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Relative z-ideals in $C(X)$

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ABSTRACT

For every two ideals $I \subseteq J$ in $C(X)$, we call I a z_J-ideal if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in J$ imply that $g \in I$. An ideal I is called a relative z-ideal, briefly a *rez*-ideal, if there exists an ideal J such that $I \subsetneqq J$ and I is a z_J-ideal. We have shown that for any ideal J in $C(X)$, the sum of every two z_J-ideals is a z_J-ideal if and only if X is an F-space. It is also shown that every principal ideal in $C(X)$ is a *rez*-ideal if and only if X is an almost P-space and the spaces X for which the sum of every two *rez*-ideals is a *rez*-ideal are characterized. Finally for a given ideal I in $C(X)$, the existence of greatest ideal J such that I to be a z_J-ideal and also for given two ideals $I \subseteq J$ in $C(X)$, a greatest z_J-ideal contained in I and the smallest z_J-ideal containing I are investigated.

1. Preliminaries

Throughout this paper, we denote by $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff space X and for terminology and notations, the reader is referred to [2,5,6]. For every $f \in C(X)$, the intersection of all maximal (minimal prime) ideals of $C(X)$ containing f is denoted by M_f (P_f). An ideal I in $C(X)$ is called a z-ideal (z°-ideal) if $M_f \subseteq I$ ($P_f \subseteq I$), $\forall f \in I$. It is easy to see that $M_f = \{g \in C(X)\colon\ Z(f) \subseteq Z(g)\}$ and $P_f = \{g \in C(X)\colon\ \operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)\}$, see also [2,3]. Equivalently I is a z-ideal (z°-ideal) if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ ($\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$) imply that $g \in I$. Clearly M_f (P_f) itself is a z-ideal (z°-ideal) for every $f \in C(X)$, which we call a basic z-ideal (z°-ideal). Note that $P_f = C(X)$ if and only if $\operatorname{int}_X Z(f) = \emptyset$. Since the sum and the intersection of z-ideals in $C(X)$ is a z-ideal, then for a given ideal I in $C(X)$ the smallest z-ideal containing I and the greatest z-ideal contained in I always exist and in the notation of Mason in [6], we denote these z-ideals by I_z and I^z respectively. The following proposition which is proved in [2] characterizes the ideals I_z and I^z in term of basic z-ideals. This proposition also gives an elementwise characterization for these ideals. For a different elementwise characterization, see [6].

Proposition 1.1. *If I is an ideal in $C(X)$, then $I_z = \{g \in C(X)\colon\ g \in M_f$ for some $f \in I\} = \sum_{f \in I} M_f$ and $I^z = \{g \in C(X)\colon\ M_g \subseteq I\} = \sum_{M_f \subseteq I} M_f$.*

An arbitrary intersection of z°-ideals is also a z°-ideal and hence the smallest z°-ideal $I_\circ$ containing a given ideal I always exists. But the sum of two z°-ideals even in $C(X)$ need not be a z°-ideal. A necessary and sufficient condition that the sum of z°-ideals in $C(X)$ be a z°-ideal is given by the following theorem due to B. de Pagter in 11.1 of [9] using different

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terminology. First we recall that a completely regular Hausdorff space X is an F-space (resp. quasi F-space) if its cozerosets (resp. dense cozerosets) are C^*-embedded. Equivalently, X is an F-space (resp. quasi F-space) if finitely generated ideals (resp. finitely generated ideals containing a nondivisor of 0) of $C(X)$ are principal. By 14.26 in [5], we have also that X is an F-space if and only if every ideal in $C(X)$ is absolutely convex. An ideal I in a partially ordered (lattice ordered) ring is called convex (absolutely convex) if, whenever $0 \leqslant x \leqslant y$ ($|x| \leqslant |y|$) and $y \in I$, then $x \in I$. For more details and properties of F-spaces and quasi F-spaces, see [4,5,9].

Theorem 1.2. *The sum of two z°-ideals of $C(X)$ is always a z°-ideal or all of $C(X)$ if and only if X is a quasi F-space.*

If I is a nonregular ideal (i.e., every member of I is a zerodivisor) in $C(X)$, then $I_\circ = \sum_{f\in I} P_f = \{g \in C(X)\colon g \in P_f$ for some $f \in I\}$ and whenever X is a quasi F-space, then the greatest z°-ideal I° contained in I exists and $I^\circ = \sum_{P_f \subseteq I} P_f = \{g \in C(X)\colon\ P_g \subseteq I\}$, see [2].

In any commutative ring, it is well known that every minimal ideal in the class of prime ideals containing a z-ideal is a z-ideal, see Theorem 1.1 in [7]. The following proposition which is proved in [2,8] by different ways, shows that the converse is also true in $C(X)$.

