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On the Continuity of the Volterra Variational Derivative

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We give a sufficient condition for the continuity of the Volterra variational derivative of a functional with respect to a fixed function. For linear functionals this condition is automatically satisfied, and so the Volterra variational derivative of a linear functional is always continuous. © 1987 Academic Press, Inc.

Throughout, all functions and functionals are real-valued. The linear space of all continuous functions on an interval $[a, b]$ is denoted by $\mathscr{C}([a, b])$. Given a Lebesgue integrable function λ on $[a, b]$, we define a linear functional L on $\mathscr{C}([a, b])$ by setting

$$L(f) = \int_a^b \lambda(x) f(x)\, dx$$

for each $f \in \mathscr{C}([a, b])$. Hamilton and Nashed proved (see [1, Theorem 2]) that if λ is bounded from below, and for each $x \in [a, b]$ the Volterra variational derivative $\delta L/\delta f(x)$ with $f = 0$ exists and equals $\lambda(x)$, then λ is continuous. Their proof utilizes in an essential way the boundedness from below of λ: an assumption which will be shown here to be unnecessary. The result itself is an interesting and potentially useful observation which deserves to be formulated in a more general setting and proved in a transparent and straightforward manner. It turns out that the added generality enhances the understanding of what is going on.

Let X be a completely regular Hausdorff space, and let μ be a locally finite Baire measure in X which assigns a positive measure to each open Baire subset of X. By $\mathscr{H}$ we denote the family of all bounded continuous functions on X which do *not change sign* and whose support is contained in a Baire set of finite measure. Thus $0 \neq \int_X h\, d\mu \neq \pm\infty$ for each nonvanishing function $h \in \mathscr{H}$. Let $\mathscr{F}$ be a collection of functions on X (not necessarily continuous) which is stable with respect to perturbations by elements of $\mathscr{H}$, i.e., if $f \in \mathscr{F}$ and $h \in \mathscr{H}$, then $f + h \in \mathscr{F}$. Finally, let J be a functional on $\mathscr{F}$.

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DEFINITION. Let $x \in X$ and $f \in \mathscr{F}$.

(i) We say that J is *continuous* at (x, f) if given $\varepsilon > 0$, we can find a $\delta > 0$ and a neighborhood U of x so that

$$|J(f+h) - J(f)| < \varepsilon$$

for each $h \in \mathscr{H}$ with $|h| < \delta$, $h(x) \neq 0$, and $\operatorname{supp} h \subset U$.

(ii) We say that J is *strongly continuous* at (x, f) if there are a $\delta > 0$ and a neighborhood U of x such that J is continuous at $(x, f+k)$ for each $k \in \mathscr{H}$ with $|k| < \delta$ and $\operatorname{supp} k \subset U$.

(iii) We say that J is *Volterra differentiable* at (x, f) if there is a real number α such that given $\varepsilon > 0$, we can find a $\delta > 0$ and a neighborhood U of x so that

$$\left| \frac{J(f+h) - J(f)}{\int_X h \, d\mu} - \alpha \right| < \varepsilon$$

for each $h \in \mathscr{H}$ with $|h| < \delta$, $h(x) \neq 0$, and $\operatorname{supp} h \subset U$.

It follows easily from the assumptions imposed on the measure μ that the number α from part (iii) of the previous definition is determined uniquely. We call it the *Volterra derivative* of J at (x, f) denoted by $\delta J/\delta f(x)$. It is immediate that J is continuous at (x, f) whenever $\delta J/\delta f(x)$ exists. In particular, the existence of $\delta J/\delta f(x)$ for all $f \in \mathscr{F}$ implies the strong continuity of J at (x, f) for any $f \in \mathscr{F}$.

THEOREM. *Let $f \in \mathscr{F}$, and let D be the set of those $x \in X$ for which $\delta J/\delta f(x)$ exists. Then the function $x \mapsto \delta J/\delta f(x)$ on D is continuous at $x_0 \in D$ whenever J is strongly continuous at (x_0, f).*

