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The correspondence between partial metrics and semivaluations

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Abstract

Partial metrics, or the equivalent weightable quasi-metrics, have been introduced in Matthews (Proc. 8th Summer Conf. on General Topology and Applications; Ann. New York Acad. Sci. 728 (1994) 183) as part of the study of the denotational semantics of data flow networks (Theoret. Comput. Sci. 151 (1995) 195). The interest in valuations in connection to Domain Theory derives from e.g. Jones and Plotkin (LICS '89, IEEE Computer Society Press, Silver Spring, MD, 1998, pp. 186–195), Jones (Ph.D. Thesis, University of Edinburgh, 1989), Edalat (LICS'94, IEEE Computer Society Press, Silver Spring, MD, 1994) and Heckmann (Fund. Inform. 24(3) (1995) 259). Connections between partial metrics and valuations have been discussed in the literature, e.g. O'Neill (in: S. Andima et al. (Eds.), Proc. 11th Summer Conf. on General Topology and Applications; Ann. New York Acad. Sci. 806 (1997) 304), Bukatin and Scott (in: S. Adian, A. Nerode (Eds.), Logical Foundations of Computer Science, Lecture Notes in Computer Science, Vol. 1234, Springer, Berlin, 1997, pp. 33–43) and Bukatin and Shorina (in: M. Nivat (Ed.), Foundations of Software Science and Computation Structures, Lecture Notes in Computer Science, Vol. 1378, Springer, Berlin, 1998, pp. 125–139). In each case, partial metrics are generated from strictly increasing valuations.

We analyze the precise relationship between these two notions. It is well known that characterizations of partial metrics in general are hard to obtain, as witnessed by the open characterization problems in the survey paper Nonsymmetric Topology (Kúnzi (Bolyai Soc. Math. Stud. 4 (1993) 303). Our approach to obtaining such a characterization involves the isolation of a "mathematically nice" class of spaces, which is sufficiently large to incorporate the quantitative domain theoretic examples involving partial metric spaces.

For these purposes we focus on the class of quasi-metric semilattices. These structures, as will be illustrated, arise naturally in quantitative domain theory and include in particular the class of totally bounded Scott domains discussed in Smyth (in: G.M. Reed, A.W. Roscoe, R.F. Wachter (Ed.), Topology and Category Theory in Computer Science, Oxford University Press, Oxford, 1991, pp. 207–229), the Baire quasi-metric spaces of (Theoret. Comput. Sci. 151 (1995) 195), the complexity spaces of Schellekens (in: Proc. MFPS 11, Electronic Notes in Theoretical Computer

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Science, Vol. I, Elsevier, Amsterdam, 1995, pp. 211–232) and the interval domain (Proc. Twelfth Ann. IEEE Symp. on Logic in Computer Science, IEEE Press, New York, 1997, pp. 248–257).

We introduce the notion of a semivaluation, which generalizes the fruitful notion of a valuation on a lattice to the context of semilattices and establish a correspondence between partial metric semilattices and semivaluation spaces.

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1. Introduction & related work

We recall that weightable quasi-metric spaces or the equivalent partial metric spaces have been introduced in [16] as part of the study of the denotational semantics of data flow networks. These structures have been applied to obtain an extensional treatment of lazy data flow deadlock in [17]. The topological study of these spaces has been continued in [15] and in the survey paper "Nonsymmetric Topology" [14]. Related structures have also been studied in [11].

Partial metric spaces typically are presented as generalized metric spaces for which the distance from a point to itself is not necessarily 0 (e.g. [16] or [18]). However, their (topological) nature is not well known, as illustrated by the open characterization problems stated in [14] which will be discussed below.

A connection between partial metrics and valuations has first been indicated in [18]. We recall that valuations have been used in computer science as part of the development of a probabilistic powerdomain (e.g. [13,12]) as well as in connection to the development of a domain theoretic treatment of integration (e.g. [6]) and in the context of the development of a lower bag domain as a semantics for nondeterministic programs [10].

In [18], valuation spaces $(X, \sqsubseteq, v)$ are defined to be consistent semilattices, i.e. meet semilattices for which every pair of elements which is bounded from above has a least upper bound and which are equipped with a strictly increasing *real*-valued valuation v. It is shown in [18] that such valuation spaces can be equipped with a partial metric p_v defined by: $p_v(x, y) = -v(x \sqcap y)$. The relationship between valuations and partial metrics has been further discussed in [3] as well as in the recent [4]. These approaches all involve partial metrics generated from strictly increasing valuations.

The idea to generate partial metrics from valuations dates back to the well-known result (e.g. [1]) that a *strictly increasing* valuation v on a lattice induces a metric $d_v(x, y) = v(x \sqcup y) - v(x \sqcap y)$ on this lattice, while arbitrary valuations are only guaranteed to induce pseudo-metrics. This approach results in a lattice equipped with an invariant (pseudo-)metric [1].

According to [4], "the existence of deep connections between partial metrics and valuations is well known in Domain Theory"; a claim which is supported in part by the examples discussed in [3,4,18].

We will show that the possibility to generate partial metrics from valuations is indeed not a coincidence and analyze the precise connections between these notions.

A stumbling block to carry out this program is the fact that the topological characterization of partial metric spaces in general poses a hard problem. Some interesting partial results for restricted classes of spaces do exist however [14,15] and will be discussed below.