Proposition 1.3. *An ideal I in $C(X)$ is a z-ideal if and only if every prime ideal minimal over I is a z-ideal.*

It follows from Proposition 1.3 that an ideal I in $C(X)$ is a z-ideal if and only if $\sqrt{I}$ is a z-ideal. We have also $I_z = (\sqrt{I})_z$, $I^z = (\sqrt{I})^z$. The corresponding statement holds for z°-ideals in $C(X)$ and for any nonregular ideal I in $C(X)$, we have $I_\circ = (\sqrt{I})_\circ$ and $I^\circ = (\sqrt{I})^\circ$, see [2]. We also cite the following simple result which will be referred to in the sequel.

Proposition 1.4. *Suppose that I is an ideal and P is a prime ideal in $C(X)$. If $I \cap P$ is a z-ideal (z°-ideal), then either I or P is a z-ideal (z°-ideal). In particular if P and Q are prime ideals which are not in a chain and $P \cap Q$ is a z-ideal (z°-ideal), then both P and Q are z-ideals (z°-ideals).*

A nonzero ideal in a commutative ring is said to be essential if it intersects every nonzero ideal nontrivially. The following proposition which topologically characterizes essential ideals of $C(X)$ is proved in [1].

Proposition 1.5. *A nonzero ideal E in $C(X)$ is an essential ideal if and only if $\bigcap Z[E] = \bigcap_{f\in E} Z(f)$ is nowhere dense (has an empty interior).*

One can easily see that every free ideal in $C(X)$ is essential and a principal ideal (f) in $C(X)$ is essential if and only if $\operatorname{int}_X Z(f) = \emptyset$. It is also easy to see that every non-maximal prime ideal in $C(X)$ is an essential ideal.

2. Relative z-ideals (z°-ideals) in $C(X)$

For every two ideals $I \subseteq J$ in $C(X)$, I is said to be a z_J-ideal if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in J$ imply that $g \in I$. In other words, I is called a z_J-ideal if $M_f \cap J \subseteq I$, $\forall f \in I$. Clearly every ideal I is a z_I-ideal and every z-ideal in $C(X)$ is a z_J-ideal for all ideals J containing I. We call an ideal I a relative z-ideal, or briefly a *rez*-ideal if there exists an ideal J in $C(X)$ such that $I \subsetneqq J$ and I is a z_J-ideal. Similarly an ideal I in $C(X)$ is called a z°_J-ideal if $I \subseteq J$ and $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$, $f \in I$ and $g \in J$ imply that $g \in I$ or equivalently if $P_f \cap J \subseteq I$, $\forall f \in I$. I is called a relative z°-ideal or briefly a *rez*°-ideal if there exists an ideal J in $C(X)$ such that $I \subsetneqq J$ and I is a z°_J-ideal. Clearly every z°_J-ideal in $C(X)$ is a z_J-ideal and every z°-ideal in $C(X)$ is a *rez*°-ideal.

According to the above definitions, the proof of the following proposition is evident. By this proposition, it turns out that for every ideal J and every z-ideal (z°-ideal) K in $C(X)$, $J \cap K$ is a z_J-ideal (z°_J-ideal).

Proposition 2.1. *Let I and J be two ideals in $C(X)$ and $I \subseteq J$.*

(a) *The following statements are equivalent*:
   (a$_1$) *I is a z_J-ideal.*
   (a$_2$) *$I_z \cap J = I$.*
   (a$_3$) *There exists a z-ideal K in $C(X)$ such that $K \cap J = I$.*
(b) *The following statements are equivalent*:
   (b$_1$) *I is a z°_J-ideal.*
   (b$_2$) *$I_\circ \cap J = I$.*
   (b$_3$) *There exists a z°-ideal K in $C(X)$ such that $K \cap J = I$.*

Whenever J is a z-ideal, then every z_J-ideal $I \subseteq J$ is also a z-ideal. In fact, if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in J$ for J is a z-ideal. Now since I is a z_J-ideal, then $g \in I$, i.e., I is a z-ideal. On the other hand in any ideal J there are many z-ideals, for example if $f \in J$, then $O_{Z(f)} = \{g \in C(X)\colon\ Z(f) \subseteq \operatorname{int}_X Z(g)\}$ is a z-ideal contained in J. The following proposition shows the existence of z_J-ideals in any given ideal J which are not z-ideals.

