Lateral stability of bending non-prismatic thin-walled beams using orthogonal series

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Abstract

The lateral stability of bending non-prismatic thin walled beams is carried out using orthogonal Chebyshev series. The considerations apply to a system with variable geometrical parameters. The problem leads to fourth order coupled partial differential equations with variable coefficients. Equations were solved using orthogonal Chebyshev series. The presented method of solution is based on the theorem leads to an infinite system of algebraic equations. In order to verify the results were compared with results obtained by FEM and other authors.

Keywords: Lateral torsional buckling; Non-prismatic thin walled beam; Chebyshev series; Recurrence relations; Analytical solutions.

1. Introduction

Although the lateral torsional buckling of thin-wall beams has been studied by many authors, only a few studies are referred to here because of the limited space. The subject of this analysis is the lateral torsional buckling of a monosymmetric thin-walled nonprismatic beam. The displacement equations describing the problem were taken from [1] and [2], where they had been derived using the virtual work principle and solved using an approximation method based on classic power series. In this paper the method described by Paszkowski in [6] was used to solve the lateral torsional buckling problem. In this method orthogonal Chebyshev series are used to approximate the solution.

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Examples of the use of the method can be found in the earlier works ([3,4,5]) by the present authors. By applying the theorem described in [6] one gets an infinite recursive system of algebraic equations for determining the sought coefficients of the series. Calculation cases were solved to verify the derived equations. The lateral torsional buckling of simply supported beams with a monosymmetric and bisymmetric cross section and that of cantilever beams were analyzed. The obtained critical moments and forces were compared with the results reported by the authors of [1] and with the values obtained using FEM.

2. Problem formulation

The lateral torsional buckling (LTB) of a thin-walled nonprismatic beam with an open cross section, under load \( q_z \) symmetric to plane \( xz \) is considered. The bending moments produced by this load are denoted as \( M_y \) and assumed to be known. The coordinate system adopted in the analysis and the denotations of the displacements are shown in Fig. 1.

![Diagram showing coordinate system and denotations of displacements.](image)

Fig. 1. Adopted coordinate system and denotations of displacements.

In order to derive the equations in [1] the following assumptions concerning the components of the displacement of point \( M(U, V, W) \) on the section contour were made:

\[
U(x, y, z) = u_0(x) - y[v(x) + z_c(x)\theta(x)]' - z[w(x) - y_c(x)\theta(x)]' - \omega(y, z)\theta'
\]

\[
V(x, y, z) = v(x) - (z - z_c(x))\theta(x)
\]

\[
W(x, y, z) = w(x) + (y - y_c(x))\theta(x)
\]

where \( y_c(x), z_c(x) \) are coordinates defining the position of shear centre \( C \) relative to cross section centre \( O \). In the case of a monosymmetric cross section, \( y_c(x) = 0 \).

The displacement equations describing the problem of the lateral torsional buckling of the monosymmetric beam are as follows (see [1]):

\[
[ EI_z v'''''] + [ EI_z z''''θ]'''' + [2 EI_z z''''θ']'''' - [M_y θ]' = q_y = 0,
\]

(2)
\[ [EI_\theta \theta''']' - [GJ \theta']' - 4(z^2EI_z \theta')' + EI_z z''' \theta - 2[2EI_z z'_z \theta'' - 2(2EI_z z'_z z'') \theta + EI_z z''']' \theta + EI_z z''' \theta'' - M_z v'' - (M_z z'''' + M'_z z'_z) \theta + [\beta_z, M_z, \theta']' = -M_\theta \theta \]

where \( I_\theta = I_\theta - z^2 I_z \), \( M_z = -e_z q_z \) (Fig. 1), \( \beta_z = \int z(y^2 + z^2) dA / I_y - 2z_z \).

