Pergamon

Appl. Math. Lett. Vol. 10, No. 1, pp. 23–29, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(96)00105-X

Necessary and Sufficient Conditions for Oscillations of Delay Equations with Impulses

AIMIN ZHAO AND JURANG YAN
Department of Mathematics, Shanxi University
Taiyuan, Shanxi 030006, P.R. China

(Received and accepted March 1996)

Communicated by D. G. M. Anderson

Abstract—In this paper, a necessary and sufficient condition for oscillation of a first-order delay differential equation with impulses

$$\begin{aligned} x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) &= 0, \qquad t \neq t_k, \\ x\left(t_k^+\right) - x(t_k) &= b_k x(t_k), \qquad k = 1, 2, \dots \end{aligned} \tag{$*$}$$

is established.

Keywords—Delay differential equation, Impulse, Oscillation, Characteristic equation.

1. INTRODUCTION

The theory of differential equations with impulses is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential equations. Moreover, such equations represent a natural framework for mathematical modeling of several real world phenomena. There exists a well-developed oscillation theory of delay differential equations [1–3]. The theory of ordinary differential equations with impulses has also been developed extensively over the past few years [4]. However, delay differential equations with impulses seem to have rarely been considered with respect to the oscillation of their solutions or the stability of their steady states [5,6].

It has been known (see [1,3]) that a necessary and sufficient condition for oscillation of all solutions of the delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0 \tag{1}$$

is that its characteristic equation

$$\lambda + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} = 0 \tag{2}$$

The authors thank D. G. M. Anderson for his very helpful comments and suggestions.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX has no real roots. In this paper, for the delay differential equation with impulses

$$\begin{aligned} x'(t) + \sum_{i=1}^{n} p_i x(t-\tau_i) &= 0, & t &\neq t_k, \\ x(t_k^+) - x(t_k) &= b_k x(t_k), & k &= 1,2,\dots, \end{aligned} \tag{3}$$

we obtain as a necessary and sufficient condition for the oscillation of all its solutions that its "characteristic equation"

$$F(\lambda) \triangleq \lambda + \sum_{i=1}^{n} p_i \xi^{\alpha_i} e^{-\lambda\tau_i} = 0 \tag{4}$$

has no real roots, where ξ and α_i are constants defined by (5) below.

Consider equation (3), where $p_i > 0$ for $i = 1,2,\dots,n$; $0 < \tau_1 < \tau_2 < \cdots < \tau_n$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{t\to\infty} t_k = \infty$, and $b_k \in R$ for $k = 1,2,\dots$. We assume that the following conditions (C) are satisfied.

(a) For the case of $n = 1$, there exists a positive integer m such that for $j = 1,2,\dots,m$; $k = 1,2,\dots,$

$$t_{km+j} = t_j + k\tau_1, \quad \text{and} \quad b_{km+j} = b_j.$$

(b) For the case $n > 1$, the quotients τ_i/τ_1 are rational numbers for $i = 2,3,\dots,n$, that is, there exist positive integers q_i and r_i which are coprime such that

$$\frac{\tau_i}{\tau_1} = \frac{q_i}{r_i},$$

and there exists a positive integer m such that for $j = 1,2,\dots,m$; $k = 1,2,\dots,$

$$t_{km+j} = t_j + kT \quad \text{and} \quad b_{km+j} = b_j,$$

where $T = \tau_1/r$ and r is the least common multiple of $r_1, r_2, \dots, r_n$.

For $\sigma \geq t_1 + \tau_n$, a function $x : [\sigma - \tau_n, \infty) \to R$ is said to be a solution of (3), if it is left continuous on $[\sigma - \tau_n, \infty)$, and is differentiable on $[\sigma, \infty)\backslash\{t_k\}$ and satisfies (3).

As is customary, we shall say that a nontrivial solution of (3) is nonoscillatory if it is eventually positive or eventually negative, and otherwise it will be called oscillatory.