Proposition 2.2. *Suppose that J is an ideal in $C(X)$ which is not a z-ideal. Then there exists an ideal $I \subsetneqq J$ which is a z_J-ideal but not a z-ideal.*

Proof. Since J is not a z-ideal, then there exist $k \in J$ and $h \in C(X)$ such that $Z(k) \subseteq Z(h)$ and $h \notin J$. Consider $g \in C(X)$, where $Z(g) \cap Z(h) = \emptyset$, $gh \neq 0$ and take $I = M_g \cap J$. By Proposition 2.1, I is a z_J-ideal and since $0 \neq gh \in I$, $k \in J$ and $k \notin M_g$, then $(0) \neq I \subsetneqq J$. Now it is enough to show that I is not a z-ideal. In fact we have $g^2k \in I$ and $Z(g^2k) \subseteq Z(g^2h)$ but $g^2h \notin I$. For otherwise if $g^2h \in I$, then $k^2h \in J$ implies that $(g^2 + k^2)h \in J$. But $g^2 + k^2$ is a unit and hence $h \in J$, a contradiction. □

Examples 2.3. (a) Every nonessential ideal in $C(X)$ is a *rez*-ideal. If I is a nonessential ideal in $C(X)$, then there exists an ideal K in $C(X)$ such that $I \cap K = (0)$. If we let $J = I + K$, obviously $I \subsetneqq J$. We show that I is a z_J-ideal. Let $f \in I$ and $g \in J$ such that $Z(f) \subseteq Z(g)$. Hence $g = i + k$, where $i \in I$, $k \in K$ and $Z(f) \subseteq Z(i + k)$. Now we have $Z(f^2 + i^2) \subseteq Z(k)$, so $X = Z(0) = Z(k(f^2 + i^2)) \subseteq Z(k)$ which implies that $k = 0$. Therefore $g = i \in I$, i.e., I is a z_J-ideal and hence I is a *rez*-ideal. We note that every nonessential ideal in $C(X)$ is not necessarily a z-ideal.

(b) If P and Q are prime ideals in $C(X)$ such that P is not a z-ideal and Q is a z-ideal. Then by Proposition 2.1, $I = P \cap Q$ is a z_P-ideal. Whenever P and Q are not in a chain, then by Proposition 1.4, I is not a z-ideal and hence I will be a *rez*-ideal, for $I \neq P$. Similarly, if we consider Q as a prime z°-ideal not in a chain with P, then I will be a *rez*$^\circ$-ideal.

(c) Finally we show that a principal ideal (f) in $C(X)$ is a *rez*-ideal if and only if $\operatorname{Ann}(f) \neq (0)$ $(\operatorname{int}_X Z(f) \neq \emptyset)$. If $\operatorname{int}_X Z(f) \neq \emptyset$, then by Proposition 1.5, (f) is a nonessential ideal and by example (a), (f) is a *rez*-ideal. Conversely, suppose there exists $J \supsetneqq (f)$ such that (f) is a z_J-ideal and suppose that $\operatorname{Ann}(f) = (0)$. By Proposition 2.1, we have $M_f \cap J = (f)_z \cap J = (f)$. Take $g \in J - M_f$, such g exists for otherwise $J \subseteq M_f$ implies $M_f \cap J = J = (f)$ which contradicts $(f) \subsetneqq J$. Therefore $Z(f) \nsubseteq Z(g)$ and hence there exists $x_0 \in Z(f)$ such that $g(x_0) \neq 0$. Clearly $gf^{1/3} \in M_f \cap J = (f)$ and consequently there exists $k \in C(X)$ such that $gf^{1/3} = kf$. Now if $x \notin Z(f)$, we have $g(x) = k(x)f^{2/3}(x)$. But $x_0 \in Z(f)$ and $\operatorname{int}_X Z(f) = \emptyset$ imply that there exists a net (x_α) in $X \setminus Z(f)$ such that $x_\alpha \to x_0$. But $g(x_\alpha) = k(x_\alpha)f^{2/3}(x_\alpha) \to 0$ which contradicts $g(x_0) \neq 0$. This means that $\operatorname{int}_X Z(f) \neq \emptyset$ or $\operatorname{Ann}(f) \neq (0)$.

By example (c) above, we have the following corollary. We recall that a space X is an almost P-space if every nonempty zeroset (or every nonempty G_δ-set) in X has a nonempty interior.

Corollary 2.4. *Every principal ideal in $C(X)$ is a rez-ideal if and only if X is an almost P-space.*

The concepts "*rez*-ideal" ("*rez*$^\circ$-ideal") and "z-ideal" ("z°-ideal") coincide for prime ideals of $C(X)$. Moreover, if I is a z_J-ideal, then J is contained in every non-z-ideal prime ideal minimal over I. This shows that whenever P is a prime ideal minimal over I which is not a z-ideal and I is a z_P-ideal, then P is the greatest member of $\{J\colon\ I$ is a z_J-ideal$\}$. In this case, P is the only prime ideal minimal over I which is not a z-ideal.

Proposition 2.5.

(a) *Every prime rez-ideal in $C(X)$ is a z-ideal.*
(b) *Suppose that P is a prime ideal in $C(X)$ which is not a z-ideal and it is minimal over a z_J-ideal I. Then $J \subseteq P$. In case I is not a z-ideal, then there exists at most one prime ideal P minimal over I such that I is a z_P-ideal.*
(c) *If Q is a semiprime (absolutely convex) ideal in $C(X)$, then every z_Q-ideal is also a semiprime (absolutely convex) ideal.*

Proof. (a) If P is a prime *rez*-ideal, then there exists an ideal J in $C(X)$ such that $P \subsetneqq J$ and $P_z \cap J = P$. This shows that either $J \subseteq P$ which implies that $P = J$, a contradiction or $P_z \subseteq P$ which implies that $P = P_z$, i.e., P is a z-ideal.