One of the reasons for the intractability of the problems stated in the survey paper [14], seems to be that partial metric spaces do not embody as yet enough of the structure of the examples arising in the applications. [1] Hence, in our study of these structures we aim to isolate a "mathematically nice" subclass of partial metric spaces, which is sufficiently large to incorporate the domain theoretic examples involving partial metric spaces.

In particular, the class should include the Baire partial metric spaces of [16] as well as the complexity spaces of [21] (cf. also [20]). To illustrate that the class is sufficiently general, we also show that it incorporates the Scott Domains, whether they be represented as totally bounded quasi-metric spaces, as in [24], or via 0–1 valued quasi-metrics (e.g. [23] or [2]). We also show that the class includes the well-known interval domain [7].

We show that each of these examples forms a quasi-metric semilattice; i.e. a semilattice equipped with a quasi-metric with respect to which the semilattice operation is quasi-uniformly continuous. Quasi-metric lattices are defined in a similar way, where the definition generalizes the classical definition of a uniform lattice (e.g. [25] or [26]). The paper introduces the notion of a semivaluation, which, as we will show, forms a natural generalization of the notion of a valuation on a lattice to the context of semilattices. The central result of the paper establishes a bijection between partial metric semilattices and semivaluation spaces in the context of quasi-metric semilattices.

2. Background

The following notation is used throughout: $\mathcal{N}$ denotes the set of natural numbers, $\mathcal{R}$ denotes the set of real numbers, $\mathcal{R}^+ = (0, \infty)$, $\mathcal{R}_0^+ = [0, \infty)$, while $\bar{\mathcal{R}} = \mathcal{R} \cup \{-\infty, \infty\}$, $\overline{\mathcal{R}^+} = \mathcal{R}^+ \cup \{\infty\}$ and $\overline{\mathcal{R}_0^+} = \mathcal{R}_0^+ \cup \{\infty\}$.

A *complete partial order* (cpo) is a partially ordered set $(P, \sqsubseteq)$ with a least element $\perp$ and such that every increasing sequence $(x_n)_n$ has a supremum $\sqcup_n x_n$. An element x of a cpo $(P, \sqsubseteq)$ is *finite* (or *compact*) iff for every increasing sequence $(x_n)_n$ in P, $x \sqsubseteq \sqcup_n x_n \Rightarrow x \sqsubseteq x_n$ for some n. A cpo is *algebraic* if every element is expressible as the supremum of an increasing sequence of finite elements. A cpo is *ω-algebraic* if it is algebraic and has at most countably many finite elements. A cpo is *bounded-complete* if every subset that is bounded from above has a least upper bound. A *Scott domain* is a bounded-complete ω-algebraic cpo.

[1] H.P. Künzi, private communication.

Example. As in Example 4 of [24], let Σ^∞ denote the set of all finite and infinite sequences ("words") over a countable alphabet Σ. Σ^* denotes the set of all finite sequences over Σ. Given a sequence $s \in \Sigma^\infty$, of length $L \geqslant 1$, then for any natural number n between 1 and L, $s(n)$ denotes the nth element. An *initial subsequence* of a sequence s is either the sequence s or any finite subsequence of the form $s(1), \ldots, s(k)$, where $k \geqslant 1$. The prefix order $\sqsubseteq$ on Σ^∞ is defined by: $s \sqsubseteq s' \Leftrightarrow s$ is an initial subsequence of s'. $(\Sigma^\infty, \sqsubseteq)$ is an example of a Scott Domain, with Σ^* as set of finite elements.

A function $d : X \times X \to \mathscr{R}_0^+$ is a *quasi-pseudo-metric* iff
(1) $\forall x \in X.\, d(x,x) = 0$
(2) $\forall x, y, z \in X.\, d(x, y) + d(y,z) \geqslant d(x,z)$.

A *quasi-pseudo-metric space* is a pair (X,d) consisting of a set X together with a quasi-pseudo-metric d on X.

In case a quasi-pseudo-metric space is required to satisfy the T_0-separation axiom, we refer to such a space as a *quasi-metric* space.

In that case, condition (1) and the T_0-separation axiom can be replaced by the following condition:
(1′) $\forall x, y. d(x, y) = d(y,x) = 0 \Leftrightarrow x = y$.

The *associated partial order* $\leqslant_d$ of a quasi-metric d is defined by $x \leqslant_d y$ iff $d(x, y) = 0$.

We write that a quasi-metric space *encodes* a partial order when $\forall x, y \in X. d(x, y) \in \{0, 1\}$. In that case, we also write that the *encoded partial order* is the partial order $(X, \leqslant_d)$. Conversely, for a given partial order $(X, \leqslant)$, one can define a quasi-metric space $(X, d_\leqslant)$ which encodes the order, in the obvious way.

A quasi-pseudo-metric space is T_0 iff the associated order of the space is a partial order (e.g. [8]). We will work under the assumption that all spaces satisfy the T_0 separation axiom; that is we will solely refer to quasi-metric spaces in the following.

The *conjugate* d^{-1} of a quasi-metric d is defined to be the function $d^{-1}(x, y) = d(y,x)$, which is again a quasi-metric (e.g. [8]). The conjugate of a quasi-metric space (X,d) is the quasi-metric space (X, d^{-1}). The (*pseudo-*)*metric* d^* induced by a quasi-(pseudo-)metric d is defined by $d^*(x, y) = \max\{d(x, y), d(y,x)\}$.

