International Journal of Solids and Structures 49 (2012) 3255–3262

Contents lists available at SciVerse ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

On the fundamental equations of piezoelasticity of quasicrystal media

Gülay Altay [a], M. Cengiz Dökmeci [b,*]

[a] Faculty of Engineering, Boğaziçi University, Bebek, 34342 Istanbul, Turkey
[b] Istanbul Technical University, PTT- Gümüşsuyu, P.K.9, 34437 Istanbul, Turkey

ARTICLE INFO

Article history:
Received 17 October 2011
Received in revised form 2 April 2012
Available online 10 July 2012

Dedicated to late Günseli Güler Hanimefendi, with full respect to her nobility, her unique gift to science (A.M.Celal Sengör), and her motivation to support academicians around her

Keywords:
Quasicrystals
Piezoelastic quasicrystals
Variational principles
Uniqueness theorem

ABSTRACT

The three-dimensional fundamental equations of elasticity of quasicrystals with extension to quasi-static electric effect are expresses in both differential and variational invariant forms for a regular region of quasicrystal material. The principle of conservation of energy is stated for the regular region and the constitutive relations are obtained for the piezoelasticity of material. A theorem is proved for the uniqueness in solutions of the fundamental equations by means of the energy argument. The sufficient boundary and initial conditions are enumerated for the uniqueness. Hamilton's principle is stated for the regular region and a three-field variational principle is obtained under some constraint conditions. The constraint conditions, which are generally undesirable in computation, are removed by applying an involutory transformation. Then, a unified variational principle is obtained for the regular region, with one or more fixed internal surface of discontinuity. The variational principle operating on all the field variables generates all the fundamental equations of piezoelasticity of quasicrystals under the symmetry conditions of the phonon stress tensor and the initial conditions. The resulting equations, which are expressible in any system of coordinates and may be used through simultaneous approximation upon all the field variables in a direct method of solutions, pave the way to the study of important dislocation, fracture and interface problems of both elasticity and piezoelasticity of quasicrystal materials.

1. Introduction

Quasicrystals[1], another form of solid matter beside the crystalline and the amorphous, refer to a new class of functional and structural materials, which are metallic alloys with quasi-periodic translational symmetry and non-crystallographic rotational symmetry (i.e., quasiperiodicity). This type of materials was discovered by a seminal observation on non-crystallographic symmetry in an alloy by Dan Shechtman on April 8, 1982, and later was reported by Shechtman et al. (1984). Soon after this experimental paper, a purely theoretical paper was published by Levine and Steinhardt (1984) who dubbed the novel alloy as "quasicrystal". Recently, Hargittai (2010) presented an account of the importance of structures beyond crystals and the discovery and recognition of quasiperiodic materials. Over the past quarter of a century, one-, two- and three-dimensional quasicrystals with a high degree of structural features were observed and investigated intensively both experimentally and theoretically. The one (or two) -dimensional quasicrystals are the ones in which the arrangement of materials is quasiperiodic in one (or two) direction, while they are periodic in other two (or one) directions, and the three-dimensional quasicrystals present quasiperiodicity in all three directions. Quasicrystals are sensitive to mechanical, thermal, electrical and similar effects, and their linear and non-linear elastic and plastic properties, the thermal properties, and the electronic structure were extensively investigated among their important properties. The independent and non-vanishing first order piezoelectric, piezomagnetic, pyromagnetic, photoelastic and magnetoelectric coefficients were obtained (e.g., Yang et al., 1995; Hu et al., 1997; Rama Mohana Rao et al., 2007). The reader is referred, for instance, to the treatises (Fujiwara and Ishii, 2008; Suck et al., 2010) and a recent comprehensive book by Fan (2011) for the development of quasicrystals, including material properties, theories of elasticity and some applications.

Several researchers in continuum physics attempted to establish a mathematical modeling of elasticity of quasicrystals under mechanical effect soon after their discovery (e.g., Bak, 1985a,b; Levine et al., 1985; Lubensky et al., 1985; De and Pelcovits, 1987; Socolar et al., 1986; Yang et al., 1993; Ding et al., 1993; Peng and Fan, 2000; Hu et al., 2000; Shi, 2005; Fan et al., 2009; Chatzopoulos and Trebin, 2010). The phenomenological theory of Landau and Lifshitz (1958) on the elementary excitation of condensed matters was essentially taken as the physical basis of elasticity of quasicrystals, and two types of excitations, phonons and phasons, were considered for quasiperiodicity of materials. The phonon field is similar to that in crystals, and its gradient describes changes of a cell in volume and shape. The gradient of the phason field characterizes

* Corresponding author. Tel.: +90 212 2856415; fax: +90 212 2856454.
E-mail addresses: dokmeci@itu.edu.tr, cengiz.dokmeci@itu.edu.tr (M. Cengiz Dökmeci).

[1] Nobel prize for Chemistry 2011.

0020-7683/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijsolstr.2012.06.016

the local rearrangement of atoms in a cell, which is a new concept with two different arguments: Bak's (1985a,b) argument and the argument of Lubensky et al. (1985). According to the former, the phason field describes particular structural disorders or structure fluctuation in quasicrystals. The latter considers the phason field as diffusive with very large diffusive time (Fan and Mai, 2004). Levine et al. (1985) and Lubensky et al. (1985) studied the elasticity of quasicrystal materials and, they derived the equations governing the elasticity of three-dimensional (icosahedral) quasicrystals. Also, Yang et al., (1993) presented the linear elasticity theory of three-dimensional (cubic) quasicrystals. De and Pelcovits (1987) dealt with the elasticity of two-dimensional (pentagonal) quasicrystals, and Peng and Fan (2000) and Liu et al. (2004) with the elasticity of one-dimensional, hexagonal and orthorhombic, quasicrystals, respectively. A generalized theory applicable to the elasticity of all quasicrystals was established by Ding et al. (1993) on the basis of the laws of motion in continuum physics. The fundamental equations of quasicrystals were established in differential form, and the associated boundary value or initial-boundary value problems were well posed. Various static and dynamic problems were solved for the elasticity of one-, two- and three-dimensional quasicrystals (e.g., Liu et al.,2004; Gao et al., 2008; Gao, 2009, 2010; Fan, 2011, and references therein). Moreover, the uniqueness and existing of solutions were investigated in the elasticity equations of quasicrystals (Fan, 2011).

The fundamental equations of piezoelasticity of quasicrystals may be grouped, following Mindlin (1974) and Deresiewicz et al. (1989), as the divergence equations (i.e., Cauchy's equations of motion for the phonon and phason fields and Maxwell's equation for the quasi-static electric field), the gradient equations, the constitutive relations and the boundary and initial conditions to supplement the internal consistency (i.e., existence, uniqueness and stability) of solutions. The divergence equations were established in integral (global) form on the basis of the well-known axioms of continuum physics, and then, in differential form under certain regularity and local differentiability of the field variables. The constitutive relations were modeled in differential form under certain rules and requirements of mechanics. The gradient equations and the boundary and initial conditions were expressed in differential form. In addition to their differential form, some of the fundamental equations were expressed in variational form by introducing an energy functional for quasicrystals (Fan, 2011). The energy functional generates only the divergence equations and the associated stress boundary conditions, and has the rest of the fundamental equations as its constraint conditions. The constraint conditions are usually undesirable in computation since they prevent the free and simple choice of trial (coordinate, shape) functions in an approximate direct method of solutions. Nevertheless, a unified variational principle operating on all the phonon and phason field variables was stated to be unavailable in the open literature (Dökmeci and Altay, 2011). More importantly, the fundamental equations of elasticity are in need of further development in order to include some other (e.g., piezoelectric, pyroelectric and piezomagnetic) effects, which were reported in quasicrystals (e.g., Hu et al., 1997; Rama Mohana Rao et al., 2007; Suck et al., 2010).

