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The Effect of a Finite Roll Rate on the Miss-Distance of a Bank-to-Turn Missile

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Abstract—We consider a three-dimensional pursuit-evasion situation where a highly maneuverable evader, which we model as a "pedestrian" á la Isaacs, is engaged by a faster-pursuer. The pursuer has limited maneuverability, that is, the pursuer has a minimal turning radius, and in order to change the spatial direction of his velocity vector, he must first re-align his thrust vector in a similar manner to a bank-to-turn missile. The state space of the ensuing differential game is three-dimensional and its complexity is intermediate between Isaac's [1] classical "Homicidal Chauffeur" and "Two Car" differential games. This new DG is solved as a game of kind, and a capture criterion for a faster but less maneuverable pursuer is analytically established in terms of the game parameters.

1. INTRODUCTION

Aerodynamically controlled bank-to-turn missiles suffer from the inherent drawback of an inevitable delay in their application of the required control. This is due to the fact that a required change in the direction of the velocity vector is contingent upon an application of lift in the same direction. Since lift acts generally in a direction normal to the wing planform, the application of a lift force in the required direction will be preceded by a roll maneuver with an attendant delay in control application.

The present work should be considered as a first step in the direction of quantifying the miss-distance dependence on the missile roll rate in this new differential game. In our analysis, we employ the intellectually satisfying paradigm of optimal pursuer and evader behavior; in other words, we give a differential game formulation. Naturally, we consider a three-dimensional encounter stiuation between a highly maneuverable evader (which we model as "pedestrian" á la Isaacs [1]) and a faster pursuer hampered by limited maneuverability, that is, the pursuer has a minimal turning radius R [2]. The (constant) speeds of the pursuer and evader are denoted by V_p and V_E, respectively, and the pursuer (bounded) roll rate is ω_s. Capture is affected whenever the pursuer-evader instantaneous separation is less than ℓ_c.

We thus obtain a novel differential game, which we believe is sufficiently simple to afford a close form solution; its equations are derived in Section 2. In Section 3, we determine the "useable part" of the target set and indicate an approach to the complete solution of the present differential game. A closed-form analytic solution for the optimal trajectories and costate variables is presented in Section 4. The problem of capturability is discussed in Section 5, where we also establish the critical capture radius ℓ_c, below which a capture of the evader is always guaranteed. The functional dependence of the critical capture radius on the problem parameters

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(V_p, V_E, ω_s, R) is obtained, which provides us with an analytical tool of assessing the general performance capabilities of a bank-to-turn missile.

Depending on the relative magnitude of his roll-rate, the optimal pursuer may enforce unconditional capture by executing a two-stage maneuver consisting of two hard-turns in opposite directions combined with a unidirectional roll maneuver. Alternatively, for larger roll rates, the optimal play is a three-stage maneuver consisting of a zero-roll (ω_s-universal curve) hard-turn, following the previously mentioned two-stage maneuver, along which the dynamic system is being steered into the target set. The critical roll rate ω_c, distinguishing between a two-stage ($\omega_s < \omega_c$) and a three-stage ($\omega_s > \omega_c$) optimal maneuver, is also analytically established for this particular game. Still another interesting result of the preceding analysis is the existence of a minimum roll-rate, ω_m, below which capture of a highly maneuverable slower evader is not possible. The case of a pursuer's infinite roll-rate is also of particular interest since clearly $\ell_c(\omega_s, V_p, V_E, R) > \ell_c(\infty, V_p, V_E, R)$, and we are interested in the degradation of the game performance index ℓ_c as a function of the final roll-rate ω_s. It is shown that by letting $\omega_s \to \infty$, the present differential game degenerates into the familiar "Homicidal Chauffeur" game for which the capture condition has been already established [1]. Finally, concluding remarks are made in Section 6.

The main result of this paper is presented in Figure 4 where the dimensionless capture-radius ℓ_c (normalized with respect to the pursuer's radius R) is plotted against the non-dimensional pursuer's roll-rate (normalized with respect to V_p/R) and the player's speed ratio $\gamma = V_E/V_p < 1$. For the sake of brevity, we choose to graphically depict the results only for one particular speed ratio, i.e., $\gamma = 0.5$. Such a solution may be considered as a first attempt towards generalizing the 2-D capturability criteria of [1,3–5], obtained for the "Homicidal Chauffeur" and "Two Car" games for a genuine 3-D pursuit-evasion encounter.

2. NEW DIFFERENTIAL GAME

As a reference frame, we employ the pursuer's body-fixed axes and describe the three-dimensional relative motion of the evader with respect to the pursuer (reduced space formulation) as depicted in Figure 1. The radius vector $\vec{r}_{EP}$ denotes the instantaneous position of E (evader) relative to P (pursuer):

$$\vec{r}_{EP} = (x, y, z). \tag{1}$$

E's inertial velocity vector (assumed constant) resolved in P's body axes is:

$$\vec{V}_E = V_E(\cos\theta\cos\psi, \cos\theta\sin\psi, -\sin\theta), \tag{2}$$

where θ and ψ are the evader's two heading controls.

