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Finite Groups with Sylow 2-Subgroups Isomorphic to $T/Z(T)$, where T is of Type M_{24}

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1. INTRODUCTION

A Sylow 2-subgroup T of the Mathieu group M_{24} has order 2^{10} and a center $Z(T)$ of order 2. The factor group $T/Z(T)$ is a split extension of its unique (elementary) abelian subgroup of order 2^6 (Lemma 3.1) by a dihedral group of order 2^3. We prove the following result.

THEOREM. *Let G be a finite group with a Sylow 2-subgroup S isomorphic to $T/Z(T)$. Assume $O(G) = 1$. Then the unique abelian subgroup of order 2^6 in S is a normal subgroup of G.*

This result is interesting in comparison with the situation for a Sylow 2-subgroup T_1 of Conway's simple group Co_3. Here again $|T_1| = 2^{10}$, $|Z(T_1)| = 2$, and $T_1/Z(T_1)$ is a split extension of an elementary abelian normal subgroup of order 2^6 by a dihedral group of order 2^3. However, the infinitely many simple groups A_{12}, A_{13}, $Sp(6, 2)$, $\Omega(7, q)$ with $q \equiv \pm 3 \pmod 8$ have Sylow 2-subgroups isomorphic to $T_1/Z(T_1)$. See [11–15] for related characterizations.

Two recent results have facilitated our investigation. The first (Lemma 2.1) due to R. Solomon provides us with a convenient conjugation family (in the sense of Alperin). In our situation the local conjugating sets can be shown to lie in $C_G(Z(S))$ and $N_G(A)$ where A is the abelian subgroup of order 2^6 in S. This enables us to prove that A is strongly closed in S with respect to G. At this point an application of D. M. Goldschmidt's classification of finite groups with an abelian strongly closed 2-subgroup (Lemma 2.5) completes the proof of the theorem.

In this way we obtain an independent proof of the main statement in [11, Key Theorem]:

COROLLARY. *Let H be a finite group with a Sylow 2-subgroup T of type*

M_{24}. *Assume* $Z(T) \subseteq Z(H)$ *and* $O(H) = 1$. *Then the unique normal extra-special subgroup of order* 2^7 *of* T *is normal in* H.

For the sake of readability we have decided to include a fair amount of detail in the following presentation. Notation, however, is standard and will not be explained.

2. General Results

In this section we state some general results to be used later. In all cases J denotes a finite group.

Lemma 2.1 [13, Lemma 3.1, and 2]. *Let p be a prime and P a fixed Sylow p-subgroup of J. Consider the set $\mathscr{H}$ of subgroups H of P that satisfy the following conditions*:

(1) *H is a tame Sylow intersection with P, i.e., there is a Sylow p-subgroup Q of J with $H = P \cap Q$ such that $N_P(H)$ and $N_Q(H)$ are Sylow p-subgroups of $N_J(H)$*;

(2) $C_P(H) \subseteq H$;

(3) *H is a Sylow p-subgroup of* $O_{p',p}(N_J(H))$;

(4) *$H = P$ or $N_J(H)/H$ is p-isolated.*

Form the set $\mathscr{S}$ of all pairs (H, N) with $H \in \mathscr{H}$ and

$$N = N_J(H), \qquad \text{if} \quad H = C_P\Omega_1 Z(H),$$
$$N = N_J(H) \cap C_J\Omega_1 Z(H), \qquad \text{if} \quad H \subset C_P\Omega_1 Z(H),$$

and the set $\mathscr{S}'$ of pairs $(H, C_J(H))$ where H satisfies (1), *but not all of* (2)–(4). *Then $\mathscr{S} \cup \mathscr{S}'$ is a conjugation family* w.r.t. *P in J. In particular, for elements x, y of P conjugate in J there exist $(H_i, N_i) \in \mathscr{S}$ $(i = 1,\ldots, m)$ and elements $x_i \in H_i$, $n_i \in N_i$ such that*

$$x = x_1, \qquad x_i^{n_i} = x_{i+1} \qquad \text{for} \quad 1 \leqslant i \leqslant m-1, \qquad x_m^{n_m} = y.$$

By a fundamental theorem of H. Bender [3] a 2-isolated group L has Sylow 2-subgroups with just one involution or else L has normal subgroups $L_1 \supseteq L_2$ such that L/L_1 and L_2 have odd order and L_1/L_2 is isomorphic to one of the simple groups $PSL(2, 2^n)$, $Sz(2^n)$, $PSU(3, 2^n)$ for suitable $n \geqslant 2$, a so-called simple group of Bender type.

Lemma 2.2. *The only simple group of Bender type involved in $GL(5, 2)$ is $PSL(2, 4) \cong A_5$.*

Proof. This follows from a comparison of group orders except for the case of $PSL(2, 8)$. A Sylow 3-subgroup of $PSL(2, 8)$ is cyclic of order 9 [8, p. 196] whereas $GL(5, 2)$ has an elementary abelian Sylow 3-subgroup of order 9 which is contained in $GL(4, 2) \simeq A_8$.