The relations describing the boundary conditions, needed to solve the problem, are as follows:

\[ EI_z v'' + EI_z z'' \theta + 2EI_z z'_z \theta' - M_z \theta = 0 \quad \text{or} \quad \delta v = 0 \]

\[ -[EI_z v''']' - [EI_z z'' \theta']' [2EI_z z'_z \theta'' + (M_z \theta')' = 0 \quad \text{or} \quad \delta v = 0 \]

\[ EI_\theta \theta'' = 0 \quad \text{or} \quad \delta \theta = 0 \]

\[ -[EI_\theta \theta''']' + GJ \theta' + 4z^2 EI_z \theta' + 2EI_z z'_z \theta'' - \beta_z M_z \theta' = 0 \quad \text{or} \quad \delta \theta = 0 \]

Considering that

\[ M_z = EI_z v'' + EI_z z'' \theta + 2EI_z z'_z \theta' - M_z \theta = 0 \]

the following was determined

\[ v'' = -z'_z \theta - 2z'_z \theta' + \frac{M_z \theta}{EI_z} \]

and after relation (9) was substituted into equation (3) the following was ultimately obtained

\[ [EI_\theta \theta''']' - [GJ \theta']' - 2M_z z'_z \theta - (M_z z'''' + M'_z z'_z) \theta - \frac{M_z^2}{EI_z} \theta + [\beta_z M_z, \theta']' + M_\theta \theta = 0 \]

**3. Solution**

The method described in [6] and in the authors’ earlier works [3,4,5] was used to solve differential equation with variable coefficients (10). According to this method, the solution is sought in the form of the Chebyshev series

\[ \theta(x) = \sum_{i=0}^{\infty} a_i[\theta] T_i(x) = \sum_{i=0}^{\infty} \theta_i T_i(x), \]

where \( \sum_{i=0}^{\infty} a_i = a_0 / 2 + a_1 + a_2 + a_3 + \ldots \). The coefficients of expansion in the case of the 4-th order equation

\[ \sum_{m=0}^{4} P_m(x) \theta^{(4-m)}(x) = 0 \]

satisfy the following recursive relations (see [6], theorem pp. 231 and 323)
\[ \sum_{k=1}^{\infty} \left[ 8(k^2 - 9)(k^2 - 4)(k^2 - 1) k (a_{k-1}(Q) + a_{k-1}(Q)) + 4(k^2 - 9)(k^2 - 4)(k^2 - 1) (a_{k-1-2}(Q) + a_{k-1-2}(Q) - a_{k-1-3}(Q) - a_{k-1+1}(Q)) + 2(k^2 - 9)(k^2 - 4)((k + 1)(a_{k+1-2}(Q) + a_{k-1-2}(Q)) - 2k (a_{k-1}(Q) + a_{k+1}(Q)) + (k-1)(a_{k-1-2}(Q) + a_{k+1-2}(Q))) + (k^2 - 9) ((k + 1)(k + 2) (a_{k-1-3}(Q) + a_{k-1+1}(Q)) - 3(k - 1)(k + 2) (a_{k-1-2}(Q) + a_{k-1+1}(Q)) + 3(k + 1)(k - 2) (a_{k-1-2}(Q) + a_{k+1-3}(Q)) + (k - 1)(k - 2)(a_{k+1-3}(Q) + a_{k+1+1}(Q)))) + \frac{1}{2} ((k + 1)(k + 2)(k + 3) (a_{k-4-4}(Q) + a_{k+1-4}(Q))) - 4(k + 3)(k^2 - 4)(a_{k-2-2}(Q) + a_{k+2-2}(Q)) + 6(k^2 - 9) (a_{k-1-3}(Q) + a_{k+1-3}(Q)) + 4(k - 3)(k^2 - 4)(a_{k-2-2}(Q) + a_{k+2-2}(Q)) + (k - 1)(k - 2)(k - 3)(a_{k-3-3}(Q) + a_{k+3+3}(Q))) \right] a_{[\theta]} = 0 , \quad k = 0, 1, 2, 3, ... \\

\text{where } a_{[\theta]} = \theta , \text{ and functions } Q_n \text{ are defined by the formulas} \\
\[ Q_n(x) = \sum_{j=0}^{m} (-1)^{n+j} \left( \begin{array}{c} 4 - j \\ m - j \end{array} \right) P_j^{(m-j)} , \quad m = 0, 1, ..., 4 \] (14)

