0(m'_0)$ (same proof as that of Theorem 1.5 in [13]).

The hypotheses of (ii) imply that $v = \sigma^{k+1}(m)\,\sigma^k(m')\,t$, where $t \in A^*$, $t < \sigma^k(b')$, and $m = \sigma^{v-k-1}(m'_v) \cdots \sigma^0(m'_{k+1})$, where $(m'_i, a'_i)_{k < i \leqslant v}$ is 1-admissible.

Now, using the above expression of t in which $a = b'$, we define $m'_k = m'$, $a'_k = b'$ and we claim that $(m'_i, a'_i)_{0 \leqslant i \leqslant v}$ is 1-admissible. Indeed by (i) of this lemma $a'_{k+1} = b$, and thence $m'_k a'_k = m'b' \leqslant \sigma(b) = \sigma(a'_{k+1})$.

Thus $v = \sigma^v(m'_v) \cdots \sigma^0(m'_0)$ and by unicity of 1-admissible writing one has $m_k(v) = m'$.

2.2. *Representations of Real Numbers*

For the next theorem we suppose that θ, the maximum eigenvalue of M, is such that $\theta > 1$ and that for any $a \in A$, the limit $\varepsilon(a) = \lim_{n \to \infty} \theta^{-n} |\sigma^n(a)|$ exists and $\varepsilon(a) > 0$. We write $\varepsilon(\omega) = 0$ and $\varepsilon(a_1 \cdots a_k) = \sum_{i=1}^{k} \varepsilon(a_i)$; remark that for any $a \in A$, $\varepsilon(\sigma(a)) = \theta\varepsilon(a)$.

THEOREM 2.2.1. *If* $a \in A$, $x \in [0, \varepsilon(a)[$, *there exists an unique sequence* $(m_i, a_i)_{i \geqslant 1}$ *of elements of* $A^* \times A$ *such that*

(i) $x = \sum_{i \geqslant 1} \varepsilon(m_i)\,\theta^{-i}$,

(ii) $m_1 a_1 \leqslant \sigma(a)$, $i \geqslant 2 \Rightarrow m_i a_i \leqslant \sigma(a_{i-1})$,

(iii) $\forall I \in \mathbb{N}, \exists i > I, m_i a_i \neq \sigma(a_{i-1})$.

For the proof see [13].

3. A First Expression for $S(N)$

In this section we give an expression of $S(N)$ related to the digits m_i of N.

3.1. *Some Notations*

In Sections 3 and 4, f is a vector $(f(a))_{a \in A}$ such that $f \cdot A = 0$; we write $f(\omega) = 0$ and $f(a_1 \cdots a_k) = \sum_{i=1}^{k} f(a_i)$. $\{\theta_i / 1 \leqslant i \leqslant \delta\}$ denotes the set of distinct eigenvalues of the matrix M of σ; for sake of simplicity we assume that $\theta_i \in \mathbb{R}$. We assume that $\theta_1 = \theta > 1$, $\theta_2 = 1$, $j \geqslant 3 \Rightarrow |\theta_j| < 1$ and define $f_i, \varepsilon_i, \lambda_i(a)$ for $1 \leqslant i \leqslant \delta$ by $f = \sum_{i=1}^{\delta} f_i$, ${}^t M f_i = \theta_i f_i$ $(i \geqslant 2)$, ${}^t(1, 1, ..., 1) = \sum_{i=1}^{\delta} \varepsilon_i$, ${}^t M \varepsilon_i = \theta_i \varepsilon_i$,

$$ {}^t(0, ..., 0, 1, 0, ..., 0) = \sum_{i=1}^{\delta} \lambda_i(a) $$

$$ (1 \text{ in the place of } a), \qquad {}^t M \lambda_i(a) = \theta_i \lambda_i(a). $$

The components of the vector $\lambda_i(a)$ are written $\lambda_i(a, b)$ $(b \in A)$. The components of f_i and ε_i are written, respectively, $f_i(a)$ and $\varepsilon_i(a)$.