In this work we develop the fundamental equations of piezoelasticity of quasicrystals, which are still lacking among the publications reported by the "ISI-Web of Science", as indicated in a recent technical report (Dökmeci and Altay, 2011). To begin with, the divergence equations of piezoelasticity are recorded for a regular region of quasicrystals in the next section. Also, the constitutive relations of quasicrystals are obtained by the principle of conservation of energy, the gradient equations and the boundary and initial conditions were given in Section 2. In Section 3, we examine the uniqueness in solutions of the fundamental equations and prove a theorem of uniqueness by means of the energy argument, and enumerate the boundary and initial conditions sufficient to the uniqueness. In Section 4, Hamilton's principle is expressed for a regular region of quasicrystals, and a three-field variational principle is obtained. The variational principle is modified through an involutory (Legendre, Friedrichs) transformation and a nine-field variational principle is derived for the region of quasicrystals. This variational principle operating on all the field variables has only Cauchy's second law of motion for the phonon stress field and the initial conditions as its constraint conditions. The variational principle is further developed for the regular region with one or more fixed internal surface of discontinuity. Some special cases, concluding remarks and further needs of research are indicated in the last section.

We note that standard tensor notation is freely used in a Euclidean 3-D space Ξ in the text. Accordingly, Einstein's summation convention is implied for all repeated subscripts or superscript Latin indices (1–3) and Greek indices (1,2), unless the indices are enclosed with parentheses. The θ^i – system in the space is identified with a fixed, right-handed system of general orthogonal coordinates. A comma and a semicolon stand for partial differentiation and covariant differentiation with respect to an indicated space coordinate θ^i, respectively, and a superposed dot for time differentiation. An asterisk is used to denote a prescribed initial or boundary quantity. A regular region of quasicrystals with its smooth boundary surface, closure and a fixed internal surface of discontinuity is denoted by $\Omega \cup \partial\Omega \cup S$, $\partial\Omega$, $\bar{\Omega}(=\Omega \cup \partial\Omega)$ and S, respectively. The unit vectors normal to the surfaces $\partial\Omega$ and S are denoted by n_i and v_i, the time interval by $T = [t_0, t_1)$ and the Cartesian product of the region and the time interval by $\Omega \times T$. Also, the bold-face brackets, $[\chi] = \chi_2 - \chi_1$, is adopted in order to indicate the jump of a quantity across the surface of discontinuity S. Further, $C_{\alpha\beta}$ refers to a class of functions with its derivatives of order up to and including α and β with respect to the space coordinate θ^i and time t. The letters (m), (p) and (e) are used to indicate a quantity belonging to the phonon field, the phason field and the quasi-static electric field, respectively.

2. Fundamental equations of piezoelasticity of quasicrystal materials

Now, we present a summary of the time- domain, fundamental three-dimensional equations of elasticity (Fan, 2011) with an extension to quasi-static electric field (Dökmeci and Altay, 2011) for a region of quasicrystal continua. The finite and bounded regular region of quasicrystals in Kellogg's, (1946) sense, $\Omega + \partial\Omega$ with its smooth boundary surface $\partial\Omega$, a fixed internal surface of discontinuity S and closure $\bar{\Omega}$ at time $t = t_0$ is referred to by a fixed system of orthogonal coordinates θ^i in the Euclidean 3-D space Ξ. The fundamental equations are stated in terms of the phonon, phason and quasi-static electric field quantities, excluding all polar, non-local, thermal and similar effects. The phonon field is analogous to the displacement of crystals and its gradient is symmetric. The phason field describes the local arrangement of quasicrystals, and its gradient is not symmetric. The total displacement field is the direct sum of the displacement fields. The fundamental equations are stated in differential form at the time interval $T[t_0, t_1)$, namely

2.1. Divergence equations

$$L_m^i = \sigma^{ji}_{;j} + f^i - \rho a^i = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.1}$$

$$L_i^m = \varepsilon_{ijk}\sigma^{jk} = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.2}$$

$$L_p^i = H^{ij}_{;j} + g^i - \rho b^i = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.3}$$

$$L_e = D^i_{;i} - \rho_e = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.4}$$

Here, the quasi-static electric field is taken to be independent of the mechanic field as in classical piezoelectricity. In these equations, $\sigma^{ij} \in C_{10}$, $u^i \in C_{12}$, $f^i \in C_{00}$ and $\rho \in C_{00}$ stand for the symmetric phonon stress tensor, the phonon displacement vector, the conventional body force vector per unit volume of continuum, and the mass density of quasicrystal, respectively, ε_{ijk} for the alternating tensor and $a^i(=\ddot{u}^i)$, and also, $H^{ij} \in C_{10}$, $g^i \in C_{00}$ and $w^i \in C_{12}$ for the asymmetric phason stress tensor, which is symmetric for cubic quasicrystals only, the generalized body force vector and the phason displacement vector, respectively, and $b^i(=\ddot{w}^i)$. Also, $D^i \in C_{10}$ denotes the electric displacement (i.e., electric flux density or electric induction) vector and $\rho_e \in C_{00}$ the electric charge density.

2.2. Gradient equations

$$L^m_{ij} = \varepsilon_{ij} - \frac{1}{2}(u_{i;j} + u_{j;i}) = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.5}$$

$$L^p_{ij} = w_{ij} - w_{ij} = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.6}$$

$$L^e_i = -(E_i + \phi_{,i}) = 0 \tag{2.7}$$

Here, $\varepsilon_{ij} \in C_{00}$, $w_{ij} \in C_{00}$, $E_i \in C_{00}$ and $\phi \in C_{10}$ denote the symmetric phonon deformation tensor (or the phonon strain tensor), the asymmetric phason deformation tensor (or the phason strain tensor), the electric field vector and the electric potential, respectively.

2.3. Constitutive relations of quasicrystal continua

We state the principle of conservation of energy in an adiabatic condition of the bounded regular region $\Omega + \partial\Omega$ in the form

$$\frac{\partial}{\partial t}\int_\Omega (k+u)dV = \int_{\partial\Omega}(T^i\dot{u}_i + h^i\dot{w}_i - n_i\phi\dot{D}^i)dS + \int_\Omega (f^i\dot{u}_i + g^i\dot{w}_i - \dot{\rho}_e\phi)dV \tag{2.8}$$

where u is the postulated internal energy density and k is the kinetic energy density, including the phonon and phason displacements, in the form

$$k = \frac{1}{2}\rho(\dot{u}^i\dot{u}_i + \dot{w}^i\dot{w}_i) \tag{2.9}$$

Here, the wavelengths near the elastic waves, which are much shorter than the electromagnetic wavelength of the same frequency, are considered, and hence, the electric kinetic energy is omitted. This assumption is certainly as accurate as necessary for a quasistatic electric field, and it is obviously well borne out experimentally (e.g., Tiersten, 1969; Nelson, 1978). Eq. (2.9) states that the rate of increase of total (kinetic plus internal) energy is equal to the rate at which the work done by the conventional traction vector $T^i(=n_j\sigma^{ji})$ and the generalized traction vector $h^i(=n_jH^{ij})$ where n_i is the unit outward vector normal to the boundary surface $\partial\Omega$, less the flux of electric energy outward across the surface $\partial\Omega$. In Eq. (2.8), the differentiation is performed with respect to time by simply permuting the integral and differential operations because of the fixed end points, the surface tractions is replaced by the components of the phonon and phason stress tensors and the rate of the kinetic energy by the components of the phonon and phason field vectors. Then, the divergence theorem is applied to the regular region with the result

$$\begin{aligned}\int_\Omega [\rho(\ddot{u}^i\dot{u}_i + \ddot{w}^i\dot{w}_i) + \dot{u}]\,dV = \int_\Omega \left[\left(\sigma^{ij}\dot{u}_j\right)_{;i} + \left(H^{ij}\dot{w}_j\right)_{;i} - \left(\phi\dot{D}^i\right)_{;i}\right]dV \\ + \int_\Omega [\rho(f^i\dot{u}_i + g^i\dot{w}_i) - \dot{\rho}_e\phi]dV\end{aligned} \tag{2.10}$$

Here, the conservation of mass (i.e., $\delta(\rho dV) = 0$) is considered. This equation is valid for any volume of continuum, and hence it may be expressed by

$$\begin{aligned}\dot{u} = \left[\sigma^{ij}_{;i} + \rho(f^j - a^j)\right]\dot{u}_j + \sigma^{ij}\dot{u}_{j;i} + \left[H^{ij}_{;i} + \rho(g^j - b^j)\right]\dot{w}_j \\ + H^{ij}\dot{w}_{j;i} - \phi\left(\dot{D}^i_{;i} - \dot{\rho}_e\right) - \dot{D}^i\phi_{,i}\end{aligned} \tag{2.11}$$

which is reduced to

$$\dot{u} = \sigma^{ij}\dot{u}_{j;i} + H^{ij}\dot{w}_{j;i} - \phi_{,i}\dot{D}^i \tag{2.12}$$

after considering the divergence equations (2.1)–(2.4). Replacing the gradient of the electric potential by the electric field and considering the symmetry of the phonon tensor in Eq. (2.2) together with Eqs. (2.5) and (2.6), one finally obtains

$$\dot{u} = \sigma^{ij}\dot{\varepsilon}_{ij} + H^{ij}\dot{w}_{ij} + E_i\dot{D}^i \tag{2.13}$$

This relation represents the first law of thermodynamics for quasicrystal continua.