(b) Let P be a prime ideal minimal over I which is not a z-ideal. Since $I_z \cap J = I \subseteq P$, then either $I_z \subseteq P$ or $J \subseteq P$. $I_z \subseteq P$ implies that P is a z-ideal by Proposition 1.3 which contradicts our hypothesis, hence $J \subseteq P$. If P and Q are two prime ideals minimal over I such that I is a z_P-ideal and is a z_Q-ideal, then clearly P and Q are not z-ideals, for $I = I_z \cap P = I_z \cap Q$ and I is not a z-ideal. Now by first half of this part, $P \subseteq Q$ and $Q \subseteq P$ imply that $P = Q$.

(c) Since $I_z \cap Q = I$, then I is a semiprime (an absolutely convex) ideal. □

In the following proposition, we observe that for any semiprime ideal I, the collection $\{J\colon\ I$ is a z_J-ideal$\}$ has a largest member. We call an ideal I an almost z-ideal if in every representation of $\sqrt{I}$ as an intersection of prime ideals, there exists at least one prime z-ideal.

Proposition 2.6.

(a) *Every rez-ideal in $C(X)$ is an almost z-ideal.*
(b) *For every semiprime ideal Q in $C(X)$, there exists a greatest ideal J containing Q such that Q is a z_J-ideal. Moreover, a semiprime ideal is a rez-ideal if and only if it is an almost z-ideal.*
(c) *If I is a rez-ideal, then $\sqrt{I}$ is also a rez-ideal.*

Proof. (a) If I is a *rez*-ideal, then there exists an ideal $J \supsetneqq I$ such that $I_z \cap J = I$. Suppose that I is not an almost z-ideal, then $\sqrt{I} = \bigcap_{\alpha \in S} P_\alpha$, where P_α is a non-z-ideal prime ideal minimal over I, $\forall \alpha \in S$. Now by Proposition 2.5(b), $J \subseteq P_\alpha$, $\forall \alpha \in S$ and hence $J \subseteq \sqrt{I}$. But $J = (\sqrt{I})_z \cap J = I_z \cap J = I$ contradicts $J \supsetneqq I$. Therefore I is an almost z-ideal.

(b) Let Q be a semiprime ideal, A be the collection of all non-z-ideals prime ideals minimal over Q and $J = \bigcap_{P \in A} P$. Clearly $Q \subseteq J$, moreover $Q_z \cap J = Q$, for Q_z is the intersection of all prime z-ideals minimal over Q and hence $Q_z \cap J$ is the intersection of all minimal prime ideals over Q. This implies that Q is a z_J-ideal. Whenever K is an ideal containing Q and Q is a z_K-ideal, then $K \subseteq P$, $\forall P \in A$, by Proposition 2.5(b), i.e., $K \subseteq J$. This means that J is the greatest ideal such that Q is a z_J-ideal. The proof of the second part of (b) is evident by part (a).

(c) Since I is a *rez*-ideal, then I is an almost z-ideal by part (a). Therefore $\sqrt{I} \subsetneqq \bigcap_{P \in A} P = J$, where A is the collection of all non-z-ideals prime ideals minimal over I. Now $(\sqrt{I})_z \cap J = I_z \cap J = \sqrt{I}$ implies that $\sqrt{I}$ is a *rez*-ideal. □

Not only for semiprime ideals, but for every ideal I in $C(X)$, where X is an F-space, there exists a greatest ideal J such that I is a z_J-ideal.

Proposition 2.7. *If X is an F-space, then for every ideal I in $C(X)$, the collection $\{J\colon\ I \text{ is a } z_J\text{-ideal}\}$ has a greatest member.*