Lemma 3.1.1. (i) *For any* $n \in \mathbb{N}$, $a \in A$,

$$ f(\sigma^n a)) = \sum_{i=1}^{\delta} f_i(a)\, \theta_i^n, \qquad f_1(a) = 0, $$

$$ |\sigma^n(a)| = \sum_{i=1}^{\delta} \varepsilon_i(a)\, \theta_i^n, $$

$$ L_b(\sigma^n(a)) = \sum_{i=1}^{\delta} \lambda_i(a, b)\, \theta_i^n \qquad \text{for} \quad b \in A. $$

(ii) $\varepsilon_1(a) = \lim_{n \to \infty} \theta^{-n} |\sigma^n(a)|$. *We write* $\varepsilon_1(a) = \varepsilon(a)$.

(iii) $(\varepsilon(a))^{-1} \lambda_1(a, b) = \lim_{n \to \infty} |\sigma^n(a)|^{-1} L_b(\sigma^n(a)) = \lambda_b$ (λ_b *defined in* (H_1)).

Proof. (i) We have $f(\sigma^{n+1}(a))_{a \in A} = {}^t M f(\sigma^n(b))_{b \in A}$; thence the first relation in (i) is a direct consequence of the properties of f_i. The proof is the same for the other relations. $f_1 \equiv 0$ is a consequence of $f \cdot A = 0$ (see [13, Lemma 2.2]).

(ii) Consequence of the second relation in (i).

(iii) The first relation is an easy consequence of (i). Moreover the two vectors $(\varepsilon(a))_{a \in A}$ and $(\lambda_1(a, b))_{a \in A}$ are eigenvectors for ${}^t M$ and for θ and then they are homothetic. Thus $\varepsilon(a) > 0$ and $(\varepsilon(a))^{-1} \lambda_1(a, b) = \lambda'_b$.

But one can show that $(\lambda'_b)_{b\in A}$ is an eigenvector for M and θ such that $\sum_{b\in A}\lambda'_b=1$. Thus $\lambda'_b=\lambda_b$ (by (H_1) in the Introduction).

3.2. *Expression for* $S(N)$

Now $N\geqslant 1$ is a *fixed integer*; let $(m_i, a_i)_{0\leqslant i\leqslant v}$ be its 1-admissible representation $(m_v\neq\omega)$.

We want to compute $S(N)=\sum_{n<N} f(u_1\cdots u_n)$. If $n<N$ we write $(m_i(n), a_i(n))_{0\leqslant i\leqslant v}$ as the 1-admissible representation for n, (possibly with $m_v(n)=\omega$). For $0\leqslant k\leqslant v$ we define $S_k=\sum_{n<N} f(\sigma^k(m_k(n)))$. Then, clearly, $S(N)=\sum_{0\leqslant k\leqslant v} S_k$.

LEMMA 3.2.1. *Let* $S'_v=0$, *and for* $0\leqslant k\leqslant v-1$,

$$S'_k=\sum_{(m,c)\in E(k)} f(\sigma^k(m))\,|\sigma^k(c)|$$

with $N_k=\sigma^{v-k-1}(m_v)\cdots\sigma^0(m_{k+1})$ *and*

$$E(k)=\{(m,c)\in A^*\times A/\exists(m',b)\in A^*\times A,\, m'b\leqslant N_k,\, mc\leqslant\sigma(b)\}$$

$$S''_k=\sum_{mb\leqslant m_k} f(\sigma^k(m))\,|\sigma^k(b)|$$

$$S'''_k=f(\sigma^k(m_k))\,|\sigma^{k-1}(m_{k-1})\cdots\sigma^0(m_0)|.$$

Then, for $0\leqslant k\leqslant v$, $S_k=S'_k+S''_k+S'''_k$.

Proof. Each $n<N$ belongs to exactly one of the segments of $\mathbb{N}$,

$$[|\sigma^{k+1}(m')\,\sigma^k(m)|,\, |\sigma^{k+1}(m')\,\sigma^k(m)|+|t|[,$$

with $m'b\leqslant N_k$, $mc\leqslant\sigma(b)$, $t=\sigma^k(c)$; or $m'=N_k$, $mb\leqslant m_k$, $t=\sigma^k(c)$; or $m'=N_k$, $m=m_k$, $t=\sigma^{k-1}(m_{k-1})\cdots\sigma^0(m_0)$.

By Lemma 2.1.3, for such an n, $m_k(n)=m$ in the two first cases because $N_k\leqslant\sigma^{v-k}(1)$, and by unicity of the 1-admissible representation, in the last case, $m_k(n)=m_k$.

