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A Functional Inequality

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In this paper we consider a functional inequality of the form $f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2)$, for each $(x_i, y_i) \in C$, $i = 1, 2$, where $f: C \to R$ and C is some cone in R^2. If the function f satisfies some conditions we obtain the general solution. © 1996 Academic Press, Inc.

A function $f: R \to R$ is said to be additive if it satisfies the Cauchy functional equation:

$$f(x + y) = f(x) + f(y), \qquad \forall x, y \in R.$$

Under some smoothing restrictions (measurability or Baire property) the only form of additive functions, as is well known, is that of cx.

The two-dimensional case of the Cauchy equation, i.e.,

$$f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_2, y_2), \qquad \forall (x_1, y_1), (x_2, y_2) \in R^2,$$

or similar ways have the solution $f(x, y) = c_1 x + c_2 y$.

We define locally (on $C \subset R^2$) the subadditive function f, $f: C \to R$ iff:

$$f(x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x_2, y_2), \qquad \forall (x_1, y_1), (x_2, y_2) \in C, \quad (1)$$

where C is some cone in R^2. For the definition of a cone in an arbitrary vector space see [2]. Let us call the class of all such functions on C LS_C.

Our task in this paper is to "solve" functional inequality (1), i.e., to give an explicit form of $f \in LS_C$.

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We begin with the following results:

PROPOSITION 1. *If* $f_k \in LS_{C_k}$, $k = 1, 2, \ldots, n$, *then*

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = f \in LS_C, \tag{2}$$

where $C = \bigcap_{k=1}^{n} C_k$ *and* $c_1, c_2, \ldots, c_n$ *are arbitrary positive constants.*

The proof follows immediately from the definition (1) of locally subadditive functions and the fact that the intersection of any family of cones is a cone.

So, if f_k are "solutions" of functional inequality (1), then we can call their linear (positive) combination a "general solution" of (1).

PROPOSITION 2. *If* $g(t)$ *is a convex function defined for* $t \in (a, b)$ *then:*

$$x \cdot g\left(\frac{y}{x}\right) = f(x, y) \in LS_C,$$

where $C := \{(x, y); a < y/x < b, x > 0\}$ *is a cone in* R^2.

Proof. According to the definition of a convex function $g(t)$, $t \in (a, b)$,

$$g(pr + qs) \le pg(r) + qg(s) \tag{3}$$

for each $r, s \in (a, b)$ and each $p, q > 0$, $p + q = 1$, and since

$$(x_1, y_1), (x_2, y_2) \in C$$

implies that $(x_1 + x_2, y_1 + y_2) = (x_1, y_1) + (x_2, y_2) \in C + C \subset C$, we have

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &= (x_1 + x_2) g\left(\frac{y_1 + y_2}{x_1 + x_2}\right) \\ &= (x_1 + x_2) g\left(\frac{x_1}{x_1 + x_2} \cdot \frac{y_1}{x_1} + \frac{x_2}{x_1 + x_2} \cdot \frac{y_2}{x_2}\right) \\ &\le (x_1 + x_2)\left(\frac{x_1}{x_1 + x_2} g\left(\frac{y_1}{x_1}\right) + \frac{x_2}{x_1 + x_2} g\left(\frac{y_2}{x_2}\right)\right) \\ &= x_1 g\left(\frac{y_1}{x_1}\right) + x_2 g\left(\frac{y_2}{x_2}\right) \\ &= f(x_1, y_1) + f(x_2, y_2), \end{aligned}$$

i.e., $f \in LS_C$.

Remark 1. Because $0 \notin C$, it follows that subset C from Proposition 2 is not a cone, but it has all the properties of a cone for $\lambda \neq 0$.

We can conclude that system of functions $g_i(t)$ convex for $t \in (a, b)$ produces a system of subadditive functions $f_i(x, y)$ over $C \subset R^2$ (in notation, $g(t) \rightrightarrows f(x, y)$) so, according to Proposition 1, we obtain a solution of (1) in the form:

$$f(x, y) = \sum_{i=1}^{n} c_i f_i(x, y), \qquad c_i > 0, (x, y) \in C.$$

Conversely to Proposition 2, we have the following:

PROPOSITION 2′. *If the function $f \in LS_C$, where C is the same subset of R^2 as in Proposition 2 and $f(\alpha x, \alpha y) = \alpha f(x, y)$ for every $\alpha \in R^+$, then there exists the convex function φ such that*

$$f(x, y) = y\varphi\left(\frac{x}{y}\right).$$

Proof. The function $\varphi(y) = f(1, y)$ is convex. Indeed,

$$\begin{aligned}\varphi\left(\frac{y_1 + y_2}{2}\right) &= f\left(1, \frac{y_1 + y_2}{2}\right)\\ &= f\left(\tfrac{1}{2} + \tfrac{1}{2}, \tfrac{1}{2}y_1 + \tfrac{1}{2}y_2\right)\\ &\leq f\left(\tfrac{1}{2}, \tfrac{1}{2}y_1\right) + f\left(\tfrac{1}{2}, \tfrac{1}{2}y_2\right)\\ &= \tfrac{1}{2}f(1, y_1) + \tfrac{1}{2}f(1, y_2)\\ &= \frac{\varphi(y_1) + \varphi(y_2)}{2}.\end{aligned}$$

Now, for $\alpha = 1/x$ we obtain

$$\frac{1}{x} f(x, y) = f\left(\frac{1}{x} \cdot x, \frac{y}{x}\right) = f\left(1, \frac{y}{x}\right) = \varphi\left(\frac{y}{x}\right),$$

i.e., $f(x, y) = x\varphi(y/x)$. This shows Proposition 2′.