A function f from a quasi-metric space (X,d) to a quasi-metric space (X', d') is quasi-uniformly continuous iff $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in X.\ d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$.

A quasi-metric space (X,d) is *totally bounded* iff $\forall \varepsilon > 0\, \exists x_1 \ldots x_n \in X\, \forall x \in X\, \exists i \in \{1, \ldots, n\}. d^*(x_i, x) < \varepsilon$.

A preorder $(X, \leqslant)$ is *directed* iff $\forall x, y \in X\ \exists z \in X. z \geqslant x$ and $z \geqslant y$.

A quasi-metric space is *directed* iff its associated partial order is directed. A quasi-metric space *has a maximum* (*minimum*) iff the associated partial order has a maximum (minimum).

Let $(P, \sqsubseteq_1)$ and $(Q, \sqsubseteq_2)$ be partial orders. A function $f : P \to Q$ is *increasing* (*decreasing*) $\Leftrightarrow \forall x, y \in P. x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y) (f(y) \sqsubseteq_2 f(x))$.

We recall [22, Lemma 5] that quasi-metrics satisfy the following property, which we refer to as the "*Monotonicity Lemma*": if (X,d) is a quasi-metric space then $\forall x, y, z \in X. (x' \leqslant_d x$ and $y' \geqslant_d y) \Rightarrow d(x', y') \leqslant d(x, y)$.

We discuss a few examples of quasi-metric spaces.

The function $d_{\mathcal{R}} : \mathcal{R}^2 \to \mathcal{R}_0^+$, defined by $d_{\mathcal{R}}(x, y) = y - x$ when $x < y$ and $d_{\mathcal{R}}(x, y) = 0$ otherwise, and its conjugate are quasi-metrics. We refer to $d_{\mathcal{R}}$ as the "left distance" and to its conjugate as the "right distance". These quasi-metrics correspond to the nonsymmetric versions of the standard metric m on the reals, where $\forall x, y \in \mathcal{R}. m(x, y) = |x - y|$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}. x \leqslant_{d_{\mathcal{R}}^{-1}} y \Leftrightarrow x \leqslant y$, while for the left distance we have $\forall x, y \in \mathcal{R}. x \leqslant_{d_{\mathcal{R}}} y \Leftrightarrow x \geqslant y$.

The function $d_c : (\bar{\mathcal{R}} - \{0\})^2 \to \mathcal{R}_0^+$, defined by $d_c(x, y) = 1/y - 1/x$ when $y < x$ and 0 otherwise, and its conjugate are quasi-metrics.

The *complexity space* (C, d_C) has been introduced in [21] (cf. also [22,20]). Here

$$C = \left\{ f : \omega \to \bar{\mathcal{R}}^+ \,\middle|\, \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty \right\}$$

and d_C is the quasi-metric on C defined by

$$d_C(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right],$$

whenever $f, g \in C$. The complexity space (C, d_C) is a quasi-metric space with a maximum $\top$, which is the function with constant value ∞.

The *dual complexity space* is introduced in [20] as a pair (C^*, d_{C^*}), where $C^* = \{ f : \omega \to \mathcal{R}_0^+ \mid \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty \}$, and d_{C^*} is the quasi-metric defined on C^* by $d_{C^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0]$, whenever $f, g \in C^*$. We recall that (C, d_C) is isometric to (C^*, d_{C^*}) by the isometry $\Psi : C^* \to C$, defined by $\Psi(f) = 1/f$ (see [20]). Via the analysis of its dual, several quasi-metric properties of (C, d_C), in particular Smyth completeness and total boundedness, are studied in [20].

In case the associated order of a quasi-metric space is a linear order we refer to the space as a *linear* quasi-metric space.

A *join* (*meet*) *semilattice* is a partial order $(X, \leqslant)$ such that every two elements $x, y \in X$ have a supremum $x \sqcup y$ (infimum $x \sqcap y$) in X. A *lattice* is a partial order which is both a join and a meet semilattice.

We write that a quasi-metric space is a (semi)lattice iff the associated order is a (semi)lattice.

In fact, with slight abuse of terminology, we will refer to quasi-metric spaces for which the associated order is a semilattice, simply as *semilattices*, and a similar convention holds for the case of lattices.

In this context, the quasi-metric space is referred to as the *underlying quasi-metric space*. This will not lead to any confusion since the results are developed almost exclusively in the context of the theory of quasi-metric spaces, with the exception of Section 5 where the meaning of the terminology will be clear from the context.

The terminology of *quasi-metric* (*semi*)*lattice* is reserved for quasi-metric spaces which are (semi)lattices for which the operations are quasi-uniformly continuous. This is in accordance with the terminology used for the theory of uniform lattices (e.g. [25,26]).

3. Semivaluations

We generalize the notion of a valuation on a lattice to the context of semilattices.

We recall the definition of a valuation on a lattice $(L, \sqsubseteq)$.

A function $f : L \to \mathscr{R}_0^+$ is a *valuation* iff

(1) *f is increasing*,

(2) $\forall x, y \in L.\, f(x \sqcap y) + f(x \sqcup y) = f(x) + f(y)$.

In case the function f is decreasing and satisfies (2), we refer to f as a *co-valuation*.

If f only satisfies (2) we say that f satisfies the *modularity law*, or also that f is *modular*.