LEMMA 3.2.2. *Let* E *be* $\{(b, m, c)\in A\times A^*\times A/mc\leqslant\sigma(b)\}$ *and*

$$\alpha=\theta^{-1}\sum_{(b,m,c)\in E}\lambda_b f_2(m)\,\varepsilon(c).$$

Then, there exists for each $a\in A$ *a real number* $\mu(a)$ *such that if* $\mu(a_1\cdots a_n)=\sum_{1\leqslant i\leqslant n}\mu(a_i)$, *one has*

$$\sum_{k=0}^{v-1} S'_k=\alpha\sum_{i=1}^{v} i\varepsilon(m_i)\,\theta^i+\sum_{i=1}^{v}\mu(m_i)\,\theta^i+O(v^2+|\theta\theta_3|^v).$$

Proof. We have, by the definition of S'_k in Lemma 3.2.1 and the definition of E, for $0 \leqslant k \leqslant v-1$,

$$S'_k = \sum_{(b,m,c) \in E} L_b(N_k)\, f(\sigma^k(m))\, |\sigma^k(c)|.$$

But $L_b(N_k) = \sum_{i=k+1}^{v} L_b(\sigma^{i-k-1}(m_i))$. Now, we can apply Lemma 3.1.1, which leads to

$$S'_k = \sum_{i=k+1}^{v} \sum_{\substack{j_1, j_2, j_3 \in \Delta^3 \\ j_2 \neq 1}} (\theta_{j_1}^{i-k-1})(\theta_{j_2}\theta_{j_3})^k\, \alpha(j_1, j_2, j_3, m_i)$$

with $\Delta = \{1, ..., \delta\}$ and $\alpha(j_1, j_2, j_3, w) = \sum_{(b,m,c) \in E} \lambda_{j_1}(w, b)\, f_{j_2}(m)\, \varepsilon_{j_3}(c)$. Now, we remark that

$$\sum_{k=0}^{i-1} \theta_{j_1}^{i-k-1}\, (\theta_{j_2}\theta_{j_3})^k = [(\theta_{j_2}\theta_{j_3})^i - \theta_{j_1}^i](\theta_{j_2}\theta_{j_3} - \theta_{j_1})^{-1}$$

$$\text{if} \quad \theta_{j_1} \neq \theta_{j_2}\theta_{j_3}, \text{ and} = i\theta_{j_1}^{i-1} \text{ if } \theta_{j_1} = \theta_{j_2}\theta_{j_3}.$$

If $j_1 \neq 1$, $(j_2, j_3) \neq (2, 1)$, this expression is in $O(i + |\theta\theta_3|^i)$. If $(j_1, j_2, j_3) = (1, 2, 1)$, we have by Lemma 3.1.1 and the definition of α,

$$\alpha(1, 2, 1, w) = \alpha\theta\varepsilon(w).$$

Thus, the term of $\sum_{k=0}^{v-1} S'_k$ corresponding to $(j_1, j_2, j_3) = (1, 2, 1)$ is the first term of the result in Lemma 3.2.2. For the terms corresponding to the cases $j_1 = 1$, $(j_2, j_3) \neq (2, 1)$, we can define

$$\mu'(a) = \varepsilon(a) \sum_{\substack{(j_2, j_3) \neq (2,1) \\ j_2 \neq 1}} (\theta - \theta_{j_2}\theta_{j_3})^{-1} \sum_{(b,m,c) \in E} \lambda_b f_{j_2}(m)\, \varepsilon_{j_3}(c),$$

and for the terms corresponding to $j_1 \neq 1$, $(j_2, j_3) = (2, 1)$, we define

$$\mu''(a) = \sum_{j \neq 1} (\theta - \theta_j)^{-1} \sum_{(b,m,c) \in E} \lambda_j(a, b)\, f_2(m)\, \varepsilon(c).$$

The lemma is proved by letting $\mu(a) = \mu'(a) + \mu''(a)$.