Now, we define the piezooelastic enthalpy density H by

$$H = u - E_iD^i \tag{2.14}$$

and after differentiating this equation with respect to time, we obtain

$$\dot{H} = \dot{u} - E_i\dot{D}^i - \dot{E}_iD^i \tag{2.15}$$

The rate of internal energy from Eq. (2.13) is replaced into Eq. (2.15) with the result

$$\dot{H} = \sigma^{ij}\dot{\varepsilon}_{ij} + H^{ij}\dot{w}_{ij} - \dot{E}_iD^i \tag{2.16}$$

This equation evidently implies that the piezoelastic enthalpy density is a function of the phonon and phason strain tensors and the electric field vector in the form

$$H = H(\varepsilon_{ij}, w_{ij}, E_i). \tag{2.17}$$

The time differentiation of Eq. (2.17) is given by

$$\dot{H} = \frac{\partial H}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial H}{\partial w_{ij}}\dot{w}_{ij} + \frac{\partial H}{\partial E_i}\dot{E}_i \tag{2.18}$$

and combining this equation with Eq. (2.16), we have

$$\left(\sigma^{ij} - \frac{\partial H}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} + \left(H^{ij} - \frac{\partial H}{\partial w_{ij}}\right)\dot{w}_{ij} - \left(D^i + \frac{\partial H}{\partial E_i}\right)\dot{E}_i = 0 \tag{2.19}$$

which holds for arbitrary quantities, $(\dot{\varepsilon}_{ij}, \dot{w}_{ij}, \dot{E}_i)$, and leads to the constitutive relations of the form

$$L^{ij}_{mc} = \sigma^{ij} - \frac{\partial H}{\partial \varepsilon_{ij}} = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.20}$$

$$L^{ij}_{pc} = H^{ij} - \frac{\partial H}{\partial w_{ij}} = 0 \quad \text{in } \bar{\Omega} \times T \tag{2.21}$$

and

$$L^i_{ec} = -\left(D^i + \frac{\partial H}{\partial E_i}\right) = 0 \quad \text{in } \bar{\Omega} \times T. \tag{2.22}$$

Also, in a reversible process, the constitutive relations can be expressed by a Legendre transformation $W\,(\sigma^{ij}, H^{ij}, D^i)$ of the piezoelastic enthalphy density $H\,(\varepsilon_{ij}, w_{ij}, E_i)$, that is, the complementary enthalpy density W is defined by

$$W = \left(\sigma^{ij}\varepsilon_{ij} + H^{ij}w_{ij} + D^iE_i\right) - H(\varepsilon_{ij}, w_{ij}, E_i) \tag{2.23}$$

The time derivative of the complementary enthalpy density is written as

$$\dot{W} = \frac{\partial W}{\partial \sigma^{ij}}\dot{\sigma}^{ij} + \frac{\partial W}{\partial H^{ij}}\dot{H}^{ij} + \frac{\partial W}{\partial D^i}\dot{D}^i \quad (2.24)$$

which gives the alternative constitutive relations for a case when the Hessian of the piezoeleastic enthalpy density function H does not vanish as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \sigma^{ij}} + \frac{\partial W}{\partial \sigma^{ji}}\right), \quad w_{ij} = \frac{\partial W}{\partial H^{ij}}, \quad E_i = -\frac{\partial W}{\partial D^i} \quad \text{in } \bar{\Omega}\times T \quad (2.25)$$

Likewise, the constitutive relations of quasicrystals can be obtained for the thermoelasticity and electromagnetoelasticity of quasicrystals (Mindlin, 1967, 1968; Altay and Dökmeci, 2010; Dökmeci and Altay, 2011).

Now, we take a Taylor expansion of the piezoelastic enthalphy function H in the neighbourhood of $(\varepsilon_{ij} = 0,\ w_{ij} = 0,\ E_i = 0)$ as

$$H = \frac{1}{2}\left(C^{ijkl}\varepsilon_{ij}\varepsilon_{kl} + K^{ijkl}w_{ij}w_{kl} - e^{ij}E_iE_j + R^{ijkl}\varepsilon_{ij}w_{kl} + R^{klij}w_{ij}\varepsilon_{kl}\right) - e^{ijk}E_i\varepsilon_{jk} - \varepsilon^{ijk}E_iw_{jk} \quad (2.26)$$

Here, the expansion is truncated in quadratic terms, since the symmetry relations of higher order terms are complicated due to various non-crystallographic symmetries of the quasicrystal materials, as indicated by Shi (2005, 2007). Moreover, Hu et al. (1997) showed that the piezoelectric effect may be due to either one of the phonon and phason fields or both of the two fields.

The density function is inserted into Eqs. (2.20)–(2.22) and the linear constitutive relations are obtained as

$$L^{ij}_{mc} = \sigma^{ij} - \left(C^{ijkl}\varepsilon_{kl} + R^{ijkl}w_{kl} - e^{ijk}E_k\right) = 0 \quad \text{in } \bar{\Omega}\times T \quad (2.27)$$

$$L^{i}_{pc} = H^{ij} - \left(K^{ijkl}w_{kl} + R^{klij}\varepsilon_{kl} - \varepsilon^{ijk}E_k\right) = 0 \quad \text{in } \bar{\Omega}\times T \quad (2.28)$$

$$L^{i}_{ec} = D^{i} - \left(e^{ij}E_j + e^{ijk}\varepsilon_{jk} + \varepsilon^{ijk}w_{jk}\right) = 0 \quad \text{in } \bar{\Omega}\times T \quad (2.29)$$

where, C^{ijkl}, K^{ijkl}, R^{ijkl}, e^{ijk} and ε^{ijk} stand for the phonon elastic, phason elastic, phonon-phason coupling moduli, and e^{ij} for the dielectric permittivity, respectively, and they satisfy the usual symmetry conditions of the form

$$C^{ijkl} = C^{jikl} = C^{ijlk} = C^{klij}, \quad R^{ijkl} = R^{jikl}, \quad K^{ijkl} = K^{klij},$$
$$e^{ijk} = e^{ikj}, \quad \varepsilon^{ijk} = \varepsilon^{ikj}, \quad e^{ij} = e^{ji} \quad \text{in } \bar{\Omega}\times T \quad (2.30)$$

and the positive-semidefinite conditions, namely

$$e^{ij}\eta_i\eta_j \geqslant 0, \quad C^{ijkl}\eta_{ij}\eta_{kl} \geqslant 0, \quad K^{ijkl}\eta_{ij}\eta_{kl} \geqslant 0 \quad \text{in } \bar{\Omega}\times T \quad (2.31)$$

for non-zero vector η_i and non-zero tensors η_{ij} as well.