LEMMA 2.3. *Let P be a Sylow subgroup of J and P_1 a weakly closed subgroup of P w.r.t. J. If J acts on a set Ω, then J-conjugate elements of Ω that are fixed by P_1 are already conjugate under $N_J(P_1)$.*

Proof. O. Grün [7] has shown that this is a consequence of Sylow's theorems. This fundamental lemma is also a consequence of a more powerful, but elementary result of J. L. Alperin based on Sylow's theorems [1].

The following technical lemma on fusion has been suggested by B. Waldmüller; it will be applied in the proof of Lemma 4.2.

LEMMA 2.4. *Suppose we have a subset B, a subgroup U, and elements x, y, g of J such that*

(i) $1 \in B$ *and* $B \cap U = 1$;

(ii) *U is a 2-subgroup* $\neq 1$;

(iii) *no two distinct elements of U are conjugate in J*;

(iv) *no element of $B^{\#}$ is conjugate to an element of $U^{\#}$ in J*;

(v) $Bx \cap Uy = \varnothing$;

(vi) *g has odd order, $x^g = y$, and $Bx \cup Uy$ is invariant under g.*

Then $|U| = 2$.

Proof. Assume by way of contradiction that $|U| = 2^m$ with $m \geqslant 2$. We first show that

(vii) *there is precisely one element $z \in Uy$ with $z^g \in Bx$.*

The statement is clear if $B = \{1\}$; note that $\{x\} \cup Uy$ is g-invariant by (vi) and $x \neq y$ by (v). If $B \neq \{1\}$ let $b \neq 1$ in B and suppose $(bx)^g \in Uy$. Then $uy = (bx)^g = b^g y$ for some $u \in U$, hence $u = b^g$ against (iv). Therefore $(bx)^g \in Bx$ for all $b \neq 1$ in B; (vii) follows.

(viii) *Suppose $y_0 \in Uy$ and $g: y_0 \to u_1 y_0 \to u_2 y_0 \to \cdots$ with distinct elements $1, u_1, u_2$ of U. Then g fixes $u_1 \neq 1$.*

In fact, $u_1^g \cdot u_1 y_0 = (u_1 y_0)^g = u_2 y_0$, hence $u_1^g = u_2 u_1^{-1} \in U$ and, by (iii), $u_1^g = u_1$. We have $g: x \to y \to y^g \to \cdots$ where $y^g \in Uy$ or $y^g \in Bx$. Assume that $y^g \in Bx$ so that $z = y$. Since we assume $m \geqslant 2$, the set $V = Uy \backslash \{y\}$ is not empty. Clearly V is invariant under g. Either g has an orbit in V

of length at least three or g fixes at least two elements of V. In the first case (viii) applies and g fixes an element $u \neq 1$ of U. In the second case, if g fixes the distinct elements $u_1 y$ and $u_2 y$ of V, then $u_1 y \cdot (u_2 y)^{-1} = u_1 u_2^{-1} = u \in U^{\#}$ serves the same purpose. In any case $g\colon uy \to uy^g \to \cdots$. If $uy^g \in Bx$, then $z = uy$ against $z = y$ and $u \neq 1$. If $uy^g \in Uy$, then $y^g \in Uy \cap Bx = \varnothing$. This contradiction shows that

(ix) $y^g \in Uy$.

Set $y^g = vy$. We have to consider $(vy)^g$. If $(vy)^g \in Bx$, then $z = vy = y^g$, $W = Uy \backslash \{y, y^g\}$ is invariant under g and, as above, there is an element $u \in U^{\#}$ fixed by g. One gets $g\colon uy \to uy^g \to u(vy)^g \to \cdots$. Either $u(vy)^g \in Bx$, whence $z = uy^g$ against $z = y^g$ and $u \neq 1$, or $u(vy)^g \in Uy$, whence $(vy)^g \in Uy \cap Bx = \varnothing$. This contradiction shows that

(x) $(vy)^g \in Uy$.

It follows from (viii) that g fixes v. Hence

$$g\colon y \to vy \to v^2 y \to \cdots \to v^n y = y$$

where n is the order of v and the length of the g-orbit of y. This number is a power of 2 by (ii) and odd by (vi). Hence $n = 1$ and $v = 1$, the final contradiction.

LEMMA 2.5 [6]. *Let S be a Sylow 2-subgroup of J, A an abelian subgroup of S strongly closed in S* w.r.t. J. *Set $M = \langle A^J \rangle$ and $\bar{J} = J/O(M)$. Then $\bar{A} = O_2(\bar{M})\,\Omega_1(\bar{S}_0)$ for a Sylow 2-subgroup S_0 of M containing A; and M is a central product of an abelian 2-group and groups L such that $L = L'$ and $N = L/Z(L)$ is a simple group of one of the following types:*

(a) *N is of Bender type (see remark following Lemma* 2.1);

(b) $N \cong PSL(2, q)$, $q \equiv 3, 5 \pmod 8$, $q > 3$;

(c) *N is of type Janko-Ree, i.e. N has an involution t in the center of a Sylow 2-subgroup such that $C_N(t) = \langle t \rangle \times N_0$ with $N_0 \cong PSL(2, q)$*, $q \equiv 3, 5 \pmod 8$ (*see* [6] *for references*).