Having substituted the coefficients of the expansion of functions \( Q_n \) into Chebyshev series into equation (13), one gets the following infinite system of algebraic equations

\[ \sum_{l=0}^{\infty} \left( E_{k,l} + G_{k,l} + \lambda (B_{k,l} + Z_{k,l} + T_{k,l}) + \lambda^2 N_{k,l} \right) \theta_j = 0 , \quad k = 0, 1, 2, ... \] (15)

Coefficient \( \lambda \) in equation (15) is a parameter specifying the value of bending moment \( M_y = \lambda M_{0y} \), where \( M_{0y} \) defines a certain adopted reference system. At this stage in the solution, terms \( E_{k,l} , G_{k,l} , B_{k,l} , Z_{k,l} , T_{k,l} , N_{k,l} \) of equation (15) contain the coefficients of the expansion of the functions being the coefficients of equation (10), as well as the coefficients of the expansion of the derivatives of the functions. When terms \( E_{k,l} , G_{k,l} , B_{k,l} , Z_{k,l} , T_{k,l} , N_{k,l} \) are transformed using the formula ([6], p. 124)

\[ a_l = \left( a_{l-1}^{(1)} - a_{l-1}^{(-1)} \right) / 2l , \quad l \neq 0 \] (16)

where \( a_{l} = a_{l}[f] \), \( a_{l}^{(p)} = a_{l}[\frac{d^p f}{dx^p}] \), terms \( E_{k,l} , G_{k,l} , B_{k,l} , Z_{k,l} , T_{k,l} , N_{k,l} \) of equation (15) ultimately assume the form

\[ E_{k,l} = 8(k^2 - 9)(k^2 - 4) l \left[ (k + 1)(l - 1) e_{k-1} - 2 \sum_{j=1}^{l-1} (k - l + 2 j) e_{k-1-2j} + (k - 1)(l - 1) e_{k+1} \right] \] (17)

\[ G_{k,l} = -2 (k^2 - 9) l \left[ (k + 1)(k + 2) (g_{k-1-2} - g_{k+1-2}) - 2(k^2 - 4) (g_{k-1} - g_{k+1}) + (k - 1)(k - 2) (g_{k-1+2} - g_{k+1+2}) \right] , \] (18)
\[ B_{k,l} = 2 (k^2 - 9) l [(k + 1)(k + 2) (m\beta_{k-l-2} - m\beta_{k+l-2}) - 2(k^2 - 4) (m\beta_{k-l} - m\beta_{k+l}) + (k-1)(k-2) (m\beta_{k-l} - m\beta_{k+l})] \] (19)

\[ Z_{k,l} = -2 ((k + 1)(k + 2)(k + 3)((k - l - 2)(k - l - 3)mz_{k-l-2} + (k + l - 2)(k + l - 3)mz_{k+l-2}) - 2(k - 2)(k - l) ((k + 2)(k + 3) - l(k + 5)mz_{k-l}) - 2(k + 2)(k + 3)(k + l) ((k - 2)(k - 3) - l(k - 5)mz_{k+l}) + (k - 1)(k - 2)(k - l + 2)(k - l + 3)mz_{k-l-2} + (k + l - 2)(k + l + 3)mz_{k+l-2})] \] (20)

\[ T_{k,l} = \frac{1}{2} ((k + 1)(k + 2)(k + 3)(mt_{k-l-4} + mt_{k+l-4}) - 4(k + 3)(k^2 - 4)(mt_{k-l-2} + mt_{k+l-2}) + 6k(k^2 - 9)(mt_{k-l} + mt_{k+l}) - 4(k - 3)(k^2 - 4)(mt_{k+l+4} + mt_{k+l+4}) + (k - 1)(k - 2)(k - 3)(mt_{k+l+4} + mt_{k+l+4})) \] (21)

\[ N_{k,l} = \frac{1}{2} ((k + 1)(k + 2)(k + 3)(me_{k-l-4} + me_{k+l-4}) - 4(k + 3)(k^2 - 4)(me_{k-l-2} + me_{k+l-2}) + 6k(k^2 - 9)(me_{k-l} + me_{k+l}) - 4(k - 3)(k^2 - 4)(me_{k+l+4} + me_{k+l+4}) + (k - 1)(k - 2)(k - 3)(me_{k+l+4} + me_{k+l+4})) \] (22)