2.4. Boundary and initial conditions

$$L^{j}_{*m\sigma} = T^{j}_{*} - n_i\sigma^{ij} = 0 \quad \text{on } \partial\Omega_t\times T, \quad L^{*mu}_{i} = u_i - u^{*}_{i} = 0 \quad \text{on } \partial\Omega_u\times T \quad (2.32)$$

$$L^{j}_{*ph} = h^{j}_{*} - n_iH^{ji} = 0 \quad \text{on } \partial\Omega_h\times T, \quad L^{*ph}_{i} = w_i - w^{*}_{i} = 0 \quad \text{on } \partial\Omega_w\times T \quad (2.33)$$

$$L^{*}_{ed} = D_{*} - n_iD^{i} = 0 \quad \text{on } \partial\Omega_d\times T, \quad L^{*}_{e\phi} = \phi - \phi_{*} = 0 \quad \text{on } \partial\Omega_\phi\times T \quad (2.34)$$

and

$$L^{m*}_{i} = u_i\left(\theta^j, t_0\right) - \nu^{*}_{i}(\theta^j) = 0, \quad L^{m}_{*i} = \dot{u}_i\left(\theta^j, t_o\right) - \alpha^{*}_{i}(\theta^j) = 0 \quad \text{in } \Omega(t_o) \quad (2.35)$$

$$L^{p*}_{i} = w_i\left(\theta^j, t_0\right) - \beta^{*}_{i}(\theta^j) = 0, \quad L^{p}_{*i} = \dot{w}_i(\theta^j, t_o) - \gamma^{p*}_{i}(\theta^j) = 0 \quad \text{in } \Omega(t_o) \quad (2.36)$$

$$L^{*}_{e} = \phi\left(\theta^i, t_o\right) - \alpha_{*}(\theta^i) = 0 \quad \text{in } \Omega(t_o) \quad (2.37)$$

where an asterisk indicates the prescribed quantities.

The fundamental equations, (2.1)–(2.7), (2.20)–(2.22), (2.27)–(2.29), comprise the forty-three equations governing the forty-three dependent variables, that is, the phonon field $\Lambda_m = \{u_i, \varepsilon_{ij}, \sigma^{ij}\}$, the phason field $\Lambda_p = \{w_i, w_{ij}, H^{ij}\}$ and the quasi-static electric field $\Lambda = \{\phi, E_i, D^i\}$, and hence, they are deterministic and define an initial and mixed boundary value problem for the piezoelasticity of a quasicrystal continuum.

2.5. Discontinuity in the field variables

Now, consider the regular region $\Omega + \partial\Omega$ with a fixed internal surface of discontinuity S, which splits the region into the subregions $\Omega_\alpha + \partial\Omega_\alpha + S$ with their boundary surface $\partial\Omega_\alpha + S$. The sub-regions are perfectly bonded, and hence, all the field variables undergo a jump (interface) across the internal surface S as follows

$$L^{j}_{dm} = \nu_i[\sigma^{ij}] = 0, \quad L^{j}_{dp} = \nu_i[H^{ij}] = 0, \quad L_{ed} = \nu_i[D^i] = 0 \quad \text{on } S\times T \quad (2.38)$$

$$L^{dm}_{i} = [u_i] = 0, \quad L^{dp}_{i} = [w_i] = 0, \quad L_{de} = [\phi] = 0 \quad \text{on } S\times T \quad (2.39)$$

where ν_i is the unit outward vector normal to S and pointing to Ω_2, and it is assumed that the subregions have different quasicrystal materials, and the bold-face brackets, $[\chi] = \chi_2 - \chi_1$ is adopted in order to indicate the jump of a quantity across a surface of discontinuity S.

3. Uniqueness of solutions

Now, we investigate the uniqueness in solutions of the fundamental equations of piezoelasticity of quasicrystals given in the previous section. The fundamental equations have a unique e solution under the boundary and initial conditions (2.32)–(2.37). To prove the uniqueness, we consider the possibility of two distinct sets of admissible solutions, namely,

$$\Lambda^{(\alpha)} = \Big(\sigma^{ij} \in C_{10}, u_i \in C_{12}, \varepsilon_{ij} \in C_{00}; H^{ij} \in C_{10}, w_{ij} \in C_{00}, w_i \in C_{12}; D^i \in C_{10}, E_i \in C_{00}, \phi \in C_{11}\Big)^{(\alpha)} \quad (3.1a)$$

The two sets of solutions, together with their derivatives are assumed to exist and to be continuous functions of the coordinate θ^i and time t in $\bar{\Omega}\times T$, and each of the solutions, which is initially zero, independently satisfies the linear fundamental equations. We denote the difference set of solutions in the form

$$\Lambda = \Lambda^{(2)} - \Lambda^{(1)} \quad (3.1b)$$

The difference set of solutions evidently satisfies the homogeneous fundamental equations due to their linearity, that is, the homogeneous divergence equations (2.1)–(2.4) by

$$L^{j}_{m} = \sigma^{ij}_{;i} - \rho a^j = 0, \quad L_i = \varepsilon_{ijk}\sigma^{jk} = 0 \quad \text{in } \bar{\Omega}\times T \quad (3.2)$$

$$L^{j}_{p} = H^{ji}_{;i} - \rho b^j = 0 \quad \text{in } \bar{\Omega}\times T \quad (3.3)$$

$$L_e = D^{i}_{;i} = 0 \quad \text{in } \bar{\Omega}\times T \quad (3.4)$$

the gradient equations (2.5)–(2.7), the constitutive relations (2.27)–(2.29), and also, the homogeneous boundary conditions (2.32)–(2.34) as

$$L^{j}_{*m\sigma} = n_i\sigma^{ij} = 0 \quad \text{on } \partial\Omega_t\times T, \quad L^{*mu}_{i} = u_i = 0 \quad \text{on } \partial\Omega_u\times T \quad (3.5)$$

$$L^{j}_{*ph} = n_iH^{ij} = 0 \quad \text{on } \partial\Omega_h\times T, \quad L^{*pw}_{i} = w_i = 0 \quad \text{on } \partial\Omega_w\times T \quad (3.6)$$

$$L^*_{ed} = n_i D^i = 0 \quad \text{on } \partial\Omega_d \times T, \quad L^*_{e\phi} = \phi = 0 \quad \text{on } \partial\Omega_\phi \times T \tag{3.7}$$

Also, the homogeneous initial conditions (2.35)–(2.37) as

$$L^{m*}_i = u_i(\theta^i, t_o) = 0, \quad L^m_{*i} = \dot{u}_i\left(\theta^j, t_0\right) = 0 \tag{3.8}$$

$$L^{p*}_i = w_i(\theta^i, t_o) = 0, \quad L^p_{*i} = \dot{w}_i\left(\theta^j, t_0\right) = 0 \tag{3.9}$$

$$L^*_e = \phi\left(\theta^i, t_o\right) = 0 \tag{3.10}$$

are satisfied by the difference set of solutions.

With the help of the divergence equations (3.3)–(3.5), we form an integral by non-zero field variables as follows

$$I = \int_T dt \int_\Omega \left[\left(\sigma^{ij}_{;i} - \rho\ddot{u}^j\right)\dot{u}_j + \left(H^{ij}_{;i} - \rho\ddot{w}^j\right)\dot{w}_j - \dot{D}^i_{;i}\phi\right] dV = 0 \tag{3.11}$$

which may be written as

$$I = \int_T dt \int_\Omega \left[-\sigma^{ij}\dot{u}_{j;i} - H^{ij}\dot{w}_{j;i} + \dot{D}^i\phi_{,i} + \left(\sigma^{ij}\dot{u}_j + H^{ij}\dot{w}_j - \dot{D}^i\phi\right)_{;i} - \rho\left(\ddot{u}^i\dot{u}_i + \ddot{w}^i\dot{w}_i\right)\right] dV = 0 \tag{3.12}$$

We apply the divergence theorem to the regular region $\Omega + \partial\Omega$ and obtain the relation as

$$I = I_\Omega - I_{\partial\Omega} = 0 \tag{3.13}$$

with the denotations of the form

$$I_\Omega = \int_T dt \int_\Omega \left(\sigma^{ij}\dot{\varepsilon}_{ij} + H^{ij}\dot{w}_{ij} + E_i\dot{D}^i + \dot{k}\right) dV \tag{3.14}$$

$$I_{\partial\Omega} = \int_T dt \int_{\partial\Omega} n_i\left(\sigma^{ij}\dot{u}_j + H^{ij}\dot{w}_j - \dot{D}^i\phi\right) dS \tag{3.15}$$