Proof. Let $x_0 \in D$ be such that J is strongly continuous at (x_0, f), and let $\varepsilon > 0$. Find a $\delta > 0$ and an open neighborhood U of x_0 so that

$$\left| \frac{J(f+h) - J(f)}{\int_X h \, d\mu} - \frac{\delta J}{\delta f(x_0)} \right| < \frac{\varepsilon}{2} \tag{1}$$

for each $h \in \mathscr{H}$ with $|h| < 2\delta$, $h(x_0) \neq 0$, and $\operatorname{supp} h \subset U$. Using the strong continuity of J at (x_0, f), we may assume that J is continuous at $(x_0, f+k)$ for each $k \in \mathscr{H}$ with $|k| < \delta$ and $\operatorname{supp} k \subset U$. If $x \in D \cap U - \{x_0\}$, choose a $k_0 \in \mathscr{H}$ so that $|k_0| < \delta$, $k_0(x) \neq 0$, $\operatorname{supp} k \subset U$, and

$$\left| \frac{J(f+k_0) - J(f)}{\int_X k_0 \, d\mu} - \frac{\delta J}{\delta f(x)} \right| < \frac{\varepsilon}{2}.$$

Let $\gamma > 0$ be such that

$$\left| \frac{J(f+k_0) - J(f) + \alpha}{\int_X k_0 \, d\mu + \beta} - \frac{\delta J}{\delta f(x)} \right| < \frac{\varepsilon}{2} \tag{2}$$

whenever $|\alpha| < \gamma$ and $|\beta| < \gamma$. As J is continuous at $(x_0, f + k_0)$, there are $\delta^* > 0$ and a Baire neighborhood V of x_0 such that $V \subset U$, $\mu(V) < +\infty$, and

$$|J(f + k_0 + h) - J(f + k_0)| < \gamma \tag{3}$$

for each $h \in \mathscr{H}$ with $|h| < \delta^*$, $h(x_0) \neq 0$, and $\operatorname{supp} h \subset V$. Let u be a continuous function on X such that $0 \leqslant u \leqslant 1$, $u(x_0) = 1$, and $\operatorname{supp} u \subset V$. Find a positive $\eta < \min(\delta, \delta^*)$ so that $\eta \int_X u \, d\mu < \gamma$, and let $h_0 = \eta u \operatorname{sign} k_0(x)$. Then both h_0 and $h = h_0 + k_0$ belong to $\mathscr{H}$, $|h_0| < \delta$, $|h| < 2\delta$, $h_0(x_0) \, h(x_0) \neq 0$, $\operatorname{supp} h_0 \subset \operatorname{supp} h \subset U$, and $|\int_X h_0 \, d\mu| < \gamma$. Thus by (1)–(3),

$$\begin{aligned} &\left| \frac{\delta J}{\delta f(x)} - \frac{\delta J}{\delta f(x_0)} \right| \\ &\quad \leqslant \left| \frac{\delta J}{\delta f(x)} - \frac{J(f+k_0) - J(f) + [J(f + k_0 + h_0) - J(f + k_0)]}{\int_X k_0 \, d\mu + \int_X h_0 \, d\mu} \right| \\ &\quad\quad + \left| \frac{J(f+h) - J(f)}{\int_X h \, dm} - \frac{\delta J}{\delta f(x_0)} \right| < \varepsilon, \end{aligned}$$

and the theorem is proved.

COROLLARY. *Let $\mathscr{F}$ be a linear space and let J be a linear functional. Choose an $f \in \mathscr{F}$ and denote by D the set of those $x \in X$ for which $\delta J/\delta f(x)$ exists. Then the function $x \mapsto \delta J/\delta f(x)$ is continuous on D.*

Indeed, for a linear functional J the existence of $\delta J/\delta f(x)$ for some $f \in \mathscr{F}$ implies the existence of $\delta J/\delta f(x)$ for all $f \in \mathscr{F}$; and of course, the value of $\delta J/\delta f(x)$ is independent of the choice of $f \in \mathscr{F}$.

Now with no restriction on λ, the result of Hamilton and Nashed is an immediate consequence of the Corollary. The second example in [1, Section 5.4] does not contradict the Corollary, as it is easy to see that in this example $\delta J/\delta y(0)$ exists only when 0 is an isolated point of $[-1, 1]$.

EXAMPLE. Let $X = [0, 1]$, $\mathscr{F} = \mathscr{C}([0, 1])$, and for $f \in \mathscr{F}$, set $J(f) = \int_0^1 f(x) \, dx$ if $f(0) \neq 0$, and $J(f) = 0$ otherwise. Letting $f \equiv 0$, it is easy to see that J is not strongly continuous at $(0, f)$. We also have $\delta J/\delta f(0) = 1$, and $\delta J/\delta f(x) = 0$ for each $x \in (0, 1]$.

REFERENCE

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