The following denotations: \( e_i = a_i[E/\alpha] \), \( g_i = a_i[GJ] \), \( m\beta_i = a_i[\beta_i M_i] \), \( mz_i = a_i[M_i z_i] \), \( mt_i = a_i[M_i] \), \( me_i = a_i[M_i^2/EJ_i] \) were used in formulas (17)-(22). The first four equations in system (15) are identically satisfied. They are replaced with four equations describing the boundary conditions. In order to formulate the equations the following identities specifying the values of the Chebyshev polynomials and their derivatives in points \( x = \pm 1 \) are used:

\[
T_n(1) = 1, \quad T_n'(1) = n^2, \\
T_n^2(1) = \frac{(n^2 - 1)^2}{3}, \quad T_n'''(1) = n^2(n^2 - 1)(n^2 - 4)/15, \\
T_n^{(m)}(-1) = (-1)^{n-m} T_n^{(m)}(1). \] (23)

4. Calculation cases

In order to verify the algorithm it was used to solve calculation cases. The first two cases were taken from [1].

Case 1

The problem of the lateral torsional buckling (LTB) of a simply supported nonprismatic beam with the following parameters: \( L = 4.0 \) m, \( L = 6.0 \) m, \( E = 210 GPa \), \( v = 0.3 \), \( h_{\text{min}} = \alpha h_{\text{max}} \) is analyzed. The beam is loaded at its ends with concentrated moments \( M_i = \lambda \) and \( M_r = \lambda \psi \) (Fig. 2). Two variants are considered: 1) the beam has a bisymmetric cross section and 2) the beam has a monosymmetric cross section. The dimensions of the beam cross sections are shown in Fig. 2.
The critical moment values calculated using the proposed method are presented in Table 1 (for the beam bisymmetric in cross section) and in Table 2 (for the beam monosymmetric in cross section). The tables also show the critical moments reported by the authors of [1] and the critical moments calculated using FEM (Sofistik) by the present authors.

### Table 1. Web tapered thin-walled beam with doubly symmetric cross-section: linear critical moments ($M_{cr}$)

<table>
<thead>
<tr>
<th>$L$(m)</th>
<th>$\alpha$</th>
<th>The critical bending moments (kNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\psi = 0.25$</td>
<td>$\psi = 0.50$</td>
</tr>
<tr>
<td>4</td>
<td>0.6</td>
<td>206.5</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>219.6</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>232.8</td>
</tr>
<tr>
<td>6</td>
<td>0.6</td>
<td>118.4</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>123.4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>128.2</td>
</tr>
</tbody>
</table>

### Table 2. Web tapered thin-walled beam with singly symmetric cross-section: linear critical moments ($M_{cr}$)

<table>
<thead>
<tr>
<th>$L$(m)</th>
<th>$\alpha$</th>
<th>The critical bending moments (kNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\psi = 0.25$</td>
<td>$\psi = 0.50$</td>
</tr>
<tr>
<td>4</td>
<td>0.6</td>
<td>67.3</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>67.3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>67.5</td>
</tr>
<tr>
<td>6</td>
<td>0.6</td>
<td>47.9</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>47.7</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>47.6</td>
</tr>
</tbody>
</table>
Case 2

A cantilever beam loaded at its free end with concentrated force $P = \lambda$ applied to cross section shear centre $C$ is analyzed. Two types of bisymmetric beams: 1) a cantilever with constant flange width $b_{\text{max}}$ and linearly variable web height $h_{p,\text{min}} = \alpha h_{p,\text{max}}$ and 2) a cantilever with constant height $h_{p,\text{max}}$ and linearly variable flange width $b_{\text{min}} = \beta b_{\text{max}}$ are analyzed. The dimensions of the cross sections of the analyzed beams are shown in Fig. 3. Different beam lengths $L$ are assumed in the calculations. The other parameters take on the same values as in Case 1. The calculated values of critical force $P_{cr}$ are presented in Table 3 and similarly as in Case 1, they are compared with the ones reported by the authors of [1] and with the results determined using FEM.