Proof. We put $J_\circ = \{f \in C(X)\colon\ M_g \cap (f) \subseteq I,\ \forall g \in I\}$ and show that $J_\circ$ is an ideal. First we prove that whenever $M_g \cap (f) \subseteq I$, then $M_g \cap (|f|) \subseteq I$. To see this let $h \in M_g \cap (|f|)$, then $Z(g) \subseteq Z(h)$ and there exists $k \in C(X)$ such that $h = k|f|$ and $Z(g) \subseteq Z(h) = Z(kf)$. Since X is an F-space, then I is absolutely convex and so $|h| = |kf|$ and $kf \in M_g \cap (f) \subseteq I$ imply that $h \in I$, i.e., $M_g \cap (|f|) \subseteq I$. Next suppose that $f_1, f_2 \in J_\circ$ and $g \in I$. Since X is an F-space, then $M_g \cap (f_1 + f_2) \subseteq M_g \cap (f_1, f_2) = M_g \cap (|f_1| + |f_2|)$, see Theorem 14.25 in [5]. Now if $h \in M_g \cap (|f_1| + |f_2|)$, then $Z(g) \subseteq Z(h)$ and $h = k(|f_1| + |f_2|)$ for some $k \in C(X)$. Thus $Z(g) \subseteq Z(k|f_1|)$ and $Z(g) \subseteq Z(k|f_2|)$, so $k|f_1| \in M_g \cap (|f_1|)$ and $k|f_2| \in M_g \cap (|f_2|)$. On the other hand $f_1, f_2 \in J_\circ$ implies that $M_g \cap (f_1) \subseteq I$, $M_g \cap (f_2) \subseteq I$ and hence $k|f_1| \in M_g \cap (|f_1|) \subseteq I$, $k|f_2| \in M_g \cap (|f_2|) \subseteq I$, imply that $h \in I$, i.e., $M_g \cap (|f_1| + |f_2|) = M_g \cap (f_1 + f_2) \subseteq I$, so $f_1 + f_2 \in J_\circ$. Now suppose that $f \in J_\circ$ and $h \in C(X)$. For all $g \in I$, we have $M_g \cap (fh) \subseteq M_g \cap (f) \subseteq I$ and hence $fh \in J_\circ$. Therefore $J_\circ$ is an ideal. Moreover we have $M_g \cap J_\circ \subseteq I$, $\forall g \in I$, in fact $I_z \cap J_\circ = (\bigcup\{M_g\colon\ g \in I\}) \cap J_\circ = \bigcup\{M_g \cap J_\circ\colon\ g \in I\} \subseteq I$. This shows that I is a $z_{J_\circ}$-ideal. Finally suppose that there exists an ideal K containing I such that I is a z_K-ideal. Hence $M_g \cap (f) \subseteq I_z \cap K = I$, $\forall g \in I$ and $\forall f \in K$. Thus $K \subseteq J_\circ$ and this means that $J_\circ$ is the greatest member of $\{J\colon\ I \text{ is a } z_J\text{-ideal}\}$. □

It is evident that every ideal I in $C(X)$ is a z_I-ideal, but one may ask when is every subideal of a given ideal I a z_I-ideal? The following lemma and corollary show that such an ideal should be a z-ideal whose every member has an open zeroset. As an example of such ideals, consider $C_F(X) = \{f \in C(X)\colon\ X \setminus Z(f) \text{ is finite}\}$. Recall that a space X is a P-space if every zeroset (G_δ-set) in X is open or if every prime ideal in $C(X)$ is a z-ideal, see [5, 4J].

Lemma 2.8. *If $f \in C(X)$, then every subideal of the principal ideal (f) is a $z_{(f)}$-ideal if and only if $Y = X \setminus Z(f)$ is a closed P-space.*

Proof. If every subideal of (f) is a $z_{(f)}$-ideal, then (f^2) is a $z_{(f)}$-ideal and hence $(f^2)_z \cap (f) = (f^2)$, i.e., $M_f \cap (f) = (f^2)$. But $f \in M_f \cap (f) = (f^2)$ implies that $f = kf^2$ for some $k \in C(X)$. Hence $f(1 - kf) = 0$ means that $Z(f) \cup Z(1 - kf) = X$ and $Z(f) \cap Z(1 - kf) = \emptyset$ which imply that $Z(f)$ is open, so Y is closed. Now suppose that $g \in C(Y)$ and g^* is an extension of g on X, say $g^*(x) = g(x)$, $\forall x \in Y$ and $g^*(x) = 1$, $\forall x \notin Y$. Since $(g^* f) \subseteq (f)$, then $(g^* f)$ is a $z_{(f)}$-ideal, i.e., $M_{g^* f} \cap (f) = (g^* f)$. Thus ${g^*}^{1/3} f \in M_{g^* f} \cap (f) = (g^* f)$, but f is a unit on Y, therefore $g^{1/3} \in (g)$ which implies that $Z(g)$ is open and hence Y is a P-space. Conversely suppose that Y is a closed P-space and $I \subseteq (f)$. Assume that $g \in I$ and $h \in M_g \cap (f)$, then there exists $k \in C(X)$ such that $h = fk$ and $Z(g) \subseteq Z(h) = Z(fk)$. Thus $Z(g|_Y) \subseteq Z(f|_Y k|_Y) = Z(k|_Y)$ for f is a unit on Y. Since $Z(k|_Y)$ is open, then $k|_Y$ is a multiple of $g|_Y$, i.e., $t \in C(X)$ exists, such that $k|_Y = t|_Y g|_Y$. Now $kf = tgf \in I$ implies that $h = kf \in I$, i.e., $M_g \cap (f) \subseteq I$ which means that I is a $z_{(f)}$-ideal. □

Corollary 2.9. *Let I be an ideal in $C(X)$. Then every subideal of I is a z_I-ideal if and only if $Z(f)$ is open, $\forall f \in I$.*