There does not seem to be a consistent terminology in the literature. Valuations, also called evaluations, as used in computer science (e.g. [3,12]) typically satisfy (1) and (2) above. In the classical mathematical literature a valuation only needs to satisfy (2) (e.g. [1]).

It is convenient for matters of presentation to reserve the definition given above for a valuation in order to state results on connections between partial metrics and valuations as they occur in Computer Science.

Finally, a (co)valuation f on a lattice $(L, \sqsubseteq)$ is called *strictly increasing* (*strictly decreasing*) if $\forall x, y \in L. x \sqsubset y \Rightarrow f(x) < f(y)$ $(f(x) > f(y))$. What we call strictly increasing is called strongly non-degenerate in [4] and strictly increasing valuations are exactly the dimension functions as defined in [5].

Definition 1. If $(X, \leqslant)$ is a meet semilattice then a function $f : (X, \leqslant) \to \mathscr{R}_0^+$ is a meet valuation iff

$$\forall x, y, z \in X.\, f(x \sqcap z) \geqslant f(x \sqcap y) + f(y \sqcap z) - f(y)$$

and f is meet co-valuation iff

$$\forall x, y, z \in X. f(x \sqcap z) \leqslant f(x \sqcap y) + f(y \sqcap z) - f(y).$$

Definition 2. If $(X, \leqslant)$ is a join semilattice then a function $f : (X, \leqslant) \to \mathscr{R}_0^+$ is a join valuation iff

$$\forall x, y, z \in X. f(x \sqcup z) \leqslant f(x \sqcup y) + f(y \sqcup z) - f(y)$$

and f is join co-valuation iff

$$\forall x, y, z \in X. f(x \sqcup z) \geqslant f(x \sqcup y) + f(y \sqcup z) - f(y).$$

Definition 3. A function is a semivaluation if it is either a join valuation or a meet valuation. A join (meet) valuation space is a join (meet) semilattice equipped with a join (meet) valuation. A semivaluation space is a semilattice equipped with a semivaluation.

Lemma 4. *Semivaluations are increasing.*

Proof. We show this for the case of join valuations. The proof for the case of meet valuations is similar.

We assume that $(X, \leqslant)$ is a join semilattice and $f : (X, \leqslant) \to \mathscr{R}_0^+$ is a join valuation. Let $z = x$ and assume $x \leqslant y$. Since $f(x \sqcup z) \leqslant f(x \sqcup y) + f(y \sqcup z) - f(y)$, we obtain that $f(x) \leqslant f(y) + f(y) - f(y) = f(y)$ and thus $f(x) \leqslant f(y)$. $\square$

Proposition 5. *Let L be a lattice.*

(1) *A function $f : L \to R_0^+$ is a join valuation if and only if it is increasing and satisfies join-modularity, i.e.*

$$f(x \sqcup z) + f(x \sqcap z) \leqslant f(x) + f(z).$$

(2) *A function $f : L \to R_0^+$ is a meet valuation if and only if it is increasing and satisfies meet-modularity, i.e.*

$$f(x \sqcup z) + f(x \sqcap z) \geqslant f(x) + f(z).$$

Proof. We recall that join valuations are increasing. So it suffices to show that they satisfy join-modularity. For this, we replace y by $x \sqcap z$ in $f(x \sqcup z) + f(y) \leqslant f(x \sqcup y) + f(y \sqcup z)$. Then we immediately obtain the join-modularity law: $f(x \sqcup z) + f(x \sqcap z) \leqslant f(x) + f(z)$.

For the converse we assume that f is increasing and satisfies join-modularity.

We remark that $f(x \sqcup z) + f(y) \leqslant f(x \sqcup y \sqcup z) + f((x \sqcup y) \sqcap (y \sqcup z))$ (f is increasing). Slightly rewriting this last expression gives $f((x \sqcup y) \sqcup (y \sqcup z)) + f((x \sqcup y) \sqcap (y \sqcup z))$.

By join-modularity: $f((x \sqcup y) \sqcup (y \sqcup z)) + f((x \sqcup y) \sqcap (y \sqcup z)) \leqslant f(x \sqcup y) + f(y \sqcup z)$. Hence, $f(x \sqcup z) + f(y) \leqslant f(x \sqcup y) + f(y \sqcup z)$ and thus f is a join-valuation. $\square$

Corollary 6. *A function on a lattice is a valuation iff it is a join valuation and a meet valuation. A function on a lattice is a co-valuation iff it is a join co-valuation and a meet co-valuation.*

The last result clearly indicates that semivaluations provide a natural generalization of valuations from the context of lattices to the context of semilattices. Indeed, Corollary 4 allows one to express a (co-)valuation in terms of two separate semilattice conditions, via the notions of a meet-(co-)valuation and of a join-(co-)valuation.

We remark that the above results do not hold at the more basic level of modular functions!

From the second part of the proof of Proposition 5, we obtain that any function which is both (co)-join-modular and (co)-meet-modular is modular. The converse is not true, however, in general, but does hold for *monotone* functions, i.e. functions which are increasing or decreasing, as shown in the first part of the proof.

The following simple counterexample provides a modular function which is neither (co)-join-modular nor (co)-meet-modular. We leave the straightforward verifications to the reader.

Counterexample. Consider the four point lattice $L = \{x, y, x \sqcap y, x \sqcup y\}$, where $x \neq y$ and let f be a function on L defined by $f(x) = 2, f(y) = 8, f(x \sqcap y) = 3$ and

$f(x \sqcup y) = 7$. We remark that f is a modular function which is neither increasing nor decreasing. Moreover, f is neither (co)-join-modular nor (co)-meet-modular.