Moreover in case (b) *and* (c) *$Z(L)$ has odd order.*

3. THE STRUCTURE OF S

We consider a 2-group S presented by generators a_1, b_1, c_1, a_2, b_2, c_2, w, v_1, v_2 and relations indicating that these generators are involutions and transform each other according to the following table of conjugates $x^y = y^{-1}xy$. A bar indicates that $x^y = x$. It follows from Table I in [11] that $S \simeq T/Z(T)$ where T is a Sylow 2-subgroup of M_{24}. One checks with Table I that the following mappings define automorphisms α_i of S. Again bars denote elements that are left fixed.

TABLE I

Conjugates x^y

x \ y	a_1	b_1	c_1	a_2	b_2	c_2	w	v_1	v_2
a_1	—	—	—	—	—	—	—	—	—
b_1	—	—	—	—	—	—	—	—	a_1b_1
c_1	—	—	—	—	—	—	a_1c_1	$a_1b_1c_1$	—
a_2	—	—	—	—	—	—	—	—	—
b_2	—	—	—	—	—	—	—	a_2b_2	—
c_2	—	—	—	—	—	—	a_2c_2	—	$a_2b_2c_2$
w	—	—	a_1w	—	—	a_2w	—	—	—
v_1	—	—	$a_1b_1v_1$	—	a_2v_1	—	—	—	wv_1
v_2	—	a_1v_2	—	—	—	$a_2b_2v_2$	—	wv_2	—

TABLE II

Images x^{α_i} for Some Automorphisms α_i

x	a_1	b_1	c_1	a_2	b_2	c_2	w	v_1	v_2
α_1	a_2	b_2	c_2	a_1	b_1	c_1	—	v_2	v_1
α_2	—	—	—	—	—	—	wb_1	—	c_1v_2
α_3	—	a_2b_1	$a_1a_2b_2 \cdot c_1$	—	—	a_2c_2	—	—	—

We set $A = \langle a_1, b_1, c_1, a_2, b_2, c_2 \rangle$, $F = \langle a_1, a_2, b_1, b_2 \rangle$, and $D = \langle w, v_1, v_2 \rangle$. Clearly $Z(S) = \langle a_1, a_2 \rangle$, $S' = F\langle w \rangle$, and $S = AD$ with $A \cap D = 1$. The group S has 30 conjugacy classes of involutions with representatives x as listed in Table III.

LEMMA 3.1. *$|Z(M)| \leqslant 2^3$ for every maximal subgroup M of S. The elementary abelian subgroup A is the unique abelian subgroup of order 2^6 in S.*

TABLE III

S-Classes of Involutions

x	$\lvert x^S \rvert$	Remarks	$C_S(x)$
a_1	1		
a_2	1		
a_1a_2	1		
b_1	2		$A\langle w, v_1\rangle$
b_2	2	$\alpha_1: b_1 \to b_2$	
a_2b_1	2	$\alpha_3: b_1 \to a_2b_1$	
a_1b_2	2	$\alpha_3\alpha_1: b_1 \to a_1b_2$	
b_1b_2	4	$x^S = \langle a_1, a_2\rangle b_1b_2$	$A \cdot \langle w\rangle$
c_1	4	$x^S = \langle a_1, b_1\rangle c_1$	$A \cdot \langle v_2\rangle$
c_2	4	$\alpha_1: c_1 \to c_2$	
a_2c_1	4	$x^S = a_2 \cdot c_1{}^S$	$C_S(c_1)$
a_1c_2	4	$\alpha_1: a_2c_1 \to a_1c_2$	
b_2c_1	4	$x^S = \langle a_1, a_2b_1\rangle b_2c_1$	$C_S(c_1)$
b_1c_2	4	$\alpha_1: b_2c_1 \to b_1c_2$	
$a_1a_2b_2 \cdot c_1$	4	$\alpha_3: c_1 \to a_1a_2b_2 \cdot c_1$	
$a_1a_2b_1 \cdot c_2$	4	$\alpha_3\alpha_1: c_1 \to a_1a_2b_1 \cdot c_2$	
c_1c_2	8		A
$a_1a_2b_1b_2c_1c_2$	8		A
w	4	$x^S = x^C$ for $C = \langle c_1, c_2\rangle$	FD
wb_1	4	$\alpha_2: w \to wb_1$	
wb_2	4	$\alpha_2\alpha_1: w \to wb_2$	
wb_1b_1	4	$\alpha_2\alpha_1\alpha_2: w \to wb_1b_2$	
v_1	8		$\langle a_1, a_2, b_1, c_2, w, v_1\rangle$
v_2	8	$\alpha_1: v_1 \to v_2$	
a_1v_1	8		$C_S(v_1)$
a_2v_2	8	$\alpha_1: a_1v_1 \to a_2v_2$	
c_1v_2	8	$\alpha_2: v_2 \to c_1v_2$	
c_2v_1	8	$\alpha_1\alpha_2\alpha_1: v_1 \to c_2v_1$	
$a_2 \cdot c_1v_2$	8		$C_S(c_1v_2)$
$a_1 \cdot c_2v_1$	8	$\alpha_1: a_2 \cdot c_1v_2 \to a_1 \cdot c_2v_1$	