Proof. If $Z(f)$ is open, $\forall f \in I$, then I is a z-ideal and clearly every subideal of I is a z_I-ideal. Conversely, if every subideal of I is a z_I-ideal, then for all $f \in I$, (f^2) is a z_I-ideal. Now by definition, (f^2) is also a $z_{(f)}$-ideal and by first part of the proof of Lemma 2.8, $Z(f)$ is open. □

We conclude this section with the following result. Part (f) of this result shows that for every two ideals $I \subseteq J$, there exists the smallest z_J-ideal containing I which we denote by I_{z_J}. By Proposition 2.1, $I_z \cap J$ is a z_J-ideal containing I and if $K \subseteq J$ is also a z_J-ideal containing I, then $K_z \cap J = K$ and hence $I_z \cap J \subseteq K_z \cap J = K$. This means that $I_{z_J} = I_z \cap J$. A similar result may be stated for the smallest z_J°-ideal containing I.

Proposition 2.10. *Let A, B, I, J, K and I_α, $\forall \alpha \in S$ be ideals in $C(X)$.*

(a) *If $I \subseteq J \subseteq K$ and I is a z_K-ideal, then I is also a z_J-ideal.*
(b) *If $I \subseteq J \subseteq K$, I is a z_J-ideal, and J is a z_K-ideal, then I is a z_K-ideal.*
(c) *If $\sqrt{I} \subseteq J$ and I is a z_J-ideal, then $I = \sqrt{I}$. In particular, if I is a $z_{\sqrt{I}}$-ideal, then $I = \sqrt{I}$.*
(d) *If $I \subseteq J$ and I is a z_J-ideal, then $\sqrt{I}$ is a $z_{\sqrt{J}}$-ideal.*
(e) *If $A \subseteq J$, $B \subseteq K$, A is a z_J-ideal and B is a z_K-ideal, then $A \cap B$ is a $z_{J\cap K}$-ideal.*
(f) *If $I_\alpha \subseteq J$ and I_α is a z_J-ideal, $\forall \alpha \in S$, then $\bigcap_{\alpha \in S} I_\alpha$ is also a z_J-ideal.*

Proof. The proof of parts (a), (b) and (c) are evident. For parts (d), (e) and (f), we have

$$(\sqrt{I})_z \cap \sqrt{J} = I_z \cap \sqrt{J} = \sqrt{I_z} \cap \sqrt{J} = \sqrt{I_z \cap J} = \sqrt{I},$$

$$(A \cap B)_z \cap (J \cap K) = A_z \cap B_z \cap J \cap K = (A_z \cap J) \cap (B_z \cap K) = A \cap B,$$

$$\bigcap_{\alpha \in S} I_\alpha \subseteq \Big(\bigcap_{\alpha \in S} I_\alpha\Big)_z \cap J \subseteq \Big(\bigcap_{\alpha \in S} I_{\alpha z}\Big) \cap J = \bigcap_{\alpha \in S} (I_{\alpha z} \cap J) = \bigcap_{\alpha \in S} I_\alpha. \qquad \square$$

3. Sum of relative *z*-ideals (z°-ideals) in $C(X)$

We observed that if $I \subseteq J$ are two ideals in $C(X)$, then I_{z_J}, the smallest z_J-ideal containing I always exists and it is equal to $I_z \cap J$ and is equal to the intersection of all z_J-ideals containing I. In this section we want to investigate the existence of I^{z_J}, the greatest z_J-ideal contained in I for every two ideals $I \subseteq J$ in $C(X)$. Clearly, whenever the sum of every two z_J-ideals is a z_J-ideal, then I^{z_J} exists and we will show in this section that the converse is also true. In Theorem 3.4, we show that the sum of every two z_J-ideals is a z_J-ideal for all ideals J in $C(X)$ if and only if X is an F-space. First we need the following propositions and lemma.

Proposition 3.1. *Let J be an ideal in $C(X)$, then the following statements are equivalent.*

(a) *For every two z-ideals I and K in $C(X)$, $(I + K) \cap J = I \cap J + K \cap J$.*
(b) *Sum of every two z_J-ideals is a z_J-ideal.*
(c) *For every subideal I of J, I^{z_J} exists and $I^{z_J} = \sum_{M_f \cap J \subseteq I} M_f \cap J$.*