4. Weights and partial metrics

We recall the notion of a weightable quasi-metric space and the equivalence with the notion of a partial metric space.

A quasi-metric space (X,d) is *weightable* iff there exists a function $w: X \to \mathscr{R}_0^+$ such that $\forall x,y \in X. d(x,y) + w(x) = d(y,x) + w(y)$. The function w is called a *weighting function*, $w(x)$ is the *weight* of x and the quasi-metric d is *weightable by the function* w. A *weighted* space is a triple (X,d,w) where (X,d) is a quasi-metric space weightable by the function w. A *weightless point* of a weighted quasi-metric space is a point of zero weight. A space (X,d) is weightable with respect to a point $z \in X$ iff the function w defined by $\forall x \in X. w(x) = d(z,x)$ is a weighting function of (X,d).

Example. The quasi-metric space $(\mathscr{R}_0^+, d_{\mathscr{R}})$ is weightable by the identity function, $w_1(x) = x$. The quasi-metric space $(\overline{\mathscr{R}^+}, d_c)$ is weightable by the function $w_c(x) = 1/x$. The complexity space (C, d_C) is weightable by the function w_C where $\forall f \in C. w_C(f) = \sum_n 2^{-n}/f(n)$. Each of the examples is weightable with respect to a point ($0, \infty$, and $\top$, respectively).

We recall that the conjugate quasi-metric space $(\mathscr{R}_0^+, d_{\mathscr{R}}^{-1})$ is *not* weightable [22]. For more information on conjugates of weightable spaces, we refer the reader to [15].

A quasi-metric space (X,d) is *co-weightable* iff its conjugate (X,d^{-1}) is weightable. A *co-weighting function* of a quasi-metric space is a weighting function of its conjugate. A *co-weighted* space (X,d,w) is a triple consisting of a set X, a quasi-metric d on X and a co-weighting function w.

A quasi-metric space (X,d) is *bi-weightable* iff it is weightable and co-weightable. We remark that any weighted space (X,d,w) of bounded weight, where say $\forall x \in X. w(x) \leqslant K$, is co-weighted by the weighting function $K - w$ [14]. Hence, any weighted space of bounded weight is bi-weightable. Similarly, one obtains that any co-weighted space of bounded co-weight is bi-weightable.

A *partial metric space* (X,p) is a pair consisting of a set X and a function $p: X \times X \to \mathscr{R}_0^+$ such that $\forall x,y,z \in X$:

(1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(2) $p(x,x) \leqslant p(x,y)$;
(3) $p(x,y) = p(y,x)$;
(4) $p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

Example. (Cf. the example of Section 2, following the definition of a Scott domain). Let Σ^∞ be the set of countably infinite and finite sequences of elements from a given set Σ and let $\emptyset$ be the empty sequence. Define the function $p: \Sigma^\infty \times \Sigma^\infty \to \mathscr{R}_0^+$

as follows:

> $\forall x, y \in \Sigma^{\infty}. p(x, y) = 2^{-\alpha}$, where $\alpha = \max\{n \mid x(n) = y(n)\}$ when the sequences x and y have a common non-empty initial subsequence and $\alpha = 0$ otherwise.

The function p is a partial metric, the "Baire partial metric", also referred to as the "Kahn partial metric" (e.g. [16]).

The following result, which we refer to as "the Correspondence Theorem", establishes the equivalence between the weightable quasi-metric spaces and the partial metric spaces.

Correspondence Theorem (Matthews [16]).

(1) *If* (X, d, w) *is a weighted quasi-metric space then the function* p, *defined by* $\forall x, y \in X.\ p(x, y) = d(x, y) + w(x)$, *is a partial metric.*

(2) *If* (X, p) *is a partial metric space then the function* d, *defined* by $\forall x, y \in X. d(x, y) = p(x, y) - p(x, x)$, *is a quasi-metric weightable by a weighting function* w, *defined by* $\forall x \in X.\ w(x) = p(x, x)$.

Matthews's result is actually stronger, since it states that the topologies and orders induced by the two kinds of generalized metrics coincide. The version given above suffices for our purposes.

The Baire quasi-metric is by definition the weightable quasi-metric b obtained via the Correspondence Theorem from the Baire partial metric. We refer to the space $(\Sigma^{\infty}, b, w_b)$ as the Baire space.

We will show that the weighting functions of a weightable space are determined by a unique fading weighting (cf. also [15]).

A function $f : X \to \mathscr{R}_0^+$ is *fading* iff $\inf_{x \in X} f(x) = 0$.

Definition 7. A weighted quasi-metric space is of fading weight iff its weighting function is fading.

Example. The spaces $(\mathscr{R}^+, d_{\mathscr{R}}, w_1)$, $(\mathscr{R}^+, d_c, w_c)$, the complexity space (C, d_C, w_C) and the Baire space $(\Sigma^{\infty}, b, w_b)$ are weighted spaces of fading weight.

Proposition 8. *The weighting functions of a weightable quasi-metric space are strictly decreasing. The weighting functions are exactly the functions* $f + c$, *where* $c \geqslant 0$ *and where* f *is the unique fading weighting of the space.*

Proof. It is easy to verify that for any given weighting function w, the function $f_w = w - L$, where $L = \inf_{x \in X} w(x)$, is a fading weighting. This remark has originally been made in [15]. It is also remarked there that any weighting function is of the form $f_w + c$ for some positive constant c.