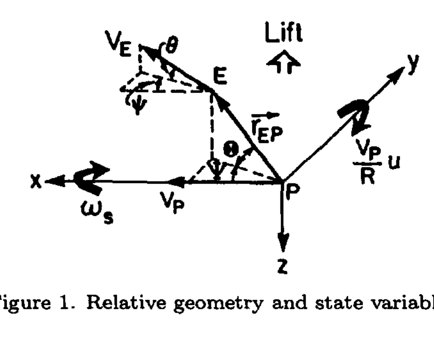

Figure 1. Relative geometry and state variables.

The angular velocity of the body-axes triad is given by:

$$\vec{\Omega} = \left(\omega_s, \frac{V_p}{R}u, 0\right), \tag{3}$$

where ω_s is the pursuer's bounded roll-rate $(-\omega_{s_{\max}} \leq \omega_s \leq \omega_{s_{\max}})$ and u is the nondimensional parameter $(-1 \leq u \leq 1)$ representing the pursuer's pitch-rate. Finally, V_p is the pursuer's constant speed, and R is his minimal curvature radius in the pitch-plane (x-z plane).

It is further assumed that the pursuer has a limited yaw rate, and for this reason it is advantageous for him to perform a bank-to-turn maneuver in order to change his direction in the three-dimensional space. This is indeed the situation with conventional aircraft and high speed missiles.

Thus, the absolute velocity of the evader, expressed in terms of the pursuer body-fixed coordinate system, is:

$$\vec{V}_E = \vec{V}_P + \dot{\vec{r}}_{EP} + \vec{\Omega} \times \vec{r}_{EP}, \tag{4}$$

where the upper dot denotes differentiation with respect to time in the inertial system. Substituting (1), (2), and (3) in (4) with $V_p(1,0,0)$, leads to the following differential system in $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= V_p - \frac{V_p}{R} uz + V_E \cos\theta \cos\psi, \\ \dot{y} &= \omega_s z + V_E \cos\theta \sin\psi, \\ \dot{z} &= \frac{V_p}{R} ux - \omega_s y - V_E \sin\theta. \end{aligned} \tag{5}$$

In solving this genuine three-dimensional pursuit-evasion game of a kind, governed by the kinematic equation (5), it is useful to first introduce the following dimensionless variables:

$$\begin{aligned} &x \longrightarrow \frac{x}{R}, \quad y \longrightarrow \frac{y}{R}, \quad z \longrightarrow \frac{z}{R}, \quad t \longrightarrow \frac{tV_p}{R}, \\ &\omega_s \longrightarrow \frac{\omega_s}{V_p/R}, \quad \omega_{s_{\max}} \longrightarrow \frac{\omega_{s_{\max}}}{V_p/R}, \quad \vec{r}_{EP} \longrightarrow \frac{\vec{r}_{EP}}{R}, \quad \gamma \triangleq \frac{V_E}{V_p}. \end{aligned} \tag{6}$$

The resulting nondimensional representation of the kinematic system (5) may be expressed in Cartesian coordinates as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -u \\ 0 & 0 & \omega_s \\ u & -\omega_s & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \gamma \begin{bmatrix} \cos\theta\cos\psi \\ \cos\theta\sin\psi \\ -\sin\theta \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

and alternatively in spherical coordinates as:

$$\begin{aligned} \dot{r} &= -\cos\Theta\cos\Psi + \gamma[\cos\Theta\cos\theta\cos(\psi - \Psi) + \sin\Theta\sin\theta], \\ \dot{\Theta} &= \frac{1}{r}\{\sin\Theta\cos\Psi - \gamma[\sin\Theta\cos\theta\cos(\psi - \Psi) - \cos\Theta\sin\theta]\} + \omega_s \sin\Psi - u\cos\Psi, \\ \dot{\Psi} &= \frac{1}{r\cos\Theta}[\sin\Psi + \gamma\cos\theta\sin(\psi - \Psi)] - (\omega_s\cos\Psi + u\sin\Psi)\tan\Theta, \end{aligned} \tag{8}$$

where γ denotes the speed ratio (6) and

$$\begin{aligned} r &= |\vec{r}_{EP}|, \\ (x, y, z) &= r(\cos\Theta\cos\Psi, \cos\Theta\sin\Psi, -\sin\Theta). \end{aligned} \tag{9}$$

Let us next define the spherical target set by:

$$r^2 = x^2 + y^2 + z^2 = \ell^2, \tag{10}$$

which implies that the three problem parameters of this new differential game are: $\gamma < 1$, $\ell > 0$, and $\omega_{s_{\max}} > 0$. One of the interesting results of the analytic solution of such a 3-D pursuit-evasion DG, is the establishment of a capture-criterion in the form of a relationship between the minimum capture-radius ℓ_c and the kinematical game parameters γ and ω_s.

3. THE USEABLE PART AND COSTATE DIFFERENTIAL EQUATIONS

The condition for a penetration of the target set by the pursuer is determined by the following inequality:

$$\max_{\theta,\psi} \vec{\nu} \cdot \dot{\vec{r}}_{EP} \leq 0, \tag{11}$$

where $\vec{\nu}$ is the outward pointing normal to the target set.