Proof. First we show that (a) and (b) are equivalent. If (a) holds and I and K are z_J-ideals, then $I + K = I_z \cap J + K_z \cap J = (I_z + K_z) \cap J = (I + K)_z \cap J$ which means that $I + K$ is a z_J-ideal (note that $(I + K)_z = I_z + K_z$ by Proposition 3.1 in [6]). Conversely, suppose (b) holds and I and K are z-ideals in $C(X)$. Take $f \in (I + K) \cap J$, then $f = f_1 + f_2$, where $f_1 \in I$, $f_2 \in K$ and $f \in J$. Thus $f^2 = ff_1 + ff_2 \in I \cap J + K \cap J$. Since $I \cap J$ and $K \cap J$ are z_J-ideals by Proposition 2.1, then $I \cap J + K \cap J$ is also a z_J-ideal by part (b). Hence $f \in M_{f^2} \cap J \subseteq I \cap J + K \cap J$, i.e., $(I + K) \cap J = I \cap J + K \cap J$. Next we show that (b) and (c) are equivalent. If (b) holds, since each $M_f \cap J$ is a z_J-ideal, then $\sum_{M_f \cap J \subseteq I} M_f \cap J$ is a z_J-ideal contained in I. Now suppose that $T \subseteq I$ and T is also a z_J-ideal. If $g \in T$, then $M_g \cap J \subseteq T \subseteq I$ and therefore $T \subseteq \sum_{M_f \cap J \subseteq I} M_f \cap J$. This means that $I^{z_J} = \sum_{M_f \cap J \subseteq I} M_f \cap J$. Conversely, suppose I^{z_J} exists, $\forall I \subseteq J$ and K and T are two z_J-ideals. Hence $K + T = K^{z_J} + T^{z_J} \subseteq (K + T)^{z_J} \subseteq K + T$, which means that $K + T$ is a z_J-ideal. $\square$

We have a similar result for the sum of z_J°-ideals as follows.

Proposition 3.2. *Suppose that X is a quasi F-space and J is an ideal in $C(X)$, then the following statements are equivalent.*

(a) *For every two z°-ideals I and K in $C(X)$, $(I + K) \cap J = I \cap J + K \cap J$.*
(b) *Sum of every two z_J°-ideals is a z_J°-ideal.*
(c) *For every subideal I of J, $I^{z_J^\circ}$ exists and $I^{z_J^\circ} = \sum_{P_f \cap J \subseteq I} P_f \cap J$.*

Proof. We note that if X is a quasi F-space, then $P_f + P_g$ is a z°-ideal or all of $C(X)$, $\forall f, g \in C(X)$. Now using basic z°-ideals instead of basic z-ideals in the proof of Proposition 3.1, the proof is similar to that of Proposition 3.1 step by step. $\square$

In the following lemma, we show that whenever J is an absolutely convex ideal, then part (a) in Proposition 3.1 holds for every two z-ideals (z_J-ideals or semiprime ideals) I and K in $C(X)$. By Proposition 3.1, this means that for an absolutely convex ideal J, the sum of two z_J-ideals is a z_J-ideal. Although part (b) of the following lemma will show that the hypothesis of absolute convexity of J is needed, we can also give an easy example showing this hypothesis cannot be omitted. To see this, take the principal ideal $J = (i)$ in $C(\mathbb{R})$, where i is the identity function in $C(\mathbb{R})$. J is convex, by 5E in [5]. Now consider two functions $f, g \in C(\mathbb{R})$ defined by $f(x) = 0$, $\forall x \leqslant 0$, $f(x) = x$, $\forall x \geqslant 0$ and $g(x) = 0$, $\forall x \geqslant 0$, $g(x) = x$, $\forall x \leqslant 0$. Therefore $(M_f + M_g) \cap J \neq M_f \cap J + M_g \cap J$ for $i = f + g \in (M_f + M_g) \cap J$ but $i \notin M_f \cap J + M_g \cap J$. In fact if $i \in M_f \cap J + M_g \cap J$, then $i = f'i + g'i$, where $[0, \infty) = Z(g) \subseteq Z(g')$ and $(-\infty, 0] = Z(f) \subseteq Z(f')$. Thus $f'(x) + g'(x) = 1$, $\forall x \neq 0$ and hence $f' + g'$ is a unit which contradicts $f'(0) = 0 = g'(0)$. Now by part (a) of Proposition 3.1, the sum of two z_J-ideals in $C(\mathbb{R})$ is not necessarily a z_J-ideal.

Lemma 3.3.

(a) *Let J be an absolutely convex ideal and I and K be two z-ideals (or z_J-ideals) in $C(X)$, then $(I + K) \cap J = I \cap J + K \cap J$.*
(b) *If J is a convex ideal in $C(X)$ and for every two z-ideals I and K in $C(X)$ we have $(I + K) \cap J = I \cap J + K \cap J$, then J is absolutely convex.*

Proof. (a) The proof resembles that of Lemma 3.1 in [10]. Let $h \in (I + K) \cap J$. Then there is an $f \in I$, $g \in K$ such that $h = f + g$. Define the following two continuous functions:

$$t(x) = \begin{cases} 0, & x \in Z(f) \cap Z(g), \\ \frac{hf^2}{f^2+g^2}, & x \notin Z(f) \cap Z(g), \end{cases} \qquad s(x) = \begin{cases} 0, & x \in Z(f) \cap Z(g), \\ \frac{hg^2}{f^2+g^2}, & x \notin Z(f) \cap Z(g). \end{cases}$$

Since $|t| \leqslant |h|$, $|s| \leqslant |h|$, $h \in J$ and J is absolutely convex, then $s, t \in J$. On the other hand I and K are z-ideals (or z_J-ideals), $Z(f) \subseteq Z(t)$, $Z(g) \subseteq Z(s)$ and $f \in I$, $g \in K$ imply that $t \in I \cap J$ and $s \in K \cap J$. Thus $h = t + s \in I \cap J + K \cap J$ and hence $(I + K) \cap J = I \cap J + K \cap J$.