THEOREM 3.2.3. (Same notations as that in Lemma 3.2.2).

$$S(N) = \alpha \sum_{i=1}^{v} i\varepsilon(m_i)\, \theta^i$$

$$+ \sum_{i=0}^{v} [\alpha(m_i) + f_2(M_i)\, \varepsilon(m_i)]\, \theta^i + O(v^2 + |\theta\theta_3|^v)$$

with $M_i = m_{i+1} m_{i+2} \cdots m_\nu$, $M_\nu = \omega$, *and for each word* m, $\alpha(m)$ *is a real number.*

Proof. Now we compute $\sum_{k=0}^{\nu} (S_k'' + S_k''')$. By Lemma 3.1.1

$$f(\sigma^k(m)) = f_2(m) + O(|\theta_3|^k)$$
$$|\sigma^k(b)| = \varepsilon(b)\,\theta^k + O(1),$$

and the theorem is proved with, for $m \in A^*$,

$$\alpha(m) = \mu(m) + \sum_{m'b \leqslant m} f_2(m')\,\varepsilon(b).$$

Remark.

$$\sum_{i=1}^{\nu} i\varepsilon(m_i)\,\theta^i = N \log_\theta N - \log_\theta\left(\sum_{i=0}^{\nu} \varepsilon(m_i)\,\theta^{i-\nu-1}\right)$$
$$- \sum_{i=0}^{\nu} (\nu+1-i)\,\varepsilon(m_i)\,\theta^i + O(\nu^2).$$

Indeed, by Lemma 3.1.1(i),

$$N = \sum_{i=0}^{\nu} \varepsilon(m_i)\,\theta^i + O(\nu), \qquad \text{thence:}$$

$$\log_\theta N = \nu + 1 + \log_\theta\left(\sum_{i=0}^{\nu} \varepsilon(m_i)\,\theta^{i-\nu-1}\right) + O(\nu\theta^{-\nu}).$$

4. A Second Expression for $S(N)$

Now we prove the main result of this paper. First we define a family of functions F_a, $a \in A$, adapted to the problem. Then we establish relations between these functions and we can prove the continuity of F_1 and its nowhere differentiability when $\alpha \neq 0$.

4.1. *Definition and Properties of the Functions* $F_a(x)$

Definition. Let a be a letter of A. For $x \in [0, \varepsilon(a)[$, let $x = \sum_{i=1}^{\infty} \varepsilon(m_i)\,\theta^{-i}$ be the a-admissible representation (cf. Section 2.2) and α, $\alpha(m)$, be as defined in Theorem 3.2.3. We define

$$F_a(x) = -\alpha x \log_\theta x + \sum_{i=1}^{\infty} (-i\alpha\varepsilon(m_i)$$
$$+ \alpha(m_i) + \varepsilon(m_i) f_2(M_i'))\,\theta^{-i}$$

with $x \log_\theta x = 0$ for $x = 0$ and $M_1' = \omega$, $M_i' = m_1 m_2 \cdots m_{i-1}$, if $i \geqslant 2$.

LEMMA 4.1.1. (i) *$a, b \in A$, $b \leqslant \sigma(a)$, $x \in [0, \varepsilon(b)[$. Then $x\theta^{-1} \in [0, \varepsilon(a)[$ and $F_a(x\theta^{-1}) = \theta^{-1}F_b(x)$*

(ii) *Let be x as in the above definition, and for $k \geqslant 1$,*

$$x_k = \sum_{i=1}^{k} \varepsilon(m_i)\,\theta^{-i}, \qquad t_k = x - x_k.$$

Then

$$F_a(x) = F_a(x_k) + \theta^{-k}F_{a_k}(\theta^k t_k) - \alpha(x \log_\theta x - x_k \log_\theta x_k)$$
$$+ \alpha t_k \log_\theta t_k + f_2(M'_{k+1})\,t_k.$$

Proof. (i) $x < \varepsilon(b) \leqslant \varepsilon(\sigma(a)) = \theta\varepsilon(a) \Rightarrow x\theta^{-1} < \varepsilon(a)$. If the b-admissible representation of x is $x = \sum_{i \geqslant 1} \varepsilon(m_i)\,\theta^{-i}$, one has $m_1 a_1 \leqslant \sigma(b)$, and thus the a-representation of $x\theta^{-1}$ is $\sum_{i \geqslant 1} \varepsilon(m_{i-1})\,\theta^{-i}$ with $m_0 = \omega$, $a_0 = b$, and an easily calculation proves (i).

(ii) We have $\theta^k t_k = \sum_{i \geqslant 1} \varepsilon(m_{k+i})\,\theta^{-i}$, $\theta^k t_k \in [0, \varepsilon(a_k)[$ (see [13, Lemma 3.1]), with $m_{k+i}a_{k+i} \leqslant \sigma(a_{k+i-1})$ for $i \geqslant 1$, and thus the right-hand side of the above equality is the a_k-admissible representation of $\theta^k t_k$. Here, too, we omit the simple computation which leads to (ii).