Proof. If $x \in Z(M)$, then x has one or two conjugates. By Table III, $Z(M) \subseteq F$, but $b_1 b_2 \notin Z(M)$. Hence $|Z(M)| \leqslant 2^3$. Suppose $A^* \neq A$ is another abelian subgroup of order 2^6. Now $|A \cdot A^*| \cdot |A \cap A^*| = |A| \cdot |A^*| = 2^{12}$. All this implies $2^5 = |A \cap A^*|$. Let $u = a \cdot d \in A^* \backslash A$ with $a \in A$ and $d \in D$. Then $A \cap A^* \subseteq C_A(u) = C_A(d)$. But Table III shows that $|C_A(d)| \leqslant 2^4$, a contradiction.

LEMMA 3.2. *The elementary abelian subgroups of order 2^5 in $K_1 = C_S(w)$ are*

$$F\langle w \rangle = \langle a_1, a_2, w, b_1, b_2 \rangle,$$

$R_1 = \langle a_1, a_2, w, b_1, v_1 \rangle$, *and* $R_2 = \langle a_1, a_2, w, b_2, v_2 \rangle$. *These groups U are normal in S, $U = C_S(U)$, with factor groups S/U of types E_{16} and $C_2 \times D_8$.*

Proof. By Lemma 3.1, $\langle a_1, a_2, w \rangle = \Omega_1 Z(K_1) \subseteq U$. We have $K_1 = FD$ so that K_1/F is dihedral. In particular, $K_1 \neq FU$. Therefore $\langle b_1, b_2 \rangle \cap U \neq 1$. If $b_1 b_2 \in U$, then $U \subseteq C_{K_1}(b_1 b_2) = K_1 \cap A\langle w \rangle = F\langle w \rangle$, hence $U = F\langle w \rangle$. If $b_1 b_2 \notin U$, we may assume $b_1 \in U$ where $C_{K_1}(b_1) = F\langle w, v_1 \rangle$. Therefore $U = \langle a_1, a_2, w, b_1, x \rangle$ for some involution $x \in \langle b_2, v_1 \rangle \backslash \langle a_2, b_2 \rangle$. Hence $x \in \{v_1, a_2 v_1\}$ and $U = R_1$.

LEMMA 3.3. *Set $S^* = S/Z(S)$. Then $Z(S^*) = \langle w^*, b_1^*, b_2^* \rangle$; the only elementary abelian subgroups of an order at least 2^5 in S^* are*

$$C = \langle Z(S^*), c_1^*, c_2^* \rangle,$$

$$C_1 = \langle Z(S^*), c_1^*, v_2^* \rangle, \qquad \textit{and} \qquad C_2 = \langle Z(S^*), c_2^*, v_1^* \rangle,$$

they are normal in S^*; $C \cup C_1 \cup C_2 \backslash \{1\}$ is the set of involutions of S^*.

Proof. Recall that $Z(S) = \langle a_1, a_2 \rangle$. By Table I, $Z(S^*) = \langle w^*, b_1^*, b_2^* \rangle$ and $S^* = Z(S^*) \cdot \langle c_1^*, c_2^* \rangle \cdot \langle v_1^*, v_2^* \rangle$. Let U be an abelian subgroup of S^* with $|U| \geqslant 2^5$ and $C \neq U$. Then U contains an involution $t = zcv$ with $z \in Z(S^*)$, $c \in \langle c_1^*, c_2^* \rangle$, $v \in \langle v_1^*, v_2^* \rangle^{\#}$, and $U \subseteq C_{S^*}(t) = C_{S^*}(cv)$. For these involutions cv one computes $C_{S^*}(cv) = C_1$ or C_2. The coset $Cv_1^* v_2^*$ consists of elements of order 4; the involutions of Cv_1^* must centralize v_1^*, hence lie in C_2; similarly the involutions of Cv_2^* lie in C_1.

We paint the involutions of S^* red and green: an involution x^* will be called red when $x^2 \neq 1$, otherwise green. Ultimately we shall be interested in green involutions only, but for the investigation of their fusion in Lemma 4.2 the red involutions have a key function.