We verify that the weighting functions are strictly decreasing (cf. also [22, Lemma 2]). Let w be a weighting function for the space (X, d). Then for $x, y \in X$ such that

$x <_d y$, we have that $d(x, y) = 0$ and $d(y,x) > 0$ and thus by the weighting equality we obtain that $w(x) - w(y) = d(y,x) > 0$ and hence $w(x) > w(y)$.

Finally, we show that fading weightings are unique. Let f_1 and f_2 be fading weighting functions for the space (X,d). Then, by the weighting equalities, we obtain that $\forall x, y \in X.\, f_1(x) - f_2(x) = f_1(y) - f_2(y)$. Hence there exists a constant c such that $f_1 - f_2 = c$. Since the functions are fading, this implies that the constant must be 0; that is f_1 and f_2 coincide. $\square$

5. The correspondence

We will discuss several examples of quasi-metric (semi)lattices which arise in Quantitative Domain Theory. In each case, the quasi-uniform continuity of the (semi)lattice operations follows from the fact that the quasi-uniformity is generated by an invariant quasi-metric.

A join semilattice (X,d) is *invariant* iff $\forall x, y, z \in X.\, d(x \sqcup z, y \sqcup z) \leqslant d(x, y)$. In that case, we also write that the quasi-metric d is invariant. An *invariant meet semilattice* is defined by replacing the join operation by the meet operation in the previous definition. An *invariant lattice* is defined in the obvious way. One can easily verify that invariant join semilattices are quasi-metric join semilattices and that similar results hold for the case of invariant meet semilattices and for invariant lattices.

It is convenient to present the following alternative characterization of invariance.

Lemma 9. *A join semilattice* (X,d) *is invariant iff* $\forall x, y \in X.\, d(x \sqcup y, y) = d(x, y)$. *A meet semilattice* (X,d) *is invariant iff* $\forall x, y \in X.\, d(x, x \sqcap y) = d(x, y)$.

Proof. We present the proof for the case of join semilattices.

If (X,d) is a join semilattice such that $\forall x, y \in X.\, d(x \sqcup y, y) = d(x, y)$, then $\forall x, y, z \in X. d(x \sqcup z, y \sqcup z) = d(x \sqcup y \sqcup z, y \sqcup z) = d(x \sqcup (y \sqcup z), (y \sqcup z)) = d(x, y \sqcup z) \leqslant d(x, y)$, where the last inequality follows by the Monotonicity Lemma.

To show the converse, we assume that (X,d) is an invariant join semilattice. Then we have that $\forall x, y \in X.\, d(x \sqcup y, y) = d(x \sqcup y, y \sqcup y) \leqslant d(x, y)$ and, by Monotonicity, $d(x, y) \leqslant d(x \sqcup y, y)$. $\square$

We say that a partial metric on a join semilattice is *invariant* iff its corresponding weightable quasi-metric is invariant. The definitions for the case of meet semilattices and lattices are similar.

We are now ready to discuss the main examples.

Example 1. It is easy to verify that any quasi-metric space which encodes a semilattice is invariant with respect to the semilattice operation. This is in particular the case for quasi-metrics which encode a Scott domain, since any bounded-complete algebraic cpo is a meet semilattice (e.g. [9]).

Not only straightforward encodings of Scott domains give rise to quasi-metric meet semilattices. We show that a main example of [24], regarding totally bounded spaces as domains of computation (Example 2), as well as the Baire partial metric spaces of [17] (Example 3), the interval domain (Example 4) and the complexity space of [21] and its dual (Example 5) correspond to quasi-metric semilattices.

Example 2. As in [24], let $(D, \sqsubseteq)$ be a Scott domain equipped with a rank function $r{:}\,F_D \to \mathcal{N}$, where $\forall n \in \mathcal{N}.r^{-1}(n)$ is a finite non empty set and F_D is the set of finite elements of D. Then the following function defines a totally bounded quasi-metric on D:

$$d_r(x, y) = \inf\{2^{-n} | e \sqsubseteq x \Rightarrow e \sqsubseteq y \text{ for every finite } e \text{ of rank } \leqslant n\}.$$

In order to verify that the resulting structure is a quasi-metric meet semilattice, we verify that d_r is invariant.

For this it suffices to show that for any natural number n and for all elements $x, y, z \in D$: if the implication $e \sqsubseteq x \Rightarrow e \sqsubseteq y$ holds for all finite elements e of rank lower than n, then the implication $e \sqsubseteq x \sqcap z \Rightarrow e \sqsubseteq y \sqcap z$ holds for all finite elements e of rank lower than n.

We assume that for all finite elements e of rank lower than n one has: $e \sqsubseteq x \Rightarrow e \sqsubseteq y$. If e is finite and of rank lower than n and $e \sqsubseteq x \sqcap z$ then $e \sqsubseteq x$ and thus $e \sqsubseteq y$. But since $e \sqsubseteq x \sqcap z$, we also have that $e \sqsubseteq z$ and thus $e \sqsubseteq y \sqcap z$. Hence $\forall x, y, z \in D.\, d_r(x \sqcap z, y \sqcap z) \leqslant d_r(x, y)$.