LEMMA 4.1.2. *If $a \in A$, $x \in [0, \varepsilon(a)[$, x_k as in Lemma 4.1.1, then*

(i) $F_a(x) = O(1)$

(ii) $|F_a(x) - F_a(x_k)| = O(k\theta^{-k})$.

Proof. (i) Clearly $\varepsilon(m_i)$ and $\alpha(m_i)$ are in $O(1)$, *and* $f_2(M'_i)$ is in $O(i)$. These imply (i), by the definition of $F_a(x)$.

(ii) We use the notation and result of Lemma 4.1.1(ii). First $t_k = O(\theta^{-k})$, and by (i) of Lemma 4.1.2, $\theta^{-k}F_{a_k}(\theta^k t_k) = O(\theta^{-k})$. Secondly $R_k = x \log_\theta x - x_k \log_\theta x_k = t_k(\log\theta)^{-1}(1 + \log c)$ with $x_k < c < x$. But, if $x_k \neq 0$, $x_k \geqslant \theta^{-k} \inf\{\varepsilon(a)/a \in A\}$ and R_k is in $O(k\theta^{-k})$. If $x_k = 0$, $R_k - t_k \log_\theta t_k = 0$. Then, using $t_k = O(\theta^{-k})$, (ii) is proved.

LEMMA 4.1.3. *If $x, y \in [0, \varepsilon(1)[$ have a finite 1-representation*

$$x = \sum_{i=1}^{k} \varepsilon(m_i)\,\theta^{-i}, \qquad y = \sum_{i=1}^{k} \varepsilon(n_i)\,\theta^{-1},$$

then

$$|F_1(y) - F_1(x)| = O(|y - x|\,k + k^2\theta^{-k} + |\theta_3|^k).$$

Proof. We define the words u and v such that $u=\sigma^{k-1}(m_1)\cdots\sigma^0(m_k)$, $v=\sigma^{k-1}(n_1)\cdots\sigma^0(n_k)$ and the integers $N=|u|$, $N'=|v|$. Using Theorem 3.2.3 and the remark just following it, the definition of F_1, and the relations $N=\theta^k x+O(k)$, $N'=\theta^k y+O(k)$, we have

$$S(N)=\alpha N\log_\theta N+\theta^k F_1(x)+O(k^2+|\theta\theta_3|^k)$$

and the same relation with N' and y, respectively, in place of N and x. But if, for instance, $N<N'$, $S(N')-S(N)=\sum_{N\leqslant n<N'} f(u_1\cdots u_n)=O((N'-N)\log N')$ because $f(u_1\cdots u_n)=\sum_{h=0}^{k-1} f(\sigma^h m_h(n))$ and by Lemma 3.1.1(i). Moreover $N'-N=\theta^k(y-x)+O(k)$, $N'\log_\theta N'-N\log_\theta N=O((N'-N)\log N')$, and $\log N'=O(k)$. All these relations imply the lemma.

Lemma 4.1.4. *If $x, y\in[0,\varepsilon(1)[$, $x\neq y$, then*

$$|F_1(y)-F_1(x)|=O(|y-x|\,(\log|y-x|)^2+|y-x|^\beta)$$

with $\beta=-\log_\theta|\theta_3|$ if $\theta_3\neq 0$, and the last term in O disappears if $\theta_3=0$. Thus F_1 is a continuous function.

Proof.

$$x=\sum_{i=1}^{\infty}\varepsilon(m_i)\,\theta^{-i},\qquad y=\sum_{i=1}^{\infty}\varepsilon(n_i)\,\theta^{-1}\qquad \text{(1-admissible writing)}.$$

Let k be the integer such that $\theta^{-1}\leqslant|y-x|\,\theta^k<1$, and

$$x_k=\sum_{i=1}^{k}\varepsilon(m_i)\,\theta^{-i},\qquad y_k=\sum_{i=1}^{k}\varepsilon(n_i)\,\theta^{-i}.$$

We have $|F_1(y)-F_1(x)|\leqslant|F_1(y)-F_1(y_k)|+|F_1(y_k)-F_1(x_k)|+|F_1(x_k)-F_1(x)|$. Moreover

$$\theta^{-k}=O(|y-x|),\ k=O(\mathrm{Log}\,|y-x|),\ y_k-x_k=O(|y-x|),$$
$$\text{and}\quad |\theta_3|^k=O(|y-x|)^\beta.$$

Then, by using Lemma 4.1.2 and 4.1.3, we prove 4.1.4.