![Fig. 3. Cantilever beam and dimensions of cross sections.](image)

Table 3. Tapered cantilevers with equal flanges: elastic critical loads

<table>
<thead>
<tr>
<th>$L$(m)</th>
<th>$\alpha$ or $\beta$</th>
<th>LTB of cantilever beam (kN)</th>
<th>Web tapered</th>
<th>Flange tapered</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.2</td>
<td>42.53</td>
<td>42.72</td>
<td>41.74</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>43.92</td>
<td>43.58</td>
<td>43.12</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>45.22</td>
<td>44.65</td>
<td>44.41</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>46.46</td>
<td>45.08</td>
<td>45.70</td>
</tr>
<tr>
<td>8</td>
<td>0.2</td>
<td>20.95</td>
<td>20.92</td>
<td>20.60</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>21.55</td>
<td>21.42</td>
<td>21.20</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>22.15</td>
<td>21.81</td>
<td>21.80</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>22.72</td>
<td>22.13</td>
<td>22.38</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>12.29</td>
<td>12.24</td>
<td>12.10</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>12.60</td>
<td>12.52</td>
<td>12.41</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>12.91</td>
<td>12.59</td>
<td>12.72</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>13.22</td>
<td>11.60</td>
<td>13.03</td>
</tr>
</tbody>
</table>
Case 3
Similarly as in Case 2, a cantilever beam with constant height $h_{\text{max}}$ and linearly variable flange width $h_{\text{min}} = \beta h_{\text{max}}$ is analyzed, but the load is different: the beam along its entire length is subjected to uniformly distributed load $q = \lambda$. Three different points of load application: the centroid, the top flange and the bottom flange are assumed. Different beam lengths $L$ are assumed. The other parameters take on similar values as in Case 2. The determined values of critical load $q_{cr}$ are shown in Table 4 and compared with the ones obtained using FEM.

<table>
<thead>
<tr>
<th>$L$(m)</th>
<th>$\beta$</th>
<th>Upper flange</th>
<th>Centroid</th>
<th>Lower flange</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Present method</td>
<td>FEM</td>
<td>SofiaStik</td>
</tr>
<tr>
<td>6</td>
<td>0.2</td>
<td>7.66</td>
<td>7.56</td>
<td>13.29</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>9.20</td>
<td>9.15</td>
<td>17.68</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>10.57</td>
<td>10.54</td>
<td>22.14</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>11.84</td>
<td>11.82</td>
<td>26.71</td>
</tr>
<tr>
<td>8</td>
<td>0.2</td>
<td>3.22</td>
<td>3.18</td>
<td>4.93</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>3.90</td>
<td>3.87</td>
<td>6.45</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>4.51</td>
<td>4.49</td>
<td>7.99</td>
</tr>
<tr>
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<td>0.8</td>
<td>5.08</td>
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<td>9.57</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
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<td>1.62</td>
<td>2.32</td>
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<tr>
<td></td>
<td>0.4</td>
<td>2.00</td>
<td>1.99</td>
<td>3.00</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>2.33</td>
<td>2.32</td>
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<td>0.8</td>
<td>2.64</td>
<td>2.63</td>
<td>4.38</td>
</tr>
</tbody>
</table>

5. Conclusion
The numerical tests validated the algorithm used to solve the problem of the lateral torsional buckling of the nonprismatic thin-walled beam with an open cross section. The comparisons showed very good agreement between the determined critical moments and critical forces and the values reported by the authors of [1] and the results obtained using FEM. Larger differences can be noticed only in Case 2 where the values obtained by the present authors are by about 0.3-12.3% higher than the ones reported in [1], but they differ only slightly from the values obtained using FEM.

References