Example 3. Any Baire partial metric space (Σ^∞, p) (cf. Section 4), gives rise to a quasi-metric meet semilattice induced by the corresponding weightable quasi-metric meet semilattice (Σ^∞, b), where $b(x, y) = p(x, y) - p(x, x)$.

It is easy to verify that the associated order of a Baire quasi-metric space (Σ^∞, b) is the prefix ordering (cf. Section 2). One can then verify that the meet operation is the binary operation which for any two sequences results in their longest common initial subsequence.

We verify that the space (Σ^∞, b) is an invariant meet semilattice, via the characterization of invariance given in Lemma 9. If $x, y \in \Sigma^\infty$ then $b(x, x \sqcap y) = p(x, x \sqcap y) - p(x, x) = p(x, y) - p(x, x) = b(x, y)$. The second equality follows from the definition of p and of $\sqcap$.

Example 4. The interval domain $(I(\mathscr{R}), p)$ consisting of the closed intervals of the reals, ordered by reverse inclusion and equipped with the partial metric p (see [19]) defined by

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}.$$

One can easily verify that the associated weighted quasi-metric space $(I(\mathscr{R}), d_p)$ is a quasi-metric meet semilattice.

Example 5. The complexity space (C, d_C) and its dual (C^*, d_{C^*}) are examples of invariant join and meet lattices respectively. We refer the reader to [22], where the invariance (optimality) of the complexity space is shown and where more general examples of invariant semilattices involving weighted function spaces are discussed.

Example 6. Any quasi-metric space for which the associated order is linear is invariant with respect to its lattice operations. We leave the verifications to the reader. Some examples are the quasi-metric space $(I, d_{\mathscr{R}}^{-1})$ considered in [24], where I is the unit interval $[0, 1]$, as well as the spaces $(\mathscr{R}_0^+, d_{\mathscr{R}})$ and $(\overline{\mathscr{R}^+}, d_c)$.

We now state the correspondence between invariant weighted quasi-metrics (invariant partial metrics) on a join semilattice and join co-valuations. A dual version, stating the correspondence between invariant co-weighted quasi-metrics on a meet semilattice and meet valuations is given below.

Theorem 10. *For every join semilattice $(X, \preccurlyeq)$, there exists a bijection between invariant weighted quasi-metrics d on X with $\leqslant_d = \preccurlyeq$ and fading strictly decreasing join co-valuations $f : (X, \preccurlyeq) \to (\mathscr{R}_0^+, \leqslant)$. The map $f \mapsto d_f$ is defined by $d_f(x, y) = f(y) - f(x \sqcup y)$. The inverse is the function which to each weighted space (X, d) associates its unique fading weighting. Similarly one can show that for every join semilattice $(X, \preccurlyeq)$, there exists a bijection between invariant co-weighted quasi-metrics d on X with $\leqslant_d = \preccurlyeq$ and fading strictly increasing join valuations $f : (X, \preccurlyeq) \to (\mathscr{R}_0^+, \leqslant)$. The map $f \mapsto d_f$ is defined by $d_f(x, y) = f(x \sqcup y) - f(y)$. The inverse is the function which to each co-weighted space (X, d) associates its unique fading co-weighting.*

Proof. (I) We first show that for every fading strictly decreasing join co-valuation $f : X \to \mathscr{R}_0^+$, the space (X, d_f) is a weighted quasi-metric space.

The function d_f is clearly positive since f is decreasing.

We verify that d_f is a quasi-metric.

Note that $\forall x \in X.\, d_f(x, x) = f(x) - f(x \sqcup x) = 0$.

We verify that $\leqslant_{d_f}$ coincides with $\preccurlyeq$. Note that $\forall x, y \in X.\, x \leqslant_{d_f} y \Leftrightarrow d_f(x, y) = 0 \Leftrightarrow f(y) - f(x \sqcup y) = 0 \Leftrightarrow y = x \sqcup y \Leftrightarrow x \preccurlyeq y$. The one but last equivalence follows from the fact that f is strictly decreasing.

Under the assumption that $d_f(x, y) = d_f(y, x) = 0$, we thus obtain that $x = y$.

To verify the triangle inequality we need to verify that $\forall x, y, z \in X.\, d_f(x, y) + d_f(y, z) \geqslant d_f(x, z)$ or equivalently that $(f(y) - f(x \sqcup y)) + (f(z) - f(y \sqcup z)) \geqslant (f(z) - f(x \sqcup z))$. This inequality holds by the co-join-modularity of f.

To verify that the quasi-metric space (X, d_f) is invariant we note that $\forall x, y \in X. d_f(x, y) = f(y) - f(x \sqcup y) = f(y) - f((x \sqcup y) \sqcup y) = d_f(x \sqcup y, y)$.

It is easy to verify that (X, d_f, f) is a weighted space.

(II) We assume that (X, d) is weightable. Let f be the unique fading weighting function of (X, d) (Proposition 8).

We remark that f is strictly decreasing (Proposition 8). So all we need to verify is that f is a join co-valuation.

Since (X, d) is invariant and weightable by f, we know that: $\forall x, y \in X. d(x, y) = d(x \sqcup y, y) = f(y) - f(x \sqcup y)$. Since $\forall x, y, z \in X.\, d(x \sqcup z, y \sqcup z) \leqslant d(x, y)$, we obtain

$f(y\sqcup z) - f(x\sqcup y\sqcup z) \leqslant f(y) - f(x\sqcup y)$ and thus $f(x\sqcup y) + f(y\sqcup z) \leqslant f(y) + f(x\sqcup y\sqcup z)$. Since f is decreasing, we obtain that f is a join co-valuation.