Here, the rate of the kinetic energy form Eq. (2.9) and Eqs. (2.5)–(2.7) for the difference system is considered. Inserting Eq. (2.12) into Eq. (3.16), one obtains

$$I_{\partial\Omega} = \int_T dt \int_{\partial\Omega} n_i\left(\sigma^{ij}\dot{u}_j + H^{ij}w_j + D^i\dot{\phi}\right) dS - \int_{\partial\Omega} n_i D^i \phi\Big|_T dS \tag{3.16}$$

The boundary and initial conditions (2.32)–(2.37) evidently render the integrands in Eq. (3.17) to zero, and hence, one writes

$$I = I_\Omega = \int_T dt \int_\Omega (\dot{u} + \dot{k}) dV = \int_T (\dot{U} + \dot{K}) dt = 0 \tag{3.17}$$

which, after integration, takes the form

$$I = U(t_1) + K(t_1) - U(t_0) - K(t_0) = E(t_1) - E(t_0) = 0 \tag{3.18}$$

Here, U, K and E are the internal, kinetic and total energy of the quasicrystal region with no singularities of any type, and they are calculated from the densities k and u by integration. The densities are positive definite, by definition, and initially zero, as are the total energy E for the difference set of solutions, namely

$$E(t_1) = E(t_0) = 0 \tag{3.19}$$

This clearly indicates a trivial solution for the difference set of solutions as

$$\Lambda = \Lambda^{(2)} - \Lambda^{(1)} = 0; \quad \Lambda^{(2)} = \Lambda^{(1)} \tag{3.20}$$

and the uniqueness is assured in solutions of the fundamental equations of the continuum. A theorem which enumerates the conditions sufficient to the uniqueness in solutions is stated as follows.

Theorem – Consider the regular, finite and bounded region $\Omega + \partial\Omega$ of a quasicrystal continuum, with a piecewise smooth boundary surface $\partial\Omega$, closure $\bar{\Omega}$ at the time interval T. The region is referred to by a fixed right-handed system of orthogonal coordinates θ^i in the three-dimensional Euclidean space Ξ, and it is set in motion by an application of assigned surface tractions and charge and prescribed displacements and electric potential. Let

$$\Lambda = \left(\sigma^{ij} \in C_{10}, u_i \in C_{12}, \varepsilon_{ij} \in C_{00}; H^{ij} \in C_{10}, w_{ij} \in C_{00}, w_i \in C_{12}; D^i \in C_{10}, E_i \in C_{00}, \phi \in C_{11}\right) \tag{3.21}$$

be an admissible state of single-valued continuous functions of θ^i and t that satisfies the divergence equations (2.1)–(2.4), the gradient equations (2.5)–(2.7), the constitutive relations (2.27)–(2.29), the boundary conditions (2.32)–(2.34), and the initial conditions (2.35)–(2.37). Also, they satisfy the symmetry of the phonon stress tensor (2.2) and the material coefficients (2.30) and (2.31) as well as the usual existence and continuity conditions of the field variables in the region. Then, if the admissible state Λ exits, it is unique, apart from a rigid-body translation, provided that the kinetic and potential energies and the mass density are positive definite.

4. Variational principles of piezoelasticity of quasicrystal continua

In this section we express the fundamental equations of piezoelasticity of quasicrystals in variational form as an alternative to their differential form given in Section 2. With their well-known features, some variational principles are formulated for a regular region of piezoelectric quasicrystals without and with one or more fixed internal surface of discontinuity. As a point of departure, we start with a general principle of physics due to Hamilton (1834, 1835) who presented it for the dynamics of a discrete mechanical system. The principle was originally deduced from D'Alembert's principle by means of integration over time, and later, was extended by Kirchhoff (1876) to a continuous media. The application of the principle to a finite region of continua always leads to a variational principle, which is either an integral type in case of conservative forces or a differential type otherwise. A Hamiltonian's type of variational principles generates the divergence equations and the associated stress boundary conditions only. The rest of the fundamental equations always remain as its constraint conditions.

We now express Hamilton's principle (e.g., Lanczos, 1986) which states that the action integral is stationary between two instants of time t_o and t_1, for the regular region $\Omega + \partial\Omega$ of quasicrystals at the time interval T, namely

$$\delta L_H\{u_i, w_i\} = -\delta \int_T L dt + \int_T \delta F dt + \int_T \delta^* W dt = 0 \tag{4.1}$$

Here, L is the Lagrangian function and $\delta^* W$ is the virtual work done by the external mechanical and electrical forces as

$$\delta L = \delta \int_\Omega \left[H(\varepsilon_{ij}, w_{ij}, E_i) - k\right] dV \tag{4.2}$$

$$\delta^* W = \int_{\partial\Omega} \left(T^i_* \delta u_i + h^i_* \delta w_i + D_* \delta\varphi\right) dS \tag{4.3}$$

$$\delta F = \int_\Omega \left(f^i \delta u_i + g^i \delta w_i - \rho_e \delta\phi\right) dS \tag{4.4}$$

An asterisk is placed upon δ^* in the above equations so as to distinguish it from the variation operator δ. We insert Eqs. (4.2)–(4.4) into Eq. (4.1), execute the variations and integrate the kinetic energy term with respect to time t, and then we finally arrive at the equation of the form

$$\delta L_H = \int_T dt \int_\Omega \left[-\rho(\ddot{u}^i\delta u_i + \ddot{w}^i\delta w_i) - \left(\frac{\partial H}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial H}{\partial w_{ij}}\delta w_{ij} + \frac{\partial H}{\partial E_i}\delta E_i \right) \right] dV + \int_\Omega \rho(\dot{u}^i\delta u_i + \dot{w}^i\delta w_i)|_T dV + \int_T \delta F dt + \int_T \delta^* W dt = 0 \tag{4.5}$$

In this equation and henceforth, the variation and integration operations are permuted for the time and volume integrals with fixed end points, and the principle of conservation of mass is considered [i.e., $\delta(\rho dV) = 0$]. The gradient equations (2.5)–(2.7) and the constitutive relations (2.20)–(2.22) are inserted into Eq. (4.5), and the symmetry condition of the phonon stress tensor (2.2) is used with the result

$$\delta L_H = \int_T dt \int_\Omega \left[-\rho\left(\ddot{u}^i\delta u_i + \ddot{w}^i\delta w_i - \sigma^{ij}\delta u_{i;j} - H^{ij}\delta w_{i;j} - D^i\delta\varphi_{,i} \right) \right] dV + \int_T \delta F dt + \int_T \delta^* W dt = 0 \tag{4.6}$$

Here, the constraint conditions of the form

$$\delta u_i = \delta w_i = 0 \quad \text{in } \Omega(t_0) \,\&\, \Omega(t_1) \tag{4.7}$$

are imposed, which is customary in the use of Hamilton's principle. The variational integral (4.6) is expressed as

$$\delta L_H = \int_T dt \int_\Omega \Big[-\rho(\ddot{u}^i\delta u_i + \ddot{w}^i\delta w_i) - (\sigma^{ij}\delta u_j)_{;i} + \sigma^{ij}_{;i}u_j - \left(H^{ij}\delta w_i\right)_{;j} + H^{ij}_{;j}\delta w_i - \left(D^i\delta\phi\right)_{;i} + \left(D^i_{;i}\delta\phi\right) \Big] dV + \int_T \delta F dt + \int_T \delta^* W dt = 0 \tag{4.8}$$

Applying the divergence theorem to the regular region $\Omega + \partial\Omega$ in Eq. (4.8), and combining the terms in the surface integral, one finally obtains a three-field variational principle as

$$\delta L_H\{\Lambda_H\} = \int_T dt \int_\Omega \Big[\left(\sigma^{ij}_{;i} + f^j - \rho\ddot{u}^j\right)\delta u_j + \left(H^{ij}_{;j} + g^i - \rho\ddot{w}^i\right)\delta w_i + \left(D^i_{;i} - \rho_e\right)\delta\phi \Big] dV + \int_T dt \int_{\partial\Omega} \Big[\left(T^i_* - n_j\sigma^{ji}\right)\delta u_i + \left(h^i_* - n_jH^{ji}\right)\delta w_i + \left(D_* - n_iD^i\right) \Big] dS \tag{4.9}$$

in a compact form by

$$\delta L_H\{\Lambda_H\} = \int_T dt \int_\Omega \left(L^i_m\delta u_i + L^i_p\delta w_i + L_e\delta\varphi \right) dV + \int_T dt \int_{\partial\Omega} \left(L^i_{*m\sigma}\delta u_i + L^i_{*ph}\delta w_i + L^*_{ed}\delta\phi \right) dS = 0 \tag{4.10}$$

with their admissible state by

$$\Lambda_H = \{u_i, w_i, \phi\} \tag{4.11}$$

in terms of the denotations defined in Eqs. (2.1)–(2.4) and Eqs. (2.32)–(2.34).