TABLE IV

S^*-Classes of Red Involutions

x^*	$\lvert x^{*S^*} \rvert$	x^2	x^{*S^*}	contained in
$(wc_1)^*$	2	a_1	$\langle b_1{}^*\rangle (wc_1)^*$	$C_1 \cap C$
$(wc_1b_2)^*$	2	a_1	$\langle b_1{}^*\rangle (wc_1b_2)^*$	$C_1 \cap C$
$(b_1v_2)^*$	4	a_1	$\langle b_2{}^*, w^*\rangle (b_1v_2)^*$	C_1
$(b_1c_1v_2)^*$	4	a_1	$\langle b_2{}^*, w^*b_1{}^*\rangle (b_1c_1v_2)^*$	C_1
x^*				$C_2 = C_1^{\alpha_1}$
$(wc_1c_2)^*$	4	a_1a_2	$\langle b_1{}^*, b_2{}^*\rangle (wc_1c_2)^*$	$C \backslash (C_1 \cup C_2)$

4. Localisation of Fusion

From now on we fix a finite group G with $O(G) = 1$ and Sylow 2-subgroup S (as described in Section 3). For the later application of Lemma 2.1 to the situation $(J, P) = (G, S)$ we wish to control the conjugating groups N appearing as $(H, N) \in \mathscr{S}$.

Lemma 4.1. *Let* $(H, N) \in \mathscr{S}$. *If* $H = C_S\Omega_1 Z(H)$, *then* $A \subseteq H$ *and* $N \subseteq N_G(A)$. *If* $H \subset C_S\Omega_1 Z(H)$, *then* $N \subseteq C_G Z(S)$.

Proof. By condition (2) of Lemma 2.1, $Z(S) \subseteq \Omega_1 Z(H)$. Therefore $N \subseteq C_G Z(S)$, if $H \subset C_S\Omega_1 Z(H)$. We now assume

$$H = C_S\Omega_1 Z(H). \tag{2'}$$

If $A \subseteq H$, then Lemma 3.1 implies $N = N_G(H) \subseteq N_G(A)$ as desired. We therefore assume

$$A \not\subseteq H \tag{5}$$

and seek a contradiction.

By (2′) and (5) there is an involution $u \in \Omega_1 Z(H) \backslash A$ with $H \subseteq C_S(u)$. Table III shows that some automorphism α of S in $\langle \alpha_1, \alpha_2 \rangle$ maps $K = C_S(u)$ onto

$$K_1 = C_S(w) \qquad \text{or} \qquad K_2 = C_S(v_1).$$

It is clear that we need only exclude the cases $H \subseteq K_1$ and $H \subseteq K_2$.

Assume $H \subseteq K_1$. By (2), $\Omega_1 Z(K_1) \subseteq \Omega_1 Z(H)$ where $\Omega_1 Z(K_1) = \langle a_1, a_2, w \rangle$

has order 2^3 and $|\Omega_1 Z(H)| \leqslant 2^5$. If $|\Omega_1 Z(H)| = 2^5$, then by Lemma 3.2 and (2') $C_S \Omega_1 Z(H) = \Omega_1 Z(H)$, $H \lhd S$, and $S/H \cong E_{16}$ or $C_2 \times D_8$ is a Sylow 2-subgroup of $N(H)/H$. By (4) in conjunction with Bender's result stated in Section 2 and Lemma 2.2 this is impossible. If $|\Omega_1 Z(H)| = 2^4$, then $\Omega_1 Z(H) = \langle a_1, a_2, w, x \rangle$ for an involution x in $\langle b_1, b_2 \rangle \cup \langle b_1, b_2 \rangle v_1 \cup \langle b_1, b_2 \rangle v_2$ with $\Omega_1 Z(H) = \Omega_1 Z(K_1 \cap C(x))$. Now $K_1 \cap C(b_1 b_2) = F\langle w \rangle$ is elementary of order 2^5, hence $x \neq b_1 b_2$. If $x = b$, then $H = K_1 \cap C(b_1) = F\langle w, v_1 \rangle \lhd S$ with $S/H \cong E_8$ against (4) as above. Similarly $x \neq b_2$. If $x \in \langle b_1 \rangle v_1$ then $H = K_1 \cap C(x) = R_1$ against $|\Omega_1 Z(H)| = 2^4$. There are no further involutions in $\langle b_1, b_2 \rangle v_1$. Similarly, $x \in \langle b_1, b_2 \rangle v_2$ is impossible. We are left with $\Omega_1 Z(H) = \langle a_1, a_2, w \rangle$, hence $H = K_1$. By (1), $N_G(H)/H$ has an elementary abelian Sylow 2-subgroup of order 4. This contradicts (4) and Lemma 2.2 as A_5 is not involved in $GL(3, 2)$.