(b) Using Theorem 5.3 in [5], it is enough to show that $|f| \in J$, $\forall f \in J$. Since $M_{f-|f|}$ and $M_{f+|f|}$ are both z-ideals, then by our hypothesis, $(M_{f+|f|} + M_{f-|f|}) \cap J = M_{f+|f|} \cap J + M_{f-|f|} \cap J$, whence $2f = (f + |f|) + (f - |f|) \in (M_{f+|f|} + M_{f-|f|}) \cap J$. So $f = s + t$, where $s, t \in J$, $Z(f + |f|) \subseteq Z(s)$ and $Z(f - |f|) \subseteq Z(t)$. But $(f + |f|)(f - |f|) = 0$ implies that $(f - |f|)s = 0$ and $(f + |f|)t = 0$. Now we have

$$|f|\big(f + |f|\big) = f\big(f + |f|\big) = s\big(f + |f|\big) + t\big(f + |f|\big) = s\big(f + |f|\big) = s\big(f - |f| + 2|f|\big) = 2|f|s.$$

Thus $|f(x)| + f(x) = 2s(x)$, $\forall x \notin Z(f)$ and since $Z(f) \subseteq Z(s)$, then $|f| + f = 2s$, i.e., $|f| + f \in J$ and hence $|f| \in J$. □

Now using preceding facts, we prove the main result of this section.

Theorem 3.4. *The following statements are equivalent.*

(a) *X is an F-space.*
(b) *For every ideal J in $C(X)$, the sum of every two z_J-ideals is a z_J-ideal.*
(c) *For every ideal J in $C(X)$, the sum of every two z_J°-ideals is a z_J°-ideal.*

Proof. If X is an F-space, then every ideal J in $C(X)$ is absolutely convex and hence by Lemma 3.3 and Proposition 3.1, (a) implies (b). Now we suppose that (b) holds and show that X is an F-space. To prove this it is enough to show that for every $f \in C(X)$, there exists $k \in C(X)$ such that $f = k|f|$, see Theorem 14.25 in [5]. Put $J = (|f|)$, since $M_{f-|f|}$ and $M_{f+|f|}$ are z-ideals, then by part (b) in Proposition 3.1, we have $(M_{f+|f|} + M_{f-|f|}) \cap J = M_{f+|f|} \cap J + M_{f-|f|} \cap J$. Now $|f| \in (M_{f+|f|} + M_{f-|f|}) \cap J$ implies that $|f| = s + t$, where $s, t \in J = (|f|)$, $Z(f + |f|) \subseteq Z(s)$ and $Z(f - |f|) \subseteq Z(t)$. Hence $|f|(f - |f|) = s(f - |f|) + t(f - |f|) = -2t|f|$, so $|f| - f = 2t$. But $t \in J = (|f|)$ implies that $t = k|f|$ for some $k \in C(X)$, therefore $f = |f| - 2k|f| = (1 - 2k)|f|$. The proof of the equivalence of parts (a) and (c) is similar. □

Finally we show that if the sum of every two *rez*-ideals in $C(X)$ is a *rez*-ideal, then X is a P-space.

Proposition 3.5. *The following statements are equivalent.*

(a) *X is a P-space.*
(b) *Every ideal in $C(X)$ is a rez-deal.*
(c) *Sum of every two rez-ideals is a rez-ideal.*

Proof. Clearly (a) implies (b) and (b) implies (c). Now suppose that (c) holds but X is not a P-space. Then there is a prime ideal P in $C(X)$ which is not a z-ideal (we recall that X is a P-space if and only if every prime ideal in $C(X)$ is a z-ideal) and hence not maximal. Let M and M' be two maximal ideals in $C(X)$ not containing P such that $M + M' = C(X)$. Since P is not maximal, then $\bigcap Z[P]$ contains at most one non-isolated point and hence by Proposition 1.5, P is an essential ideal. So $I = P \cap M$ and $K = P \cap M'$ are two nonzero z_p-ideals by Proposition 2.1. Moreover $I \subsetneqq P$ and $K \subsetneqq P$ imply that I and K are *rez*-ideals. Now by part (a) of Lemma 3.3, since the prime ideal P is absolutely convex, then $I + K = P \cap M + P \cap M' = P \cap (M + M') = P \cap C(X) = P$ and P is not a *rez*-ideal by Proposition 2.5(a), a contradiction. □

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