We define

$$\xi \triangleq \prod_{k=1}^{m}(1+b_k), \quad \eta \triangleq \prod_{k=1}^{m}\left(1+b_k^+\right), \quad \alpha_i \triangleq \frac{rq_i}{r_i}, \qquad \text{for } i = 1,2,\dots,n, \tag{5}$$

where $b_k^+ = \max\{0, b_k\}$, r, r_i, and q_i are defined in conditions (C). It is obvious that α_i are positive integers such that $\tau_i = \alpha_i T$ for $i = 1,2,\dots,n$.

2. MAIN RESULTS

Initially, we give some useful lemmas. The first lemma is an immediate conclusion from conditions (C).

LEMMA 1. *Assume that conditions (C) hold. Then, for any $t \geq t_1 + \tau_n$,*

$$\prod_{t-\tau_i \leq t_k < t} (1+b_k) = \xi^{\alpha_i} \tag{6}$$

and

$$\prod_{t-\tau_i \leq t_k < t} \left(1+b_k^+\right) = \eta^{\alpha_i}. \tag{7}$$

LEMMA 2. *Assume that $x : [\sigma - \tau_n, \infty) \to R$ is a positive function such that*

$$\begin{aligned} x'(t) &< 0, & t > \sigma, \quad t \neq t_k, \\ x\left(t_k^+\right) - x\left(t_k\right) &= b_k x\left(t_k\right), & k = 1, 2, \ldots. \end{aligned} \tag{8}$$

Then, for any $\sigma \leq t_* < t^* < \infty$,

$$x\left(t^*\right) < x\left(t_*\right) \prod_{t_* \leq t_k < t^*} \left(1 + b_k\right), \tag{9}$$

$$x\left(t_*\right) + \sum_{t_* \leq t_k < t^*} b_k x\left(t_k\right) \leq x\left(t_*\right) \prod_{t_* \leq t_k < t^*} \left(1 + b_k^+\right), \tag{10}$$

and

$$\inf_{t_* \leq t \leq t^*} x(t) \geq x\left(t^*\right) \prod_{t_* \leq t_k < t^*} \left(1 + b_k^+\right)^{-1}. \tag{11}$$

PROOF. From (8), we know that $x(t)$ is decreasing on every subinterval $(t_k, t_{k+1}]$ of $[\sigma, \infty)$. Suppose that $t_j, t_{j+1}, \ldots, t_l$ are all of the impulse points situated in $[t_*, t^*)$, that is

$$t_* \leq t_j < t_{j+1} < \cdots < t_l < t^* \leq t_{l+1}.$$

Thus, we have

$$\begin{aligned} x\left(t_*\right) \geq x\left(t_j\right) &= \left(1 + b_j\right)^{-1} x\left(t_j^+\right), \\ x\left(t_j^+\right) > x\left(t_{j+1}\right) &= \left(1 + b_{j+1}\right)^{-1} x\left(t_{j+1}^+\right), \\ &\cdots\cdots \\ x\left(t_{l-1}^+\right) > x\left(t_l\right) &= \left(1 + b_l\right)^{-1} x\left(t_l^+\right), \end{aligned}$$

and

$$x\left(t_l^+\right) > x\left(t^*\right).$$

Hence,

(i) $x(t^*) < x(t_*) \prod_{k=j}^{l} (1 + b_k) = x(t_*) \prod_{t_* \leq t_k < t^*} (1 + b_k)$;

(ii) for $k = j, j+1, \ldots, l$,

$$x\left(t_k\right) \leq x\left(t_*\right) \prod_{i=j}^{k} \left(1 + b_i\right) \leq x\left(t_*\right) \prod_{i=j}^{k} \left(1 + b_i^+\right).$$

Furthermore,

$$\begin{aligned} x\left(t_*\right) + \sum_{t_* \leq t_k < t^*} b_k x\left(t_k\right) &\leq x\left(t_*\right) + \sum_{k=j}^{l} b_k^+ \prod_{i=j}^{k} \left(1 + b_k^+\right) x\left(t_*\right) \\ &= x\left(t_*\right) \prod_{k=j}^{l} \left(1 + b_k^+\right) \\ &= x\left(t_*\right) \prod_{t_* \leq t_k < t^*} \left(1 + b_k^+\right). \end{aligned}$$

(iii) For $k = j, j+1, \ldots, l$,

$$x\left(t_k\right) \geq x(t^*) \prod_{i=k}^{l} \left(1 + b_k\right)^{-1} \geq x\left(t^*\right) \prod_{i=k}^{l} \left(1 + b_k^+\right)^{-1}.$$

Thus,

$$\inf_{t_*\le t\le t^*} x(t) = \min\{x(t_j), x(t_{j+1}), \dots, x(t_l), x(t^*)\} \ge x(t^*) \prod_{t_*\le t_k<t^*} (1+b_k^+)^{-1}.$$

The proof of Lemma 2 is complete.