It then follows from (8) that:

$$\theta_f = \Theta_f, \quad \psi_f = \Psi_f \tag{12}$$

and

$$\cos\Theta_f \cos\Psi_f \geq \gamma, \tag{13}$$

where the subscript f is used here to denote termination of the game.

The useable part (UP) is therefore a spherical cap, and the boundary of the useable part (BUP) is a circle—the circumference of the cap's basis.

Let us denote the outward pointing normal to the Barrier surface by $(\lambda_x, \lambda_y, \lambda_z)$ and express the corresponding Hamiltonian as:

$$\begin{aligned} H = \gamma \max_{\theta,\psi}(\lambda_x \cos\theta\cos\psi + \lambda_y \cos\theta \sin\psi - \lambda_z \sin\theta) \\ + \min_{|u|\leq 1}[u(\lambda_z x - \lambda_x z)] + \min_{|\omega_s|\leq \omega_{s_{\max}}}[\omega_s(\lambda_y z - \lambda_z y)] - \lambda_x = 0, \end{aligned} \tag{14}$$

which is supplemented by the following adjoint equation:

$$(\dot{\lambda_x}, \dot{\lambda_y}, \dot{\lambda_z}) = (-u\lambda_z, \omega_s\lambda_z, u\lambda_x - \omega_s\lambda_z). \tag{15}$$

The main equation (14) (ME_1) leads to

$$\lambda_x = \lambda\cos\theta\cos\psi, \quad \lambda_y = \lambda\cos\theta\sin\psi, \quad \lambda_z = -\lambda\sin\theta, \tag{16}$$

where, for convenience, we may choose $\lambda = 1$. The Hamiltonian (14) also renders the following relationships for the pursuer's optimal controls:

$$\begin{aligned} u &= \mathrm{sgn}(\lambda_x z - \lambda_z x) = \mathrm{sgn}(S_1), \\ \omega_s &= \omega_{s_{\max}} \mathrm{sgn}(\lambda_z y - \lambda_y z) = \omega_{s_{\max}} \mathrm{sgn}(S_2), \end{aligned} \tag{17}$$

where S_1 and S_2 denote the respective switching functions. Substitution of (16) and (17) into (14) yields the main equation (ME_2):

$$\gamma - \lambda_x - |S_1| - \omega_{s_{\max}}|S_2| = 0. \tag{18}$$

Solving the adjoint system (15) together with (16) results in:

$$\begin{aligned} \dot{\theta} &= \omega_s \sin\psi - u\cos\psi, \\ \dot{\psi} &= -(\omega_s\cos\psi + u\sin\psi)\tan\theta, \end{aligned} \tag{19}$$

where in deriving (19), it has been assumed that $\theta \neq \pm\frac{\pi}{2}$.

4. ANALYTIC SOLUTION FOR STATE AND COSTATE VARIABLES

In view of the optimality condition (16) and (17), the two sets of differential equations (7) and (15) may be transformed into a set of six linear and coupled differential equations, the solution of which may be expressed as:

$$\begin{aligned}\lambda_x(\tau) &= -\frac{u}{\omega}a\cos\omega\tau + \frac{u}{\omega}b\sin\omega\tau + \frac{\omega_s}{\omega}c,\\ \lambda_y(\tau) &= \frac{\omega_s}{\omega}a\cos\omega\tau - \frac{\omega_s}{\omega}b\sin\omega\tau + \frac{u}{\omega}c,\\ \lambda_z(\tau) &= a\sin\omega\tau + b\cos\omega\tau\end{aligned} \tag{20}$$

for the costate variables, where τ is the retrograde nondimensional time, ω is the total pursuer's nondimensional rate of turn, i.e.,

$$\omega^2 \triangleq u^2 + \omega_s^2, \tag{21}$$

and a, b, c are constants defined below:

$$\begin{aligned}a &\triangleq \frac{u}{\omega}\lambda_{x_f} + \frac{\omega_s}{\omega}\lambda_{y_f},\\ b &\triangleq \lambda_{z_f},\\ c &\triangleq \frac{\omega_s}{\omega}\lambda_{x_f} + \frac{u}{\omega}\lambda_{y_f}.\end{aligned} \tag{22}$$

The corresponding solution for the state variables is found as:

$$\begin{aligned}x(\tau) &= -\frac{u}{\omega}f_1(\tau)\cos\omega\tau + \frac{u}{\omega}f_2(\tau)\sin\omega\tau + \frac{\omega_s}{\omega}f_3(\tau),\\ y(\tau) &= \frac{\omega_s}{\omega}f_1(\tau)\cos\omega\tau - \frac{\omega_s}{\omega}f_2(\tau)\sin\omega\tau + \frac{u}{\omega}f_3(\tau),\\ z(\tau) &= f_1(\tau)\sin\omega\tau + f_2(\tau)\cos\omega\tau - \frac{u}{\omega^2},\end{aligned} \tag{23}$$

where

$$\begin{aligned}f_1(\tau) &= A - \gamma a\tau,\\ f_2(\tau) &= B - \gamma b\tau + \frac{u}{\omega^2},\\ f_3(\tau) &= C - \gamma c\tau + \frac{\omega_s}{\omega}\tau,\end{aligned} \tag{24}$$

and A, B, C are the following constants:

$$\begin{aligned}A &\triangleq -\frac{u}{\omega}x_f + \frac{\omega_s}{\omega}y_f,\\ B &\triangleq z_f,\\ C &\triangleq \frac{\omega_s}{\omega}x_f + \frac{u}{\omega}y_f.\end{aligned} \tag{25}$$