Lemma 4.1.5. *If $\alpha\neq 0$, F_a is a nowhere differentiable function on $]0,\varepsilon(a)[$.*

Proof. Clearly, it suffices to show that the function $G_a(x)=F_a(x)+\alpha x\log_\theta x$ is nowhere differentiable.

First, if $b \in A$ for i large enough, $|\sigma^i(b)| \geqslant 2$ (by irreducibility of M) and for infinitely many i there exists $c_i \in A$ such that the number $\tau_i = \varepsilon(c_i)\,\theta^{-i}$ has, for b-admissible representation, $\tau_i = \sum_{j=1}^{\infty} \varepsilon(m_j)\,\theta^{-j}$ with $m_i = c_i$ and $m_j = \omega$ for $j \neq i$. We have, by definition of F_b, $G_b(\tau_i) = [-i\alpha\varepsilon(c_i) + \alpha(c_i)]\,\theta^{-i}$; thus $\lim_{i\to\infty} \tau_i^{-1} G_b(\tau_i) = \infty$ (using $\alpha \neq 0$). Now suppose the existence of $a \in A$ and $x \in]0, \varepsilon(a)[$ such that G_a is differentiable for x. Let $(m_j, a_j)_{j\geqslant 1}$ be the n-admissible sequence which represents x. There exist $b \in A$ and J infinite subset of $\mathbb{N}$ such that $k \in J \Rightarrow a_k = b$. Let t be an arbitrary number in $]0, \varepsilon(b)[$, and for $k \in J$, $x_k = \sum_{j=1}^{k} \varepsilon(m_j)\,\theta^{-j}$; $x'_k = x_k + t\theta^{-k}$. Then $x'_k \in]0, \varepsilon(a)[$ and, by Lemma 4.1.1(ii),

$$[G_a(x'_k) - G_a(x_k)](x'_k - x_k)^{-1} = t^{-1}G_b(t) - \alpha k + f_2(m_1 \cdots m_k).$$

But, one can prove easily that the limit for $k \in J$, $k \to \infty$, of the lhs of the above relation is precisely $G'_a(x)$. Then

$$t^{-1}G_b(t) = G'_a(x) - \lim_{\substack{k\to\infty \\ k\in J}} [f_2(m_1 \cdots m_k) - \alpha k]$$

is independent of t, i.e., $G_b(t) = Kt$, in contradiction with the existence of $\tau_i \to 0$ such that $\lim_{i\to\infty} \tau_i^{-1} G_b(\tau_i) = \infty$.

Remark. The functions F_a could be differentiable if $\alpha = 0$. For instance, if $A = \{1, 2\}$, $\sigma(1) = 121$, $\sigma(2) = 212$, $f(1) = -f(2) = 1$, one has $S(N) = N/2 + O(1)$ and then $F_1(x) = x/2$.

4.2. *The Main Result*

THEOREM 4.2.1. *With the hypotheses and notation of Section* 3, *there exists a continuous function G defined for* $x > 0$ *such that*

(i) $x > 0 \Rightarrow G(\theta x) = G(x)$

(ii) $S(N) = \alpha N \log_\theta N + NG(N) + o(N)$.

Proof. The property $0 \leqslant x < \varepsilon(1) \Rightarrow F_1(x\theta^{-1}) = \theta^{-1}F_1(x)$ (Lemma 4.1.1(i) and $1 \leqslant \sigma(1)$) and the continuity of F_1 imply the existence of a continuous function F defined for all $x \geqslant 0$ such that F coincides with F_1 on $]0, \varepsilon(1)[$ and $F(\theta x) = \theta F(x)$ everywhere.