LEMMA 3. *Assume that conditions (C) hold and $x : [\sigma-\tau_n, \infty) \to R$ is a positive solution of (3). Then, for $i = 1, 2, \dots, n$ and $t > \sigma + (3/2)\tau_i$,*

$$x(t-\tau_i) < \left(\frac{2\eta^{\alpha_i}}{p_i\tau_i}\right)^2 x(t). \tag{12}$$

PROOF. For a given $s > \sigma + \tau_n$ and $1 \le i \le n$, suppose that $t_j, t_{j+1}, \dots, t_l$ are all of the impulse points situated in $[s, s+(\tau_i/2))$, that is

$$s \le t_j < t_{j+1} < \dots < t_l < s + \frac{\tau_i}{2} \le t_{l+1}.$$

Then, using Lemma 1 and Lemma 2, we have

$$\begin{aligned}\int_s^{s+(\tau_i/2)} x'(t)\,dt &= \left(\int_s^{t_j} + \int_{t_j}^{t_{j+1}} + \dots + \int_{t_l}^{s+(\tau_i/2)}\right) x'(t)\,dt \\ &= x\left(s+\frac{\tau_i}{2}\right) - x(s) - \sum_{k=j}^{l} b_k x(t_k) \\ &< -x(s) \prod_{s\le t_k<s+(\tau_i/2)} (1+b_k^+)\end{aligned} \tag{13}$$

and

$$\begin{aligned}\int_s^{s+(\tau_i/2)} x(t-\tau_i)\,dt &\ge \frac{\tau_i}{2} \inf_{s-\tau_i\le t\le s-(\tau_i/2)} x(t) \\ &\ge \frac{\tau_i}{2} x\left(s-\frac{\tau_i}{2}\right) \prod_{s-\tau_i\le t_k<s-(\tau_i/2)} (1+b_k^+)^{-1}.\end{aligned} \tag{14}$$

From (3) we have, for $i = 1, 2, \dots, n$,

$$\begin{aligned} x'(t) + p_i x(t-\tau_i) &\le 0, && t > \sigma, \quad t \ne t_k, \\ x(t_k^+) - x(t_k) &= b_k x(t_k), && k = 1, 2, \dots. \end{aligned} \tag{15}$$

Integrating both sides of (15) from s to $s + (\tau_i/2)$ and using (13) and (14), we obtain

$$x\left(s-\frac{\tau_i}{2}\right) < \frac{2}{p_i\tau_i} \prod_{s-\tau_i\le t_k<s-(\tau_i/2)} (1+b_k^+)^{-1} \prod_{s\le t_k<s+(\tau_i/2)} (1+b_k^+)^{-1} x(s). \tag{16}$$

For $t > \sigma + (3/2)\tau_n$, let $s = t - (\tau_i/2)$ and $s = t$, respectively, we get

$$x(t-\tau_i) < \frac{2}{p_i\tau_i} \prod_{t-(3\tau_i/2)\le t_k<t-\tau_i} (1+b_k^+)^{-1} \prod_{t-(\tau_i/2)\le t_k<t} (1+b_k^+)^{-1} x\left(t-\frac{\tau_i}{2}\right)$$

and

$$x\left(t-\frac{\tau_i}{2}\right) < \frac{2}{p_i\tau_i} \prod_{t-\tau_i\le t_k<t-(\tau_i/2)} (1+b_k^+)^{-1} \prod_{t\le t_k<t+(\tau_i/2)} (1+b_k^+)^{-1} x(t).$$

The desired inequality (12) follows by combining these two inequalities. The proof of Lemma 3 is complete.

Now we give our main theorem.