Remark 2. A method for obtaining the functions from LS_C is the following: If $\sup_{(x, y)}(f(x + a, y + b) - f(x, y)) = g(a, b)$, then $g \in LS_C$, where $f: C \to R$.

Proof. Since

$$
\begin{aligned}
g(a_1 + b_1, a_2 + b_2)
&= \sup_{(x,y)} \left(f(x + a_1 + b_1, y + a_2 + b_2) - f(x,y) \right) \\
&= \sup_{(x,y)} \left(f(x + a_1 + b_1, y + a_2 + b_2) - f(x + b_1, y + b_2) \right. \\
&\quad \left. + \left(f(x + b_1, y + b_2) - f(x,y) \right) \right) \\
&\le \sup_{(x,y)} \left(f(x + a_1 + b_1, y + a_2 + b_2) - f(x + b_1, y + b_2) \right) \\
&\quad + \sup_{(x,y)} \left(f(x + b_1, y + b_2) - f(x,y) \right) \\
&= g(a_1, a_2) + g(b_1, b_2),
\end{aligned}
$$

then $g \in LS_C$.

Another property of the subadditive function is the following:

PROPOSITION 3. *If $f \in LS_C$, then*

$$
f\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i \right) \le \sum_{i=1}^{n} f(x_i, y_i)
$$

for $(x_i, y_i) \in C$, $i = 1, 2, \ldots, n$.

This is easy to prove by induction on n, since from $(x_i, y_i) \in C$, $i = 1, 2, \ldots, n$, it follows that

$$
\begin{aligned}
\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i \right) &= \sum_{i=1}^{n} (x_i, y_i) \in \underbrace{C + C + \cdots + C}_{n} \\
&\subset \underbrace{C + C + \cdots + C}_{n-1} \subset \cdots \subset C + C \subset C.
\end{aligned}
$$

Propositions 2 and 3 are the source for obtaining all kinds of two-parameter inequalities. We illustrate this with some examples.

EXAMPLE 1. Since $\ln t \rightrightarrows -x \ln(y/x)$, $x, y > 0$, using Proposition 3, and by putting $x_i = b_i$, $y_i = a_i b_i$, $i = 1, 2, \ldots, n$, we obtain the generalized arithmetic–geometric inequality

$$
\prod_{i=1}^{n} a_i^{b_i} \le \left(\frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i} \right)^{\sum_{i=1}^{n} b_i}, \qquad a_i, b_i > 0,
$$

i.e., putting $b_i/(\Sigma_{i=1}^n b_i) = p_i$, $i = 1, 2, \ldots, n$,

$$\prod_{i=1}^{n} a_i^{p_i} \leq \sum_{i=1}^{n} a_i p_i, \qquad \forall p_i, a_i > 0;\ \sum_{i=1}^{n} p_i = 1.$$

EXAMPLE 2. Since

$$t^r \rightrightarrows \begin{cases} -x\left(\dfrac{y}{x}\right)^r & \text{for } r \in (0,1) \\ x\left(\dfrac{y}{x}\right)^r & \text{for } r \in R \setminus [0,1] \end{cases} ;\ x, y > 0$$

putting in Proposition 3 $x_i = b_i^q$, $y_i = a_i^p$, $r = 1/p$, and $1 - r = 1/q$, we obtain the generalized Hoelders inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1;\ p, q > 1,$$

and

$$\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1;\ p < 1 \text{ or } q < 1.$$

EXAMPLE 3. Since: $\ln \sin t \rightrightarrows -x \ln \sin(y/x)$, using Proposition 3 with $x_i = 1$, $i = 1, 2, \ldots, n$, we have

$$\prod_{i=1}^{n} \sin y_i \leq \sin^n\left(\frac{1}{n} \sum_{i=1}^{n} y_i\right), \qquad y_i \in (0, \pi).$$

For the n-dimensional case of locally subadditive functions we next give

PROPOSITION 4. *Function $f(x_1, x_2, \ldots, x_n)$ if LS_C if*

$$f(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \leq f(x_1, x_2, \ldots, x_n) + f(y_1, y_2, \ldots, y_n) \tag{4}$$

for each $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in C \subset R^n$ where C is a cone in R^n. Function $g(t)$, convex for $t \in (a, b)$, produces the locally subadditive function $f(\cdot)$ on $C \subset R^n$,

$$f(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} A_i x_i\right) g\left(\frac{\Sigma_{i=1}^n B_i x_i}{\Sigma_{i=1}^n A_i x_i}\right),$$

where

$$C := \left\{ (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} A_i x_i > 0, a < \frac{\sum_{i=1}^{n} B_i x_i}{\sum_{i=1}^{n} A_i x_i} < b \right\},$$

and A_i, B_i *are arbitrary constants, not all equal to zero.*

Proof. This is similar to the one from Proposition 2. Since

$$(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in C$$

imply that $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in C$, by putting in (3)

$$p = \frac{\sum_{i=1}^{n} A_i x_i}{\sum_{i=1}^{n} A_i (x_i + y_i)}, \qquad q = \frac{\sum_{i=1}^{n} A_i y_i}{\sum_{i=1}^{n} A_i (x_i + y_i)},$$

$$r = \frac{\sum_{i=1}^{n} B_i x_i}{\sum_{i=1}^{n} A_i x_i}, \qquad s = \frac{\sum_{i=1}^{n} B_i y_i}{\sum_{i=1}^{n} A_i y_i}$$

we obtain (4), i.e., that $f \in LS_C$.

It is obvious that Propositions 1 and 3 could be easily translated on R^n.

REFERENCES

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2. Y. Wong and K. F. Ng, "Partially Ordered Topological Vector Spaces," Oxford Univ. Press, Oxford, 1973.