Now, if $N = \sum_{i=0}^{\nu-1} |\sigma^i(m_{\nu-i})|$ (1-admissible representation), one has (see the proof of Lemma 4.1.3)

$$S(N) = \alpha N \log_\theta N + \theta^\nu F\left(\sum_{i=1}^{\nu} \varepsilon(m_i)\,\theta^{-i}\right) + O(\nu^2 + |\theta\theta_3|^\nu).$$

And, by using Lemma 4.1.4 and $N-\sum_{i=1}^{v}\varepsilon(m_i)\,\theta^{v-i}=O(\log N)$ we obtain

$$\theta^{-v}F(N)-F\left(\sum_{i=1}^{v}\varepsilon(m_i)\,\theta^{-i}\right)=\theta^{-v}o(N).$$

Thus, $S(N)=\alpha N\log_\theta N+F(N)+o(N)$ and the theorem is proved with $G(x)=x^{-1}F(x)$ for $x\neq 0$.

Remark. The same result remains true if some of the eigenvalues θ_i, $i\geqslant 3$, are complex numbers, by considering the conjugate eigenvalues and eigenvectors.

5. Summation Formulae for Generalized Sum of Digit Functions

Now, we consider a substitution σ on a finite alphabet A such that hypothesis (H_1) of Section 1 is true, some real numbers $f(m)$ for each word $m<\sigma(a)$ $(a\in A)$ and $S(N)=\sum_{n<N}s(n)$, where $s(n)=\sum_{i=0}^{v}f(m_i(n))$, $(m_i(n))_{i=0,\ldots,v}$, being the digits of n is this 1-admissible representation. For instance, if σ is the substitution of Proposition 2.1.2 and for each $m\in A^*$, $f(m)=|m|$, then $s(n)$ is the "ordinary" sum of digits of n in the natural system of Numeration relative to the sequence

$$G_k=|\sigma^k(1)|.$$

Once more, we assume that $\mathbb{C}^d$ has a base of eigenvectors for the matrix ${}^tM$; thus the two last relations in Lemma 3.1.1(i) remain true. We can do the calculus of $S(N)$ in the same way as we did in Sections 3 and 4. In Lemma 3.2.1 nothing is changed, but in the definition of S_k, S'_k, S''_k, S'''_k we have $f(m)$ instead of $f(\sigma^k(m))$. For the other results we have to distinguish three cases for $\theta'=|\theta_2|$:

$$\text{Case 1: }\theta'>1,\qquad \text{Case 2: }\theta'=1,\qquad \text{Case 3: }\theta'<1$$

(the last case occurs when θ is a Pisot number: for instance this is the case where σ is as in Proposition 2.1.2 (see [4])). In the notation nothing is changed, but now, in the definition of α in Lemma 3.2.2, we read $f(m)$ instead of $f_2(m)$.

Concerning the results, the modifications are the following:

In Lemma 3.2.2 the "error term" is

$$\begin{array}{ll} O(v\theta'^{v}) & \text{in Case 1} \\ O(v^2) & \text{in Case 2} \\ O(1) & \text{in Case 3.} \end{array}$$

The same modifications are used for Theorem 3.2.3, apart from Case 3, where the error is $O(\nu)$.

In Lemma 4.1.3 the right-hand side of the result is

$$O(|y-x|\,k+k(\theta'/\theta)^k) \qquad \text{in Case 1}$$

$$O(|y-x|\,k+k^2\theta^{-k}) \qquad \text{in Case 2}$$

$$O(|y-x|\,k) \qquad \text{in Case 3.}$$

In Lemma 4.1.4, $|F_1(y)-F_1(x)|$ is

$$O(|y-x|^{1-\log_\theta\theta'}\log|y-x|) \qquad \text{in Case 1}$$

$$O(|y-x|\,(\log|y-x|)^2) \qquad \text{in Case 2}$$

$$O(|y-x|\log|y-x|) \qquad \text{in Case 3}$$

(the last result generalizes (13) in [9] concerning the Fibonacci case).

In all cases the main result, i.e., Theorem 4.2.1, remains unmodified, the error term (depending on θ') being always in $o(N)$. In particular, in Case 3, using $N=\sum_{i=1}^{\nu}\varepsilon(m_i)\,\theta^{\nu-i}+O(1)$, the error term is in $O(\log N)$, in accordance with [18]. In fact, concerning this last case, we can omit the hypothesis "$\mathbb{C}^d$ has a base of eigenvectors of ${}^tM$" because we have

$$L_b(\sigma^n(a))=\lambda_b\varepsilon(a)\,\theta^n+O(\theta''^n) \qquad \text{for any} \quad \theta'' \text{ saisfying } \theta'<\theta''<1.$$