THEOREM. *Assume that conditions (C) hold. Then the following statements are equivalent.*

(a) *Equation (3) has a nonoscillatory solution.*

(b) *The characteristic equation (4) has a real root.*

PROOF. To prove (b)⇒(a), assume that λ_0 is a real root of (4) and define

$$x(t) = \prod_{t_1 \le t_k < t} (1 + b_k) e^{\lambda_0 t}, \qquad \text{for } t > t_1. \tag{17}$$

It is obvious that $x(t)$ is left continuous on $[t_1, \infty)$ and is differentiable on $(t_1 + \tau_n, \infty) \backslash \{t_k\}$. Furthermore, for $t > t_1 + \tau_n$ and $t \neq t_k$,

$$\begin{aligned} x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) &= \lambda_0 \prod_{t_1 \le t_k < t} (1 + b_k) e^{\lambda_0 t} + \sum_{i=1}^{n} p_i \prod_{t_1 \le t_k < t - \tau_i} (1 + b_k) e^{\lambda_0 (t - \tau_i)} \\ &= \prod_{t_1 \le t_k < t} (1 + b_k) e^{\lambda_0 t} \left[\lambda_0 + \sum_{i=1}^{n} p_i \prod_{t - \tau_i \le t_k < t} (1 + b_k)^{-1} e^{-\lambda_0 \tau_i} \right] \\ &= \prod_{t_1 \le t_k < t} (1 + b_k) e^{\lambda_0 t} \left[\lambda_0 + \sum_{i=1}^{n} p_i \xi^{-\alpha_i} e^{-\lambda_0 \tau_i} \right] \\ &= 0, \end{aligned}$$

and for $t_k \ge t_1 + \tau_n$,

$$x\left(t_k^+\right) - x(t_k) = b_k x(t_k).$$

Thus, $x(t)$ is a positive solution of (3).

To prove (a)⇒(b), without loss of generality, assume that $x(t)$ is an eventually positive solution of (3). So there exists $\sigma > t_1 + \tau_n$ such that $x(t) > 0$ for $t \ge \sigma - \tau_n$. Set

$$\Lambda = \{\lambda > 0 : \; x'(t) + \lambda x(t) < 0 \text{ eventually for } t \neq t_k\}.$$

From (3) and Lemma 2, we have

$$x'(t) + p_i \prod_{t - \tau_i \le t_k < t} (1 + b_k)^{-1} x(t) < 0, \qquad \text{for } t > \sigma + \tau_n \quad \text{and} \quad t \neq t_k.$$

Thus, $p_i \xi^{-\alpha_i} = p_i \prod_{t - \tau_i \le t_k < t} (1 + b_k)^{-1} \in \Lambda$. Also from Lemma 3, we get

$$\begin{aligned} 0 &= x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) \\ &\le x'(t) + \left[\sum_{i=1}^{n} p_i \left(\frac{2\eta^{\alpha_i}}{p_i \tau_i} \right)^2 \right] x(t), \qquad \text{for } t > \sigma + \frac{3}{2}\tau_i, \quad t \neq t_k. \end{aligned}$$

Therefore, $\sum_{i=1}^{n} p_i (2\eta^{\alpha_i}/p_i\tau_i)^2$ is an upper bound for Λ. Since Λ is nonempty and bounded, we may set $\lambda_0 = \sup \Lambda$.

Let $\lambda \in \Lambda$ be given and define y on $[\sigma - \tau_n, \infty)$ by $y(t) = x(t)e^{\lambda t}$. Then, there is a suitable $T_\lambda \in (\sigma, \infty)$ such that

$$y'(t) = (x'(t) + \lambda x(t)) e^{\lambda t} < 0, \qquad \text{for } t > T_\lambda \quad \text{and} \quad t \neq t_k.$$

On the other hand,

$$y\left(t_k^+\right) - y(t_k) = b_k y(t_k), \qquad \text{for } t_k > T_\lambda.$$

So by Lemma 2, we know

$$y(t - \tau_i) > y(t) \prod_{t - \tau_i \le t_k < t} (1 + b_k)^{-1} = \xi^{-\alpha_i} y(t), \qquad \text{for } t > T_\lambda + \tau_n.$$