For optimal paths emanating from the BUP, we may also express the path-equations as

$$\begin{aligned}x(\tau) &= \lambda_x(\ell - \gamma\tau) + \frac{\omega_s^2}{\omega^2}\tau + \frac{u^2}{\omega^3}\sin\omega\tau,\\ y(\tau) &= \lambda_y(\ell - \gamma\tau) + \frac{u\omega_s}{\omega^2}\tau - \frac{u\omega_s}{\omega^3}\sin\omega\tau,\\ z(\tau) &= \lambda_z(\ell - \gamma\tau) - \frac{u}{\omega^2}(1 - \cos\omega\tau).\end{aligned} \tag{26}$$

According to (17), (20), and (26), we find that for $\tau \to 0^+$, the optimal controls employed by the pursuer just prior to penetration into the BUP are given by:

$$\begin{aligned}\bar{u} &= -\operatorname{sgn}(\lambda_{z_f}) = -\operatorname{sgn}(z_f) = \operatorname{sgn}(\Theta_f),\\ \bar{\omega}_s &= \omega_{s_{\max}}\operatorname{sgn}[\lambda_y \cdot \operatorname{sgn}(-z_f)] = -\omega_{s_{\max}}\operatorname{sgn}(y_f z_f) = \omega_{s_{\max}}\operatorname{sgn}(\Theta_f \Psi_f).\end{aligned} \tag{27}$$

Thus, the pursuer always turns on the target set towards the evader by employing his maximum roll rate so as to decrease y, the lateral distance to the evader. The evader, on the other hand, runs away from the pursuer and contends to increase y. From the ME_1 (14), it is clear that for $\omega_{s_{\max}} \to \infty$, both y and λ_y must vanish. Under these circumstances, the evader is forced to choose $\psi = 0$, i.e., to remain in the plane, and uses his full velocity vector to increase the planar separation from the pursuer, which in turn employs a zero roll-rate maneuver now ($\omega_s = 0$). The resulting game degenerates into the familiar Homicidal Chauffeur game [1]. Indeed, it can be easily verified that for the planar case, in which both $\omega_s = 0$ and $y = \lambda_y = 0$, the present formulation reduces to

$$\begin{aligned} \lambda_x &= \cos(\theta_f + \bar{u}\tau), \\ \lambda_z &= -\sin(\theta_f + \bar{u}\tau) \end{aligned} \tag{28}$$

and

$$\begin{aligned} x(\tau) &= (\ell - \gamma\tau)\cos(\theta_f + \tau) + \frac{1}{\bar{u}}\sin\bar{u}\tau, \\ z(\tau) &= -(\ell - \gamma\tau)\sin(\theta_f + \tau) - \frac{1}{\bar{u}}(1 - \cos\bar{u}\tau), \end{aligned} \tag{29}$$

which are precisely the path equations derived by Isaacs [1] for the Homicidal Chauffeur game (p. 232), with a BUP given now by (13), as:

$$\cos\Theta_f = \gamma, \quad \Psi_f = 0. \tag{30}$$

5. CRITICAL CAPTURE RADIUS

As long as the distance between the pursuer and the evader is larger than some critical range, P can always keep E directly ahead of him, whatever E does. As the range between the players decreases, E must choose the right instance to break abruptly towards a new direction in the $\mathbb{R}^3$ space. As this particular break-away point, which is characterized by an initial range x_1, the initial adjoint λ_{x_1} is thus zero:

$$\lambda_{x_1} = 0, \tag{31}$$

and, in addition

$$y_1 = z_1 = 0. \tag{32}$$

A 2-D analog of such a condition has been also employed by Breakwell and Merz [3] in their quest to find a capture criterion for Isaacs' Game of Two Cars.

Starting with the terminal conditions on the BUP, the kinematic equations are solved in the retrogressive sense by moving backwards along the optimal trajectory up to the point reaching $(x_1, 0, 0)$. We thus find the critical capture radius ℓ_c as a function of the pursuer's nondimensional roll rate ω_s (actually—the ratio: roll rate to pitch rate) and the speed-ratio $\gamma(V_E/V_p)$. Without loss of generality, we consider here only those trajectories which terminate on one quarter of the BUP (see Figure 2), at which $1 \leq \epsilon \leq \frac{\pi}{2}$, where ϵ satisfies:

$$\epsilon = \tan^{-1}\left(\frac{\sin\Psi_f}{\tan\Theta_f}\right). \tag{33}$$

The existence of a Barrier surface which terminates tangentially at the BUP is guaranteed if $r_f \geq 0$. Utilizing (8), (12), (13), and (33) leads to the following equivalent condition:

$$(\ell_c \cos\epsilon)^2 + \gamma^2 \leq 1. \tag{34}$$

In the sequel, we will consider all possible cases for which (34) holds. In particular, a distinction is made between cases in which the optimal trajectory leading to the BUP consists of two- or three-stage maneuvers, where at least one player is applying a different (bang-bang) control on each segment.