(III) Finally, we need to verify that the correspondence obtained above is a bijection, i.e. that the maps $f \to d_f$ and $d \to f_d$ are inverse to each other.

(a) To verify that $d_{f_d} = d$, we remark that $d(x, y) = d(x\sqcup y, y) = f_d(y) - f_d(x\sqcup y) = d_{f_d}(x, y)$, where the last equality is the definition of d_{f_d} and the rest was already shown in part II.

(b) To verify that $f_{d_f} = f$, we remark that by part I, f is a weighting of d_f. By the uniqueness of fading weightings, $f = f_{d_f}$.

We leave the similar verifications of the second part of the Theorem to the reader. □

We present a dual version of the preceding result.

Theorem 11. *For every meet semilattice* $(X, \preccurlyeq)$, *there exists a bijection between invariant co-weighted quasi-metrics d on X with* $\leqslant_d = \preccurlyeq$ *and fading strictly increasing meet valuations* $f : (X, \preccurlyeq) \to (\mathscr{R}_0^+, \leqslant)$. *The map* $f \mapsto d_f$ *is defined by* $d_f(x, y) = f(x) - f(x\sqcap y)$. *The inverse is the function which to each weighted space* (X, d) *associates its unique fading co-weighting. Similarly, one can show that for every meet semilattice* $(X, \preccurlyeq)$, *there exists a bijection between invariant weighted quasi-metrics d on X with* $\leqslant_d = \preccurlyeq$ *and fading strictly decreasing meet co-valuations* $f : (X, \preccurlyeq) \to (\mathscr{R}_0^+, \leqslant)$. *The map* $f \mapsto d_f$ *is defined by* $d_f(x, y) = f(x\sqcap y) - f(x)$. *The inverse is the function which to each weighted space* (X, d) *associates its unique fading weighting.*

Remark. We motivate the choices for the terminology "join valuation" and "join co-valuation". We chose to reserve the terminology join co-valuation for weightings rather than for co-weightings since weightings turn out to be decreasing, while co-weightings are increasing and hence the last are in accordance with the traditional computer science convention which defines valuations as increasing functions.

Corollary 12. *For every lattice* $(X, \preccurlyeq)$, *there exists a bijection between invariant weighted quasi-metrics d on X with* $\leqslant_d = \preccurlyeq$ *and fading strictly decreasing modular functions on the lattice and there exists a bijection between invariant co-weighted quasi-metrics d on X with* $\leqslant_d = \sqsubseteq$ *and fading strictly increasing valuations on the lattice.*

We conclude with the following result in the context of lattices.

Proposition 13. *Let L be a lattice and v a strictly increasing valuation on L. Let m be the invariant metric induced by the valuation v on L* (*cf.* [1]), *defined by*

$$\forall x, y \in L.m(x, y) = v(x\sqcup y) - v(x\sqcap y).$$

The quasi-metric d_v *induced by the strictly increasing meet valuation v is such that its associated metric* d_v^* *is equivalent with m. Moreover, in case v is bounded, say* *by a constant K, then the quasi-metric d_{K-v} induced by the strictly decreasing join co-valuation $K-v$, coincides with d_v.*

Proof. Let L be a lattice and v a strictly increasing valuation on L. Let m be the invariant metric induced by the valuation v on L, as defined in the lemma. We remark that v is in particular a strictly increasing meet valuation. Without loss of generality we can assume that v is fading and thus, as in Theorem 11, we can define the quasi-metric d_v. We need to verify that d_v^* is equivalent with m.

It suffices to verify that $\forall x,y \in X.\, m(x,y) = d_v(x,y) + d_v(y,x)$, since this implies in particular that both d_v and its conjugate d_v^{-1} are bounded by m.

We remark that $\forall x,y \in X.\, m(x,y) = v(x \sqcup y) - v(x \sqcap y) = v(x) - v(x \sqcap y) + v(y) - v(x \sqcap y) = d_v(x,y) + d_v(y,x)$, where the second equality holds by the modularity of v. Hence $m = d_v + d_v^{-1}$, a metric which is clearly equivalent to $d_v^* = \max(d_v, d_v^{-1})$.

Under the assumption that v is bounded by a constant K, one can easily verify that $K-v$ is a strictly decreasing join co-valuation.

Hence, by Theorem 10, we obtain the quasi-metric d_{K-v}, defined by $\forall x,y \in X.\, d_{K-v}(x,y) = (K-v)(y) - (K-v)(x \sqcup y) = v(x \sqcup y) - v(y)$. By modularity of v we then obtain that $d_{K-v} = d_v$. □

We omit the straightforward dual version of the above lemma for the case of co-valuations.

6. Conclusion

The relevance of partial metrics and valuations to Theoretical Computer Science is well known. Connections between the two notions have been indicated in [3,4,18], where partial metrics are generated from valuations.

We have defined the new notion of a semivaluation as a natural generalization of a valuation to the context of semilattices. A bijection between invariant partial metric semilattices and semivaluation spaces has been obtained.

The result sheds new light on the nature of partial metrics and allows for a simplified representation of well known partial metric spaces, where the semivaluation involved is simply the partial metric self-distance function.

The established correspondence thus provides further motivation for the fact that (generalized) valuations provide an important tool for Quantitative Domain Theory.

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