Hence, for $t > T_\lambda + \tau_n$ and $t \neq t_k$,

$$\begin{aligned} 0 &= x'(t) + \sum_{i=1}^{n} p_i x(t-\tau_i) \\ &= x'(t) + \sum_{i=1}^{n} p_i y(t-\tau_i) e^{-\lambda(t-\tau_i)} \\ &> x'(t) + \sum_{i=1}^{n} p_i \xi^{-\alpha_i} y(t) e^{\lambda(t-\tau_i)} \\ &= x'(t) + \sum_{i=1}^{n} p_i \xi^{-\alpha_i} e^{\lambda\tau_i} x(t). \end{aligned} \tag{18}$$

This shows that $\sum_{i=1}^{n} p_i \xi^{-\alpha_i} e^{\lambda\tau_i} \in \Lambda$, and hence, $\sum_{i=1}^{n} p_i \xi^{-\alpha_i} e^{\lambda\tau_i} \leq \lambda_0$. Since $\lambda \in \Lambda$ is arbitrary, we conclude that

$$\sum_{i=1}^{n} p_i \xi^{-\alpha_i} e^{\lambda_0 \tau_i} \leq \lambda_0.$$

Therefore, $F(-\lambda_0) = -\lambda_0 + \sum_{i=1}^{n} p_i \xi^{-\alpha_i} e^{\lambda_0\tau_i} \leq 0$. Noticing that $F(+\infty) = +\infty$, we know that (4) has a real root. The proof of the theorem is complete.

REMARK. If conditions (C) hold, from the theorem we know that a necessary and sufficient condition for oscillation of all solutions of (3) is that the characteristic equation (4) has no real roots.

From the theorem and [2, Theorem 2.2.1], we can immediately obtain the following result.

COROLLARY 1. *Assume that conditions (C) hold. Then each of the following conditions is sufficient for the oscillation of all solutions of (3):*

(a)

$$\sum_{i=1}^{n} p_i \xi^{\alpha_i} \tau_i \frac{1}{e};$$

(b)

$$\left(\prod_{i=1}^{n} p_i \xi^{\alpha_i}\right)^{1/n} \left(\sum_{i=1}^{n} \tau_i\right) \frac{1}{e}.$$

Similarly, from the theorem and in [2, Theorem 2.1.1], we immediately obtain the following result.

COROLLARY 2. *Assume that conditions (C) hold. Then,*

$$\left(\sum_{i=1}^{n} p_i \xi_i^{\alpha_i}\right) \left(\max_{1\leq i\leq n} \tau_i\right) \leq \frac{1}{e} \tag{19}$$

is a sufficient condition for the existence of a nonoscillatory solution of (3).

Finally, consider the delay differential equation with impulses

$$\begin{aligned} x'(t) + px(t-\tau) &= 0, & t &\neq t_k, \\ x(t_k^+) - x(t_k) &= bx(t_k), & k &= 1, 2, \dots, \end{aligned} \tag{20}$$

which is the special case of (3). Thus by the theorem, we obtain the following result.

COROLLARY 3. *Assume that*

$$p > 0, \quad \tau > 0, \quad b > -1, \quad t_0 > 0, \quad \text{and} \quad t_k = t_0 + k\tau, \qquad \text{for } k = 1, 2, \dots. \tag{21}$$

Then a necessary and sufficient condition for the existence of a nonoscillatory solution of (20) is

$$p\tau e \leq b + 1. \tag{22}$$

REFERENCES

1. O. Arino, I. Györi and A. Jawhari, Oscillation criteria in delay equation, *J. Diff. Equas.* **53**, 115–123 (1984).
2. I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991).
3. B.R. Hunt and J.A. Yorke, When all solutions of $x' = \sum_{i=1}^{n} q_i(t)x(t - \tau_i(t))$ oscillate, *J. Diff. Equas.* **53**, 139–145 (1984).
4. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, (1989).
5. M.-P. Chen, J.S. Yu and J.H. Shen, The persistence of nonoscillatory solutions of delay differential equations under impulsive perturbations, *Computers Math. Applic.* **27** (8), 1–6 (1994).
6. K. Gopalsamy and B.G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.* **139**, 110–122 (1989).