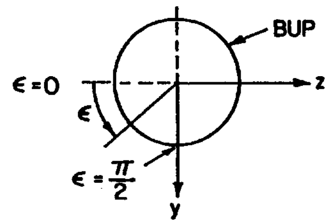

Figure 2. The BUP angle ϵ.

5.1. The Two-Stage Trajectory

An optimal trajectory which terminates on the BUP at $0 < \epsilon < \frac{\pi}{2}$, is found to consist of two stages, the first stage starting at $(x_1, 0, 0)$ and terminating at (x_s, y_s, z_s) on a switching line along which $S_1 = 0$. The second stage starts at (x_s, y_s, z_s) and ends at a point (x_f, y_f, z_f) lying on the BUP (see Figure 3). The switching time is the evolute of the smallest positive τ_s which annuls the switching function $S_1 = \lambda_x z - \lambda_z x$. Thus, τ_s satisfies the following implicit relation:

$$\tau_s = -\frac{1}{\omega_s^2} \frac{\cos\theta_f \cos(\psi_f + 2\alpha)[1 - \cos\omega\tau_s] + \sin\theta_f \cos\alpha \sin\omega\tau_s}{\sin\theta_f \cos\omega\tau_s + \cos\theta_f \cos(\psi_f + \alpha)\sin\omega\tau_s}, \tag{35}$$

where

$$\alpha \triangleq \tan^{-1}\omega_s. \tag{36}$$

We note that ω_s can take on values in the range $\omega_m \leq \omega_s \leq \omega_c$, where with $\omega_s = \omega_m$, the trajectory terminates at $\epsilon = \frac{\pi}{2}$, and for $\omega_s = \omega_c$, termination occurs at $\epsilon = 0$. In order to guarantee capture, the pursuer must therefore possess a roll-rate ω_s larger than or equal to ω_m. For roll-rates higher than ω_c, one gets a different type of optimal trajectory, which is discussed in the next paragraph. Thus, for a given ω_s ($\omega_m \leq \omega_s \leq \omega_c$), the retrogressive time τ_s, the state vector (x_s, y_s, z_s), and the costate vector $(\lambda_{x_s}, \lambda_{z_s}, \lambda_{y_s})$ are determined from the transversality conditions:

$$\begin{aligned} \vec{x}_s(\tau_s^+) &= \vec{x}_s(\tau_s^-) \triangleq \vec{x}_s = (x_s, y_s, z_s), \\ \vec{\lambda}_s(\tau_s^+) &= \vec{\lambda}_s(\tau_s^-) \triangleq \vec{\lambda}_s = (\lambda_{x_s}, \lambda_{y_s}, \lambda_{z_s}). \end{aligned} \tag{37}$$

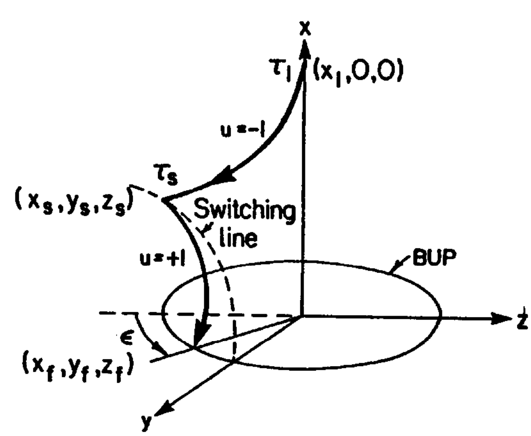

Figure 3. The two-stage trajectory.

The new "initial conditions" (actually, the terminal conditions of the first stage) are now obtained as functions of ℓ and ϵ. It should be also noted that τ_s does not explicitly depend on the miss-distance ℓ. Based on these new initial conditions, we get the path-equations for the first-stage of the trajectory and find both $\ell = \ell_c$ and ϵ by satisfying (31) and (32).

The time travelled along the first-stage is the smallest positive root τ_1 which annuls λ_{x_1}, i.e.,

$$\tau_1 = \frac{1}{\omega}\left[\pi + \sin^{-1}\left(\omega_s \frac{c}{d}\right) - \phi\right], \tag{38}$$

where

$$d \triangleq \sqrt{a^2 + b^2},$$
$$\phi \triangleq \sin^{-1}\left(\frac{a}{d}\right), \tag{39}$$

and a, b, c are given by (22) after replacing $\vec{\lambda}_f$ by $\vec{\lambda}_s$.