The variational principle (4.9) and (4.10) leads to the divergence equations and the natural boundary conditions, as its Euler–Lagrange equations. The rest of the fundamental equations remain as its constraint (subsidiary) conditions. This principle (4.10) is readily expressed in the form

$$\delta L_G\left\{\Lambda_H^{(1)} \cup \Lambda_H^{(2)}\right\} = \sum_{\alpha=1}^{2} \delta L_H^{(\alpha)} = 0 \tag{4.12}$$

with its admissible state as

$$\Lambda_H^{(\alpha)} = \{u_i, w_i, \phi\}^{(\alpha)} \tag{4.13}$$

for the regular region $\Omega + \partial\Omega$ with a fixed internal surface of discontinuity S. The region is split by the surface S into the subregions $\Omega_\alpha + \partial\Omega_\alpha + S$ with their boundary surface $\partial\Omega_\alpha + S$. Each subregion has different quasicrystal material and both the subregions are perfectly bonded at their interfaces S. The field variables undergo a jump across the internal surface S as given by Eqs. (2.38) and (2.39). The three-field variational principle (4.12) generates the divergence equations and the associated stress boundary conditions as in Eq. (4.10), and it has the constraint conditions of Eq.(4.10) and the interface conditions as well.

The constraint conditions are generally undesirable in computation. The variational principles with as few constraint conditions as possible, and especially those operating on all the field variables are of special importance. Thus, the trial (or shape, coordinate) functions can be chosen simply and freely in a direct method of approximate solutions. Some of the constraint conditions of the Hamiltonian variational principles (4.9), (4.10), and (4.12) can be removed. To remove the constraint conditions, we apply an involutory transformation and introduce the dislocation potentials (e.g., Fraeijs De Veubeke, 1973; Dökmeci, 2010). Thus, each constraint as a zero times a Lagrange multiplier is introduced in the form

$$\Delta_1^1 = \int_T dt \int_\Omega \left(\lambda^{ij}_m L^m_{ij} + \lambda^{ij}_p L^p_{ij} + \lambda^i_e L^e_i \right) dV$$

$$\Lambda_2^2 = \int_T dt \left[\int_{\partial\Omega_u} \lambda^i_m L^{*mu}_i dS + \int_{\partial\Omega_w} \lambda^i_p L^{*pw}_i dS + \int_{\partial\Omega_\phi} \lambda_e L^*_{e\phi} dS \right]$$

$$\Delta_d = \int_T dt \int_S \left(\lambda^i_{dm} L^{dm}_i + \lambda^i_{dp} L^{dp}_i + \lambda_{de} L_{de} \right) dS \tag{4.14}$$

Here, the denotations of Eqs. (2.5)–(2.7), (2.32)–(2.34), and (2.39) are used. Also, the Lagrange parameters, $(\lambda^{ij}_m, \lambda^i_m, \lambda^i_{dm})$, $(\lambda^{ij}_p, \lambda^i_p, \lambda^i_{dp})$ and $(\lambda^i_e, \lambda_e, \lambda_{de})$, are defined. The dislocation potentials are added into the variational integral (4.12) as

$$\delta L_G = \sum_{\alpha=1}^{2} \left(\delta L_H + \delta\Delta^\beta_\beta \right)^\alpha + \delta\Delta_d = 0 \tag{4.15}$$

for each of the subregions.

As before, taking the variations, considering the Lagrange multipliers as independent variables, integrating by parts with respect to time, and using the generalized version of Green's theorem of the form

$$\int_{\Omega - S} \chi^i_{;i} dV = \oint_{\partial\Omega} n_i\chi^i dS - \int_S v_i[\chi^i] dS \tag{4.16}$$

we arrive at the variational equation of the form

$$\delta L_G = \sum_{\alpha=1}^{2} \Big\{ \delta L_m\left[(u_i, \varepsilon_{ij}, \lambda^{ij}_m, \lambda^i_m) \right] + \delta L_p\left[(w_i, w_{ij}, \lambda^{ij}_p, \lambda^i_p) \right] + \delta L_e[(\phi, E_i, \lambda^e_i, \lambda_e)] \Big\}^{(\alpha)} + \delta L_s\left\{ u_i, \lambda^i_{md}; w_i, \lambda^i_{pd}; \phi, \lambda_{ed} \right\} = 0 \tag{4.17}$$

Here, the denotations of the form

$$\delta L^\alpha_m = \Big\{ \int_T dt \int_\Omega \left[-\frac{\partial H}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} + \delta\lambda^{ij}_m L^m_{ij} + \lambda^{ij}_m\delta\varepsilon_{ij} + \lambda^{ij}_{m;i}\delta u_j + \rho(f^i - a^i)\delta u_i \right] dV + \int_T dt \int_{\partial\Omega_u} \left(\delta\lambda^i_m L^{*mu}_i + \lambda^i_m\delta u_i \right) dS + \int_T dt \int_{\partial\Omega_t} \left(T^i_* - n_j\lambda^{ji}_m \right)\delta u_i dS \Big\}^{(\alpha)} \tag{4.18}$$

$$\delta L^\alpha_p = \Big\{ \int_T dt \int_\Omega \Big[-\frac{\partial H}{\partial w_{ij}}\delta w_{ij} + \delta\lambda^{ij}_p L^p_{ij} + \lambda^{ij}_p\delta w_{ij} + \lambda^{ij}_{p;j}\delta w_i + \rho(g^i - b^i)\delta w_i \Big] dV + \int_T dt \int_{\partial\Omega_w} \left(\delta\lambda^i_p L^{*pw}_i + \lambda^i_p\delta w_i \right) dS + \int_T dt \int_{\partial\Omega_h} \left(h^i_* - n_j\lambda^{ji}_p \right)\delta w_i dS \Big\}^{(\alpha)} \tag{4.19}$$

$$\delta L^\alpha_e = \Big\{ \int_T dt \int_\Omega \left[-\frac{\partial H}{\partial E_i}\delta E_i + \delta\lambda^i_e L^e_i + \lambda^i_e\delta E_i - \lambda^i_{e;i}\delta\phi + \rho_e\delta\phi \right] dV + \int_T dt \int_{\partial\Omega_\phi} \left(\delta\lambda_e L^*_{e\phi} + \lambda_e\delta\phi \right) dS + \int_T dt \int_{\partial\Omega_d} \left(D_* - n_i\lambda^i_e \right)\delta\phi dS \Big\}^{(\alpha)} \tag{4.20}$$

and

$$\delta L_s = \int_T dt \int_S \left\{ \delta\lambda^i_{md}[u_i] + \delta\lambda^i_{pd}[w_i] + \delta\lambda_{ed}[\phi] \right\} + \int_T dt \int_S \sum_{\alpha=1}^{2} + \left\{ \left(\lambda^j_{md} + v_i\lambda^{ij}_{m(\alpha)}\right)\delta u^{(\alpha)}_j + \left(\lambda^i_{pd} + v_i\lambda^{ij}_{p(\alpha)}\right)\delta w^{(\alpha)}_j + \left(\lambda_{ed} - v_i\lambda^i_{e(\alpha)}\right)\delta\phi^{(\alpha)} \right\}(-1)^{(\alpha)}. \quad (4.21)$$