As a consequence, we have

PROPOSITION 5. *If $s_G(n)$ is the sum of the digits of n relative to $G_k=|\sigma^k(1)|$, σ being as in Proposition 2.1.2, there exists a continuous function G defined for $x\geqslant 0$ such that*

(i) $x\geqslant 0 \Rightarrow G(\theta x)=G(x)$

(ii) $\sum_{n<N}s_G(n)=\alpha N\log_\theta N+NG(N)+O(\log N)$,

where θ is the dominating root of $X^d-a_1X^{d-1}-\cdots-a_d=0$ and

$$\alpha=\theta^{-1}\sum_{\substack{a\in A\\ mc\leqslant\sigma(a)}}\lambda_a\,|m|\,\varepsilon(c)$$

$$=\theta^{-1}\varepsilon(1)\sum_{i=1}^{d}\lambda_a\frac{a_i(a_i-1)}{2}+\sum_{i=1}^{d-1}\lambda_ia_i\varepsilon(i+1).$$

Moreover, G is nowhere differentiable, as consequence of Lemma 4.1.5, because $\alpha\neq 0$.

This proposition was obtained with a distinct but equivalent expression for α by P. J. Grabner and R. F. Tichy [16].

REFERENCES

1. S. AUBRY, C. GODRECHE, AND J. M. LUCK, Scaling properties of a structure intermediate between quasiperiodicity and random, *J. Statist. Phys.* **51** (1988), 1033–1075.
2. E. BOMBIERI AND J. E. TAYLOR, Which distributions of matter diffract? An initial investigation, *J. Phys. Coll. C* **3** (1986), 19, 28.
3. D. W. BOYD, J. COOK, AND P. MORTON, On sequences of ± 1's defined by binary patterns, *Dissertationes Math.* **CCLXXXIII** (1989).
4. A. BRAUER, On algebraic equations with all one root in the interior of the unit circle, *Math. Nachr.* **4** (1951), 250–257.
5. J. BRILLHART, P. ERDOS, AND P. MORTON, On sums of Rudin–Shapiro coefficients, II, *Pacific J. Math.* **107** (1983), 271–323.
6. G. CHRISTOL, T. KAMAE, M. MENDES-FRANCE, AND G. RAUZY, Suites algébriques, automates et substitutions, *Bull. Soc. Math. France* **108** (1980), 401–419.
7. A. COBHAM, Uniform tag sequences, *Math. Systems Theory* **6** (1972), 164–192.
8. J. COQUET, A summation formula related to the binary digits, *Invent. Math.* **73** (1983), 107–115.
9. J. COQUET AND P. VAN DER BOSCH, A summation formula involving Fibonacci digits, *J. Number Theory* **22** (1986), 139–146.
10. H. DELANGE, Sur la fonction sommatoire de la fonction "Somme des chiffres," *Enseign. Math.* **21** (1975), 31–47.
11. J. M. DUMONT, Formules sommatoires et systèmes de numérations liés aux substitutions, *in* "Sém. th. Nombres de Bordeaux, 1987–1988," Exp. 39.
12. J. M. DUMONT, Summation formulae for substitutions on a finite alphabet, *in* "Number Theory and Physics," Springer Proceedings in Physics, Vol. 47, pp. 185–194, Springer-Verlag, New York/Berlin, 1990.
13. J. M. DUMONT AND A. THOMAS, Systèmes de numération et fonctions fractales relatifs aux substitutions, *Theoret. Comp. Sci.* **65** (1989), 153–169.
14. A. S. FRAENKEL, Systems of numeration, *Amer. Math. Monthly* **92** (1985), 105–114.
15. C. GODRECHE, J. M. LUCK, AND F. VALLET, Quasiperiodicity and types of order: A study in one dimension, *J. Phys. Ser. A* **20** (1987), 4483–4499.
16. P. J. GRABNER AND R. F. TICHY, Contributions to digit expansions with respect to linear recurrences, *J. Number Theory* **36** (1990), 160–169.
17. A. M. OSBALDESTIN AND P. SHIU, A correlated digital sum problem associated with sums of three squares, *Bull. London Math. Soc.* **21** (1989), 369–374.
18. A. PETHÖ AND R. F. TICHY, On digit expansions with respect to linear recurrences, *J. Number Theory* **33** (1989), 243–256.
19. G. RAUZY, Des mots en arithmétique, *in* "Journées de théorie des langages et complexité des Algorithmes, Avignon, 1983."