From $z_1 = z(\tau_1) = 0$, we get:

$$\ell_c = \gamma(\tau_s + \tau_1) - \frac{a_1 \sin \omega\tau_1 + b_1 \cos \omega\tau_1}{a \sin \omega\tau_1 + b \cos \omega\tau_1}, \tag{40}$$

where τ_s is given by (35), τ_1 by (38), and a_1, b_1 are defined as:

$$a_1 \triangleq 2\frac{\omega_s^2}{\omega^3}\tau_s - \frac{\omega_s^2 - 1}{\omega^4}\sin \omega\tau_s,$$
$$b_1 \triangleq -\frac{1}{\omega^2}(2 - \cos \omega\tau_s). \tag{41}$$

Similarly, using $y_1 = y_1(\tau_1) = 0$ implies

$$\ell = \gamma(\tau_s + \tau_1) + \frac{c_1 + (\omega_s/\omega)\tau_1 - \omega_s(a_1 \cos \omega\tau_1 - b_1 \sin \omega\tau_1)}{\omega_s(a \cos \omega\tau_1 - b \sin \omega\tau_1) - c}, \tag{42}$$

where c_1 is defined as:

$$c_1 = \frac{\omega_s}{\omega^3}\left[(\omega_s^2 - 1)\tau_s + \frac{2}{\omega}\sin \omega\tau_s\right]. \tag{43}$$

By comparing (40) with (42), we finally get an implicit equation from which the BUP angle ϵ ($0 \le \epsilon \le \frac{\pi}{2}$) can be found, and consequently, the critical capture radius ℓ_c can be determined. A typical graph of ℓ_c as a function of ω_s, for a speed ratio of $\gamma = 0.5$, in the case of a two-stage maneuver, is shown in Figure 4.

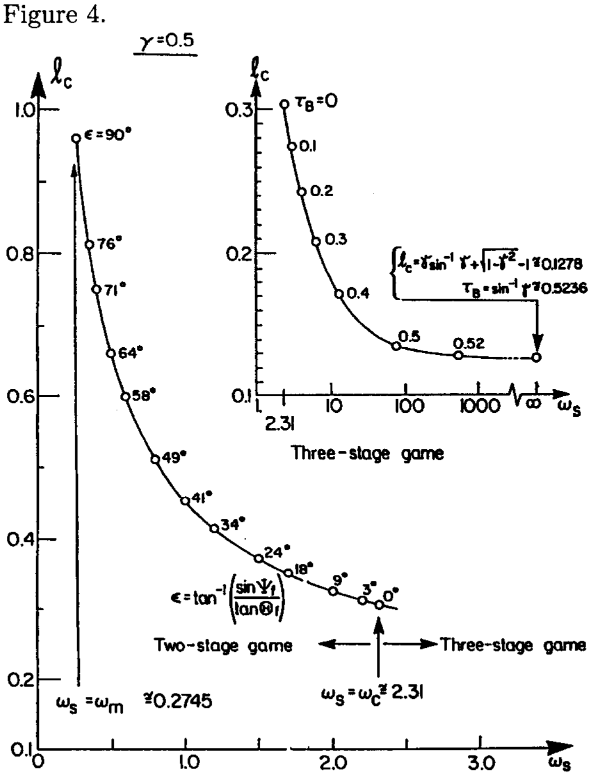

Figure 4. Capture radius (miss-distance) versus roll-rate ω_s for $\gamma = 0.5$.

5.2. The Three-Stage Trajectory

If the pursuer has a maximum roll rate which is larger than ω_c, then the optimal maneuver consists of three distinct stages—the first stage starts at $(x_1, 0, 0)$ and terminates at (x_s, y_s, z_s) on the switching line, the second stage (a tributary) starts at (x_s, y_s, z_s) and terminates at (x_B, y_B, z_B) on the ω_s-Universal Line (UL), and the third stage (a segment of the ω_s-UL) starts at (x_B, y_B, z_B) and ends on the BUP at $(x_f, 0, z_f)$ (see Figure 5).

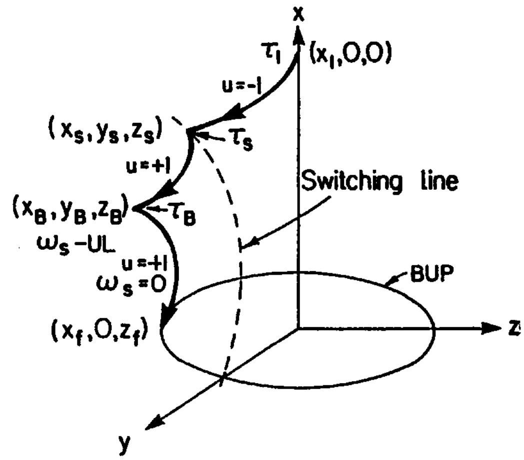

Figure 5. The three-stage trajectory.

Along a ω_s-UL, the planar descriptions of (28) and (29) prevail, from which we find the "initial conditions" for the second stage, i.e., $\vec{x}_B$ and $\vec{\lambda}_B$ for $\tau = \tau_B$, where τ_B denotes the departure time of the tributary (second stage) from the UL.