are defined. Substituting Eqs. (4.18)–(4.21) into Eq. (4.17), the Lagrangian multipliers are identified by

$$\lambda^{ij}_m = \frac{\partial H}{\partial \varepsilon_{ij}} = \sigma^{ij}, \quad \lambda^i_m = n_j\lambda^{ji}_m = n_j\sigma^{ji} = T^i, \quad \lambda^j_{md} = -v_i\lambda^{ij}_{m(\alpha)} = -v_i < \sigma^{ij} >$$

$$\lambda^{ij}_p = \frac{\partial H}{\partial w_{ij}} = H^{ij}, \quad \lambda^i_m = n_j\lambda^{ji}_p = n_j H^{ji} = h^i, \quad \lambda^j_{pd} = -v_i\lambda^{ij}_{p(\alpha)} = -v_i < H^{ij} > \quad (4.22)$$

and

$$\lambda^i_e = \frac{\partial H}{\partial E_i} = -D^i, \quad \lambda_e = -n_i\lambda^i_e = n_i D^i, \quad \lambda_{ed} = -v_i\lambda^i_{e(\alpha)} = -v_i < D^i > \quad (4.23)$$

since the volumetric variations are arbitrary in the volume and the surface variations on the boundary and interface surfaces. The Lagrange multipliers in Eq. (4.23) are inserted in Eqs. (4.18)–(4.21) and then, into Eq. (4.17) to obtain a unified variational principled of the form

$$\delta L_A\{\Lambda_A\} = \sum_{\alpha=1}^{2}\left(\delta L_m\{\Lambda_m\} + \delta L_p\{\Lambda_p\} + \delta L_e\{\Lambda_e\}\right)^{(\alpha)} + \delta\Delta_D = 0 \quad (4.24)$$

Here, we define

$$\Delta_D = \int_T dt \int_S v_i\Big\{[\sigma^{ij}] < \delta u_j > - < \delta\sigma^{ij} > [u_j] + [H^{ij}] < \delta w_i > - < \delta H^{ij} > [w_i] + [D^i] < \delta\phi > - < \delta D^i > [\phi]\Big\} dS = 0 \quad (4.25)$$

with its admissible state by

$$\Lambda_A = \Lambda_m \cup \Lambda_p \cup \Lambda_e; \quad \Lambda_m = \{u_i, \varepsilon_{ij}, \sigma^{ij}\},$$
$$\Lambda_p = \{w_i, w_{ij}, H^{ij}\}, \quad \Lambda_e = \{\phi, E_i, D^i\} \quad (4.26)$$

Also, the phonon, phason and electric quantities of the form fl

$$\delta L_m\{\Lambda_m\} = \int_T dt \int_\Omega \left\{(L^i_m\delta u_i + L^m_{ij}\delta\sigma^{ij} + L^{ij}_{mc}\varepsilon_{ij}\right\} dV + \int_T dt \int_{\partial\Omega_\sigma} L^i_{*m\sigma}\delta u_i dS + \int_T dt \int_{\partial\Omega_{mu}} L^{*mu}_i n_j\delta\sigma^{ji} dS$$

$$\delta L_p\{\Lambda_p\} = \int_T dt \int_\Omega \left\{(L^i_p\delta w_i + L^p_{ij}\delta H^{ij} + L^{ij}_{pc}w_{ij}\right\} dV + \int_T dt \int_{\partial\Omega_h} L^i_{*ph}\delta w_i dS + \int_T dt \int_{\partial\Omega_{pw}} L^{*pw}_i n_j\delta H^{ji} dS$$

$$\delta L_e\{\Lambda_e\} = \int_T dt \int_\Omega \left\{(L_e\delta\phi + L^e_i\delta D^i + L^i_{ec}E_i\right\} dV + \int_T dt \int_{\partial\Omega_d} L^*_{ed}\delta\phi dS + \int_T dt \int_{\partial\Omega_\phi} L^*_{e\phi} n_i\delta D^i dS \quad (4.27)$$

are defined in terms of the denotations in Eqs. (2.1)–(2.4), Eqs. (2.5)–(2.7), Eqs. (2.20)–(2.22), Eqs. (2.32)–(2.34) and Eqs. (2.38) and (2.39). The variational principle (4.24) leads to all the equations of each region and the jump conditions on the interface of the regions under the symmetry of the phonon stress tensor (2.2) and the initial conditions (2.35)–(2.37) for each region.

Unified variational principle- Let $\Omega + \partial\Omega + S$ denote a regular, finite and bounded region of a quasicrystal continuum, with its piecewise smooth boundary surface $\partial\Omega$ and a fixed internal surface S as

$$\partial\Omega(= \partial\Omega_u \cup \partial\Omega_t = \partial\Omega_w \cup \partial\Omega_h = \partial\Omega_d \cup \partial\Omega_\phi; \quad \partial\Omega_u \cap \partial\Omega_t = \partial\Omega_w \cap \partial\Omega_h = \partial\Omega_\phi \cap \partial\Omega_d = \emptyset)$$

and its closure $\bar{\Omega}$. Then, of all the admissible states that satisfy the initial conditions, (3.5)–(3.7), the symmetry of the phonon stress tensor (1b) and also the usual existence, continuity and differentiability conditions of the field variables, *if and only if*, that admissible state which satisfies the divergence equations ((2.1)–(2.4)), the gradient equations (2.5)–(2.7), the constitutive relations, (2.20)–(2.22), and the boundary conditions, (2.32)–(2.34) for each subregion of the regular region, is determined by the unified variational principle, $\delta L_A\{\Lambda_A\} = 0$ in (4.24), as its Euler–Lagrange equations. Conversely, if these equations are identically met, the eighteen-field variational principle is evidently satisfied.

Lastly, we extend the variational principle (4.24) for the regular region with N fixed internal surface of discontinuity as follows

$$\delta L_D\{\Lambda_D\} = \sum_{n=1}^{N}\left(\delta L_m\{\Lambda_m\} + \delta L_p\{\Lambda_p\} + \delta L_e\{\Lambda_e\}\right)^{(n)} + \delta L_{SD} = 0 \quad (4.28)$$

where

$$\delta L_S = \int_T dt \sum_{n=1}^{N-1}\Bigg(\int_S v_i\,[\sigma^{ij}] < \delta u_j > - < \delta\sigma^{ij} > [u_j] + [H^{ij}] < \delta w_j > - < \delta H^{ij} > [w_i] + [D^i] < \delta\phi > - < \delta D^i > [\phi] dS\Bigg)^{(n)} = 0 \quad (4.29)$$

which generates all the equations of each subregion and associated interface conditions under the symmetry of the phonon stress tensor and the initial conditions.

5. Some concluding remarks

In this work we generalize the theory of elasticity of quasicrystals, and we introduce a theory in order to govern the physical response of piezoelectric quasicrystals in the linear elastic range, excluding polar, non-local, thermal, magnetic and similar effects.

We develop the three-dimensional fundamental equations in differential and variational invariant forms for the piezoelasticity of a regular region of quasicrystals. The fundamental equations, which are applicable to all quasicrystals, are deterministic and the associated initial-boundary value problems are well posed. A theorem of uniqueness is proved in solutions of the fundamental equations, including the sufficient boundary and initial conditions. A three-field variational principle is obtained from Hamilton's principle under some constraint conditions, and then, the principle is modified by means of an involutory transformation so as to derive a variational principle operating on all the field variables. Also, a unified variational principle is obtained for the region of quasicrystals with one or more fixed internal surfaces of discontinuity. The unified variational principle generates, as its Euler–Lagrange equations, all the fundamental equations of piezoelasticity of quasicrystals, including the interface conditions on the discontinuity surfaces, under the symmetry condition of the phonon stress tensor and the initial conditions. The resulting equations are quite general, they agree with and recover some of earlier theories of elasticity of quasicrystals, and they can be readily reduced to the fundamental equations of piezoelasticity of one-, two- and three-dimensional quasicrystals by simply omitting some terms of the quasi-static electric field and/or the phason field (cf., Dökmeci, 1973).