Now assume $H \subseteq K_2$. Again $\langle a_1, a_2, b_1, v_1 \rangle = \Omega_1 Z(K_2) \subseteq \Omega_1 Z(H)$. If $|\Omega_1 Z(H)| = 2^5$, then $\Omega_1 Z(H) = \langle \Omega_1 Z(K_2), x \rangle$ for an involution $x \in \langle w, c_2 \rangle$. If $x = w$, then $\Omega_1 Z(H) = R_1$ which we have seen is impossible. If $x = c_2$, then $\Omega_1 Z(H) = H$ has $N_S(H) = A\langle w, v_1 \rangle$ with $N_S(H)/H \cong E_8$. As before one obtains a contradiction from (1), (2'), (4), and Lemma 2.2. We are left with $\Omega_1 Z(H) = \Omega_1 Z(K_2)$, hence $H = K_2$. Here $N_S(H) = H \cdot \langle b_2, c_1 \rangle$ with $H \cap \langle b_2, c_1 \rangle = 1$. The group $\bar{N} = N(H)/N(H) \cap C\Omega_1 Z(H)$ acts faithfully on $\Omega = \langle a_1, a_2, b_1, v_1 \rangle$ and has Sylow 2-subgroup $\langle \bar{b}_2, \bar{c}_1 \rangle \cong N_S(H)/H$. Thus, $\bar{N}$ has a normal series $\bar{N} \supseteq N_1 \supset N_2 \supseteq 1$ with $N_1/N_2 \cong A_5$ and $\bar{N}/N_1$ and N_2 of odd order. Looking at $GL(4, 2) \cong A_8$ we see that $N_1 \supseteq B$ where $B \cong A_5$ contains $\langle \bar{b}_2, \bar{c}_1 \rangle$. We study the action of B on Ω. The group $\langle \bar{b}_2, \bar{c}_1 \rangle$ produces the eight classes $\{x\}$ for $x \in \langle a_1, a_2, b_1 \rangle$ and the two classes $4v_1$ and $4a_1 v_1$ on Ω. We compute $C_\Omega(z) = \langle a_1, a_2, b_1 \rangle$ for any $z \in \langle \bar{b}_2, \bar{c}_1 \rangle^{\#}$. It follows that there is an involution $e \in \langle a_1, a_2, b_1 \rangle$ centralized by an element d of order three in $N_B(\langle \bar{b}_2, \bar{c}_1 \rangle)$. This element e has precisely 5 B-conjugates in Ω. Suppose there is also a B-orbit of length 10. Then this orbit contains at least one $\langle \bar{b}_2, \bar{c}_1 \rangle$-class $\{x\}$ with $x \in \langle a_1, a_2, b_1 \rangle$ so that $[B: C_B(x)]$ is odd, a contradiction. Consequently there are three B-classes of length 5 each in $\Omega^{\#}$. In particular, $4v_1$ fuses with some element $x \in \langle a_1, a_2, b_1 \rangle^{\#}$, and we have an element $f \in B$ of order 5 with the action

$$f\colon v_1 \to x \to x_1 \to x_2 \to x_3 \to v_1$$

where x_1, x_2, x_3 are in $4v_1$ and of the form $x_i = y_i v_1$ with $y_i \in \langle a_1, a_2, b_1 \rangle$. It follows that

$$f\colon xv_1 \to x_1 x \to x_2 x_1 \to x_3 x_2 \to v_1 x_3 \to xv_1 .$$

Clearly, xv_1 lies in $4v_1$ or $4a_1 v_1$. So the B-orbit $\{xv_1, x_1 x, x_2 x_1, x_3 x_2, v_1 x_3\}$ contains precisely one involution of $\langle a_1, a_2, b_1 \rangle$. However, the distinct elements x_2x_1 and x_3x_2 both lie in $\langle a_1, a_2, b_1\rangle$. This contradiction completes the proof of the lemma.

LEMMA 4.2. *The subgroup A is strongly closed in S with respect to $C_GZ(S)$.*

Proof. We introduce the canonical homomorphism $*: K = C_GZ(S) \to K^* = K/Z(S)$. Look at the elementary abelian subgroups of order 2^5 in S^*, they are C, C_1, C_2 (Lemma 3.3). If C_1 is conjugate to C in K^*, then as S^* is weakly closed in S^* these groups are conjugate in $N_{K^*}(S^*)$ (Lemma 2.3). However, $C = (S^*)'A^*$ is invariant under $N_{K^*}(S^*)$. Similarly, $C_1 \sim C_2$ is impossible. Therefore C, C_1, C_2 and CC_1, CC_2 are weakly closed in S^* w.r.t. K^*.

The group $C_1 \cap C = Z(CC_1)$ has just two S^*-classes of red involutions (Table IV in Section 3). If $(wc_1)^* \sim (wc_1b_2)^*$ in K^* then this happens already in $N_{K^*}(CC_1)$ (Lemma 2.3). But the orbit of $(wc_1)^*$ under this normalizer has length an odd number times

$$[S^*: C_{S^*}((wc_1)^*)] = [S^*: CC_1] = 2.$$

Consequently, the classes $2(wc_1)^*$ and $2(wc_1b_2)^*$ remain unfused in K^*. It follows that an element of odd order in $N_{K^*}(CC_1)$ centralizes all red involutions in $C_1 \cap C$. These involutions generate a subgroup of order 2^3. Therefore elements of odd order in $N_{K^*}(CC_1)$ act trivially on $C_1 \cap C$. This proves that there is no K^*-fusion of S^*-classes of green involutions in $C_1 \cap C$ (Lemma 2.3). A corresponding statement holds for the involutions in $C_2 \cap C$.