By following a similar procedure to the one used in the previous paragraph, we obtain an implicit equation from which we can determine τ_B for a given $\omega_s \geq \omega_c$ as well as the critical capture radius ℓ_c. The switching time τ_s in this case satisfies the following relation:

$$\begin{aligned}\tau_s = &-\omega_s^{-2}\{[\sin(\theta_f + \tau_B)\cos\omega\tau_s + \cos(\theta_f + \tau_B)\cos\alpha\sin\omega\tau_s]\}^{-1}\\ &\{[\gamma + \cos(\theta_f + \tau_B)\cos 2\alpha]\,[1 + \omega_s^2\cos\omega\tau_s]\\ &- \cos(\theta_f + \tau_B)[1 + \cos\omega\tau_s] + \sin(\theta_f + \tau_B)\cos\alpha\sin\omega\tau_s\},\end{aligned} \tag{44}$$

where α is defined by (36). For $\omega_s = \omega_c$, where $\epsilon = 0$, (44) and (35) can be shown to be identical.

Starting with the "initial conditions" $\vec{x}_B$ and $\vec{\lambda}_B$ at τ_B, travelling backwards along the second-stage and substituting $\tau = \tau_s$ from (44), we get the necessary "initial conditions" for the first stage, $\vec{x}_s$ and $\vec{\lambda}_s$, in terms of the parameters ℓ and τ_B.

The time travelled along the first stage is given by (38). Letting $z_1 = z(\tau_1) = 0$, (40) is replaced here by:

$$\ell_c = \gamma(\tau_B + \tau_s + \tau_1) - \frac{a_1\sin\omega\tau_1 + b_1\cos\omega\tau_1}{a\sin\omega\tau_1 + b\cos\omega\tau_1}, \tag{45}$$

where (41) is now replaced by:

$$\begin{aligned}a_1 &\triangleq 2\frac{\omega_s^2}{\omega^3}(\tau_s + \sin_B) + \frac{\omega_s^2 - 1}{\omega^2}\left[\left(\frac{\omega_s^2}{\omega^2} - \cos\tau_B\right)\sin\omega\tau_s - \frac{1}{\omega}\sin\tau_B\cos\omega\tau_s\right],\\ b_1 &\triangleq -\frac{2}{\omega^2} - \left(\frac{\omega_s^2}{\omega^2}\cos\tau_B\right)\omega\tau_s - \frac{1}{\omega}\sin\tau_B\sin\omega\tau_s,\end{aligned} \tag{46}$$

with τ_s and τ_1 given by (44) and (38), respectively. Next, we demand that $y_1 = y(\tau_1) = 0$ and as a result, equation (42) renders

$$\ell_c = \gamma(\tau_B + \tau_s + \tau_1) + \frac{c_1 + (\omega_s/\omega)\tau_1 - \omega_s(a_1\cos\omega\tau_1 - b_1\sin\omega\tau_1)}{\omega_s(a\cos\omega\tau_1 - b\sin\omega\tau_1) - c}, \tag{47}$$

with (43) replaced now by:

$$c_1 = \frac{\omega_s(\omega_s^2 - 1)}{\omega^3}(\tau_s + \sin\tau_B) - 2\frac{\omega_s}{\omega^2}\left[\left(\frac{\omega_s^2}{\omega^2} - \cos\tau_B\right)\sin\omega\tau_s - \frac{1}{\omega}\sin\tau_B\cos\omega\tau_s\right]. \tag{48}$$

As in the previous case, the coefficients a, b, and c are calculated from (22) by replacing $\vec{\lambda}_f$ by $\vec{\lambda}_s$. It is easily seen that for $\tau_B = 0$, (46) and (48) are identical with (41) and (43). Finally, by comparing (45) with (47), we get an implicit equation from which τ_B is derived, and then, from (45) or (45), the critical capture-radius ℓ_c is determined.

If the pursuer has an infinite roll-rate, it can be shown that the condition $\lim_{\omega_s\to\infty} S_1 = 0$ leads to

$$\lim_{\omega_s\to\infty} \omega\tau_s = \frac{\pi}{2}. \tag{49}$$

For this degenerate case, the coefficients a, b, c, and d are given by

$$a = \sin(\theta_f + \tau_B), \quad b = 0, \quad c = \cos(\theta_f + \tau_B), \quad d \triangleq \sqrt{a^2 + b^2} = |\sin(\theta_f + \tau_B)|. \tag{50}$$

According to (38), τ_1 exists only if the argument of $\sin^{-1}$ satisfies:

$$\lim_{\omega_s\to\infty}\left|\frac{\omega_s c}{d}\right| \le 1, \tag{51}$$

from which we get by substituting (50):

$$\lim_{\omega_s\to\infty}(\theta_f + \tau_B) = \frac{\pi}{2}. \tag{52}$$

Using the relationship $\theta_f = \cos^{-1}\gamma$ in (52), we get for $\omega_s \to \infty$:

$$\tau_B = \sin^{-1}\gamma. \tag{53}$$

Following (38), it is also clear that for $\omega_s \to \infty$, $\tau_1 = 0$. Thus, the time-to-go for an infinite roll-rate is:

$$t_{go} = \tau_B + \tau_S + \tau_1 = \tau_B = \sin^{-1}\gamma. \tag{54}$$

This result coincides, as expected, with that of the planar Homicidal Chauffeur game, into which our three-stage game degenerates when $\omega_s \to \infty$.