The fundamental equations, which are expressible in any desirable system of orthogonal curvilinear coordinates, may be used through simultaneous approximation upon all the field variables in a direct method of solutions, and they pave the way to the forthcoming study of quasicrystal structural elements, important dislocation, fracture, interface and similar problems of both elasticity and piezoelasticity of quasicrystals.

Acknowledgements

The authors are grateful to the reviewers for their very careful reading and enlightening comments on the manuscript, and they acknowledge the support of their departments, and TUBA (The Turkish Academy of Sciences). The second author (Dökmeci) always feels to be indebted to late Dr. Solomon Bicerano, and especially to late Ms. Rikocya CapeloD for their kind concern and care in his health, enthusiastic motivation and encouragement for his academic activities. Also, he is grateful to Dr. Joseph Bicerano for kindly permitting Ms. CapeloD who patiently spared her invaluable time to support and to motivate him.

References

Altay, G., Dökmeci, M.C., 2010. On the fundamental equations of electromagnetoelastic media in variational form with an application to shell/laminae equations. Int. J. Solids Struct. 47, 466–492.
Bak, P., 1985a. Phenomenological theory of icosahedral incommensurate ("quasiperiodic") order in Mn-Al alloys. Phys. Rev. Lett. 54, 1517–1519.
Bak, P., 1985b. Symmetry, stability, and elastic properties of icosahedral incommensurate crystals. Phys. Rev. B 32, 5764–5772.
Chatzopoulos, A., Trebin, H-R., 2010. Hydrodynamic structure factor of quasicrystals. Phys. Rev. B 81, 064205-1–064205-11.
De, P., Pelcovits, R.A., 1987. Linear elasticity theory of pentagonal quasicrystals. Phys. Rev. B 35, 8609–8620.
Deresiewicz, H., Bieniek, M.P., DiMaggio, F.L. (Eds.), 1989. The Collected Works of Raymond D. Mindlin, vols. I–II. Springer-Verlag, Berlin, pp. 989–1001.
Ding, D.H., Yang, W., Hu, C.Z., Wang, R., 1993. Generalized elasticity theory of quasicrytals. Phys. Rev. B 48, 7003–7010.
Dökmeci, M.C., 1973. Variational principles in piezoelectricity. Lett. Nuovo Cimento 7, 449–454.
Dökmeci, M.C., 2010. Hamilton's principle and associated variational principle in Polar thermopiezoelectricity. Physica A 389, 2966–2974.
Dökmeci, M.C., Altay, G., 2011. On the mathematical modeling of quasicrystals under coupled effects. ITU&BU-TR.
Fan, T.-Y., 2011. Mathematical Theory of Elasticity of Quasicrystals and its Applications. Science Press–Springer, Beijing.
Fan, T.-Y., Mai, Y.-W., 2004. Elasticity theory, fracture mechanics, and some relevant thermal properties of quasi-crystalline materials. Appl. Mech. Rev. 57 (5), 325–343.
Fan, T.Y., Wang, X.F., Li, W., Zhu, A.Y., 2009. Elasto-hydrodynamics of quasicrystals. Philos. Mag. 89, 501–512.
Fraeijs De Veubeke, B., 1973. Dual principles of elastodynamics, finite element applications. In: Oden, J.T., Oliveira, E.R.A. (Eds.), Lectures on Finite Eelement Methods in Continuum Mechanics. University of Alabama Press, Huntsville, pp. 357–377.
Fujiwara, T., Ishii, Y. (Eds.), 2008. Quasicrystals, Handbook of Metal Physics. Elsevier, Amsterdam.
Gao, Y., 2009. The appropriate edge conditions for two-dimensional quasicrystal semi- infinite strips with mixed edge data. Int. J. Solids Struct. 46, 1849–1855.
Gao, Y., 2010. The exact theory of one-dimensional quasicrystal deep beams. Acta Mech. 212, 283–292.
Gao, Y., Zhao, B.S., Xu, S.P., 2008. A theory of general solutions of plane problems in two-dimensional octagonal quasicrystals. J. Elasticity 93 (3), 263–277.
Hamilton, W.R., 1834. On a general method in dynamics. Philos. T. R. Soc. 124, 247–308.
Hamilton, W.R., 1835. Second essay on a general method in dynamics. Philos. T. R. Soc. 125, 95–144.
Hargittai, I., 2010. Structures beyond crystals. J. Mol. Struct. 976, 81–86.
Hu, C.Z., Wang, R.H., Ding, D.H., 2000. Symmetry groups, physical property tensors, elasticity and dislocations in quasicrystals. Rep. Prog. Phys. 63, 1–39.
Hu, C.Z., Wang, R., Ding, D.-H., Yang, W., 1997. Piezoelectric effects in quasicrystals. Phys. Rev. B 56 (5), 2463–2468.
Kellogg, O.D., 1946. Foundations of Potential Theory. Ungar, New York.
Kirchhoff, G., 1876. Vorlesungen über mathematische Physik. Mechanik, Leipzig, Germany.
Lanczos, C., 1986. The Variational Principles of Mechanics, fourth ed. Dover, New York.
Landau, L.D., Lifshitz, E.M., 1958. Theoretical Physics V: Statistical Physics. Pergamon Press, New York.
Levine, D., Steinhardt, P.J., 1984. Quasi-crystals: A new class of ordered structure. Phys. Rev. Lett. 53, 2477–2480.
Levine, D., Lubensky, T.C., Ostlund, S., Ramaswamy, S., Steinhardt, P.J., Toner, J., 1985. Elasticity and dislocations in pentagonal and icosahedral quasicrystals. Phys. Rev. Lett. 54, 1520–1524.
Liu, G.-T., Fan, T.-Y., Guo, R.-P., 2004. Governing equations and general solutions of plane elasticity of one-dimensional quasicrystals. Int. J. Solids Struct. 41 (14), 3949–3959.
Lubensky, T.C., Ramaswamy, S., Toner, J., 1985. Hydrodynamics of icosahedral quasicrystal. Phys. Rev. B 32 (11), 74447452.
Mindlin, R.D., 1967&1968. Lecture Notes on the Theory of Elasticity I & II. Columbia University, New York.
Mindlin, R.D., 1974. Equations of high frequency vibrations of thermopiezoelectric crystal plates. Int. J. Solids Struct. 10, 625–637.
Peng, Y.Z., Fan, T.Y., 2000. Elastic theory of 1D-quasiperiodic stacking of 2D crystals. J. Phys. – Condens. Mat. 12, 9381–9387.
Nelson, D.F., 1978. Theory of nonlinear electroacoustics, dielectric, piezoelectric, and pyroelectric crystals. J. Acoust. Soc. Am. 63 (6), 1738–1748.
Rama Mohana Rao, K., Hemagiri Rao, P., Chaitanya, B.S.K., 2007. Piezoelectricity in quasicrystals: A group-theoretical study. Pramana – J. Phys. 68 (3), 481–487.
Shechtman, D., Blech, I., Gratias, D., Cahn, J.W., 1984. Metallic phase with long-range orientational order and no translational symmetry. Phys. Rev. Lett. 53, 1951–1953.
Shi, W., 2005. Conservation laws of decagonal quasicrystal in elastodynamics. Eur. J. Mech. A—Solid 24, 217–226.
Shi, W., 2007. Conservation integrals of any quasicrystal and application. Int. J. Fract. 44, 61–64.
Socolar, J.E.S., Lubensky, T.C., Steinhardt, P.J., 1986. Phonons, phasons and dislocations in quasicrystals. Phys. Rev. B 34, 3345–3360.
Suck, J.-B., Schreiber, M., Haussler, P. (Eds.), 2010. Quasicrystals: An Introduction to Structure, Physical Properties and Applications. Springer-Verlag, Berlin.
Tiersten, H.F., 1969. Linear Piezoelectric Plate Vibrations. Plenum Press, New York.
Yang, W.G., Wang, R., Ding, D.-H., Hu, C.Z., 1995. Elastic strains induced by electric fields in quasicrystals. J. Phys. – Condens Mat. 7, L499–L502.
Yang, W.C., Wang, R.H., Ding, D.H., Hu, C.Z., 1993. Linear elasticity theory of cubic quasicrystals. Phys. Rev. B 48, 6999–7002.