The red involutions x^* of C fall into three categories: those in $C_1 \cap C$ have $x^2 = a_1$, those in $C_2 \cap C$ have $x^2 = a_2$ and the remaining ones form one S^*-class with $x^2 = a_1a_2$ (Table IV). There can be no fusion between distinct categories by the definition of K as $C_GZ(S)$. Therefore an element of odd order in $N_{K^*}(C)$ centralizes the four S^*-classes of length 2, the subgroup $\langle (wc_1)^*, b_1^*, b_2^*, (wc_2)^*\rangle$ they generate, hence C. By Lemma 2.3 there is no K^*-fusion of S^*-classes of green involutions in C.

We turn to C_1 for which case the work has been done in Lemma 2.4. Again the possible fusion among elements of C_1 already occurs in $N_{K^*}(C_1)$. By Table IV the S^*-classes of red involutions in C_1 have lengths 2, 2, 4, 4. First suppose that some of these classes combine to a complete $N_{K^*}(C_1)$-orbit of length 10. Then $N_{K^*}(C_1)$ contains an element g of order five that does not centralize this orbit. However, g fixes the remaining class of length 2 and the subgroup of order 4 generated by it as well as the corresponding factor group C_1 of order 2^3, so that g fixes C_1. This is a contradiction. Suppose now that there is no complete $N_{K^*}(C_1)$-class of red involutions of the form $2 \sim 4$. Then both classes of length 2 remain unfused. In particular, an element of odd order in $N_{K^*}(C_1)$ must centralize the subgroup $\langle b_1{}^*, b_2{}^*, (wc_1)^*\rangle$ generated by them and at least one of the remaining 8 red involutions; hence C_1 is centralized. This shows that there is no K^*-fusion among S^*-classes of green involutions of C_1 in this case. Finally, assume that $2 \sim 4$ occurs as a complete $N_{K^*}(C_1)$-orbit of red involutions. By Table IV the class of length 2 in this orbit has the form Bx with $B = \langle b_1{}^*\rangle$ and the class of length 4 has the form Uy with $U = \langle b_2{}^*, w^*\rangle$ or $U = \langle b_2{}^*, w^*b_1{}^*\rangle$. The elements x, y may be chosen in such a way that there is an element $g \in N_{K^*}(C_1)$ of odd order with $x^g = y$. Note that $\langle B, U\rangle \subseteq Z(S^*) = C_1 \cap C \cap C_2$ so that the hypotheses of Lemma 2.4 are satisfied. We conclude that this situation is impossible.

Now let r be a green involution extremal in S^* w.r.t. K^* and let s be an involution conjugate to r in S^*. Then there is an element $j \in K^*$ with $s^j = r$ and $C_{S^*}(s)^j \subseteq C_{S^*}(r)$ [11, Lemma 2.5]. By Lemma 3.3, s is contained in one of the subgroups C, C_1, C_2, say $s \in C_0$. This subgroup is weakly closed in S^* w.r.t. K^*, hence $C_0 = C_0{}^j \subseteq C_{S^*}(r)$ and $r \in C_{S^*}(C_0) = C_0$. As s and r both lie in C_0 we conclude from what we have shown above that s and r are already conjugate in S^*. The subgroup A^* is normal in S^*, hence A is strongly closed in S w.r.t. K.

Remark. A repeated application of Glauberman's Z^*-theorem shows that K has a normal 2-complement. However, we do not need this result here.

5. Proof of the Theorem

We conclude from Lemma 4.1 and Lemma 4.2 on the basis of Lemma 2.1 that A is strongly closed in S even w.r.t. G. Hence Goldschmidt's theorem (Lemma 2.5) describes the structure of $M = \langle A^G\rangle$. Note that $O(M) = 1$ since we assume $O(G) = 1$. So $A = O_2(M)\,\Omega_1(S_0)$ where S_0 is a Sylow 2-subgroup of M containing A. It follows that $A = \Omega_1(S_0)$ as A is elementary. We may assume $S_0 \subseteq S = AD$. Hence $S_0 = A \cdot (S_0 \cap D)$. If $S_0 \supset A$, then $1 \neq \Omega_1(S_0 \cap D) \subseteq D \cap \Omega_1(S_0) = D \cap A = 1$, q.e.a. We have shown that A is a Sylow 2-subgroup of M (and G is not simple). In particular, the normal subgroups L of M occurring in Lemma 2.5 have elementary abelian Sylow 2-subgroups. So $Z(L) = 1$ and L is a simple group of type $PSL(2, q)$, $q \equiv 0, 3, 5 \pmod 8$, or of type Janko-Ree.