One should note, however, that there is no switching surface in this particular case, and τ_1 elapses into τ_s, i.e., (see (49)):

$$\lim_{\omega_s\to\infty} \omega\tau_1 = \frac{\pi}{2}. \tag{55}$$

From (46), we readily get, for $\omega_s \to \infty$, that

$$a_1 = 1 - \cos\tau_B, \quad b_1 = 0. \tag{56}$$

Substitution of (50), (52), (54), (55), and (56) into (45) yields the known result [1, p. 237] for the capture-radius of the Homicidal Chauffeur game:

$$\ell_c = \gamma\sin^{-1}\gamma + \sqrt{1-\gamma^2} - 1, \tag{57}$$

where ℓ_c and γ have to satisfy:

$$\ell_c^2 + \gamma^2 \le 1. \tag{58}$$

This limiting relationship may be also obtained directly by setting $\epsilon = 0$ in (34). A typical graph displaying the variation of ℓ_c as a function of ω_s for the three-stage game, and for a speed ratio of $\gamma = 0.5$, is shown in Figure 4.

6. CONCLUDING REMARKS

An analysis has been presented by which the effect of a finite (bounded) roll-rate on the miss-distance of a bank-to-turn missile has been established. The theoretical model of an $\mathbb{R}^3$ encounter is based on a DG game of kind formulation of an optimal pursuit-evasion situation between a faster pursuer and a slower evader which has a maneuverability advantage. The proposed new game may also be considered as a 3-D generalization of Isaacs' Homicidal Chauffeur Game. It is rather fortunate that this spatial DG is still amenable for analytic treatment and that closed-form expressions for the 3-D optimal trajectories, as well as state and costate variables, may still be determined in a relatively simple form.

Besides the analytical formulation of this novel three-dimensional differential game, the main result of this paper is the determination of the capturability conditions. Such a criterion is obtained by postulation, following the elegant 2-D analysis of Breakwell and Merz [3], that the Barrier surface is marginally closed for a head-on or a tail-chase encounter. Using these as initial conditions, it is shown that for some initial separations ℓ_c, the faster pursuer can steer the dynamic system into the target set (thus, guaranteeing unconditional capture) by executing a two or three-stage maneuver, depending on the relative magnitude of his available roll-rate. There exists a critical value of the roll-rate above which the optimal game consists of three different trajectories and below which there are only two. The two distinct stages are characterized by two hard-turns (maximum pitch rate) of the pursuer in opposite directions, while the evader is maintaining his course direction. In spite of the abrupt change in the pitch-rate, the pursuer simultaneously employs his maximum roll-rate in the same direction. For example, the nondimensional value of the critical roll-rate (normalized with the pitch-rate V_p/R) for a speed ratio of $\gamma = V_E/V_p = 0.5$ is found (see Figure 4) to be $\omega = 2.31$.

For values of ω_s larger than ω_c, the optimal game leading to the BUP turns out to consist of three stages, where in addition to the previous two-stages, the pursuer employs a zero roll-rate ($\omega_s = 0$) hard-turn just prior to reaching the target set. Along the third-stage, which is in fact an ω_s-universal line, the pursuer maintains the same pitch-rate as in the second stage. Thus, for $\omega_s > \omega_c$, the pursuer is capable of steering the "end-game" into a 2-D plane which is his preferable pitch-plane. In many realistic pursuit-evasion situations, it is desired to force the evader into the pitch-plane just prior to game termination. Thus, the value of ω_c may prove useful in the optimal design of roll/pitch controllers of a bank-to-turn missile for a prescribed miss-distance and speed ratio.

In addition to the critical roll-rate which renders a continuous transition from a two to a three-stage maneuver, there exists also a minimum roll rate below which capture is not possible. This minimum roll-rate, which is here denoted by ω_m, is determined from the limiting value of the BUP angle, i.e., $\epsilon = 0$. In the present example ($\gamma = 0.5$), we find (see Figure 4) that $\omega_m \approx 0.2745$ and that the minimum miss-distance is $\ell_c \approx 0.96$. This value may be compared against the minimum capture radius of $\ell_c \approx 0.3$ obtained at $\omega_s = \omega_c$.

Increasing the roll-rate indefinitely ($\omega_s \to \infty$) reduces the game formulation into the Homicidal Chauffeur game, which is in fact a one-stage maneuver in which the faster pursuer uses his maximum pitch-rate while executing a zero-roll maneuver. Under these conditions, the present capture criterion degenerates into the well-known 2-D Isaacs' criterion (Figure 4), which for $\gamma = 0.5$ renders $\ell_c \approx 0.128$. Thus, increasing the roll-rate in the range $\infty > \omega_s > \omega_m$ tends to decrease the capture radius, where the smallest value is that given by the Homicidal Chauffeur criterion (57) and the maximum value is attained at $\omega_s = \omega_m$. In a subsequent paper, the present capturability condition, which for the sake of brevity has been demonstrated here only for $\gamma = 0.5$, is further extended to cover the whole range of speed ratios (i.e., $1 > \gamma > 0$).

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