Suppose a subgroup Δ of D normalizes one of these normal subgroups L of M. We show that 1 is the only element of Δ that induces an inner automorphism of L. In fact, an element $d \in D \cap N_G(L)$ also normalizes the Sylow 2-subgroup $L \cap A$ of L; if $f \in L$ induces the same automorphism $\bar{f} = \bar{d}$ of L as d, then $f \in N_L(L \cap A)$, $f = r \cdot a$ with $a \in L \cap A$ and r of odd order, $\dot{f} = \dot{r} \cdot \dot{a} = \dot{d}$, $\dot{r} = \dot{d}\dot{a}^{-1} \in S$, $\dot{r} = 1$, $\dot{d} = \dot{a}$, but $\dot{a} \mid A = 1$, i.e., $d \in C_D(A) = 1$, $d = 1$. The same argument works with L replaced by L_0 in the case of a simple group of type Janko-Ree provided that Δ normalizes L_0.

Clearly $\langle L^D \rangle$ is a direct product of simple groups isomorphic to L. This product can have at most three factors as L is simple with Sylow 2-subgroup of order at least 2^2 and $|A| = 2^6$. However, D acts on this set of direct factors [8, p. 70], so their number is 1 or 2. In the first case set $\Delta = D$. In the second case D has a normal subgroup Δ of index 2 that normalizes both factors. If L is of type Janko-Ree we choose an involution $t \in L \cap A$ that is centralized by Δ. Since L has only one class of involutions [9; 10, p. 275], $C_L(t) = \langle t \rangle \times L_0$ with $L_0 \cong PSL(2, q)$, $q \equiv 3, 5 \pmod 8$; $C_L(t)$ and L_0 are normalized by Δ. So in all cases Δ normalizes a group of type $PSL(2, q)$, $q \equiv 0, 3, 5 \pmod 8$, and is isomorphic to a group of outer automorphisms of $PSL(2, q)$, i.e., a subgroup of $P\Gamma L(2, q)/PSL(2, q)$, [4, pp. 103–104, 96–97, 91–96]. Thus, Δ is abelian which excludes the first case where $\Delta = D$. In the second case $D = \langle \Delta, v_1 \rangle$ or $D = \langle \Delta, v_2 \rangle$; we may assume $D = \langle \Delta, v_1 \rangle$. Set $U = L \cap A$, resp. $U = L_0 \cap A$ in the case of a group L of type Janko-Ree; we have $U^\Delta = U$. Then $U \cap U^{v_1} = 1$ as $L \cap L^{v_1} = 1$. On the other hand $C_A(v_1) = \langle a_1, a_2, b_1, c_2 \rangle$ where $\langle a_1, a_2, b_1 \rangle \subseteq C_A(w)$, hence U contains an element $x'c_1 = xc_2{}^i c_1$ with x', $x \in C_A(w)$, $i = 0$ or 1. Clearly $w \in \Delta$. Hence U also contains $(xc_2{}^i c_1)^w = (a_2{}^i a_1)(xc_2{}^i c_1)$ and $a_2{}^i a_1$, which contradicts $U \cap U^{v_1} = 1$. It follows that $M = A$. Q.E.D.

References

1. J. L. Alperin, On a theorem of Manning, *Math. Zeitschr.* **88** (1965), 434–435.
2. J. L. Alperin, Sylow intersections and fusion, *J. Algebra* **6** (1967), 222–241.
3. H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festläßt, *J. Algebra* **17** (1971), 527–554.
4. J. Dieudonné, "La Géométrie Des Groupes Classiques," Third edition, Springer Verlag, Berlin-Heidelberg-New York, 1971.
5. D. M. Goldschmidt, A conjugation family for finite groups, *J. Algebra* **16** (1970), 138–142.
6. D. M. Goldschmidt, 2-Fusion in finite groups, *Annals Math.* **99** (1974), 70–117.
7. O. Grün, Beiträge zur Gruppentheorie. III, *Math. Nachr.* **1** (1948), 1–24.
8. B. Huppert, "Endliche Gruppen I," Springer Verlag, Berlin-Heidelberg-New York, 1967.
9. Z. Janko, A new finite simple group with abelian Sylow 2-subgroups and its characterization, *J. Algebra* **3** (1966), 147–186.
10. Z. Janko and J. G. Thompson, On a class of finite simple groups of Ree, *J. Algebra* **4** (1966), 274–292.
11. U. Schoenwaelder, Finite groups with a Sylow 2-subgroup of type M_{24}, I, *J. Algebra* **28** (1974), 20–45.

12. U. Schoenwaelder, Finite groups with a Sylow 2-subgroup of type M_{24}, II, *J. Algebra* **28** (1974), 46–56.
13. R. Solomon, Finite groups with Sylow 2-subgroups of type $\mathfrak{A}_{12}$, *J. Algebra* **24** (1973), 346–378.
14. R. Solomon, Finite groups with Sylow 2-subgroups of type $\Omega(7, q)$, $q \equiv \pm 3$ (mod 8), *J. Algebra* **28** (1974), 174–181.
15. R. Solomon, Finite groups with Sylow 2-subgroups of type .3, *J. Algebra* **28** (1974), 182–198.