JOURNAL OF MULTIVARIATE ANALYSIS 11, 185–198 (1981)

Multiple Autoregressive Models with Random Coefficients

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Communicated by E. J. Hannan

This paper derives conditions for the stationarity of a class of multiple autoregressive models with random coefficients. The models considered include as special cases those previously discussed by Andel (*Ann. Math. Statist.* **42** (1971), 755–759; *Math. Operationsforsch. Statist.* **7** (1976), 735–741).

1. INTRODUCTION

The statistical literature relating to time series modelling has been primarily concerned with linear models, and, in particular, the autoregressive moving average models, more recently extended to include exogenous variables. The theory for the estimation of such models is now essentially complete and interest has shifted to computationally efficient procedures for the estimation of such models, and procedures for order determination. Recent research has been directed towards models other than linear models, such as linear models with random coefficients.

This paper will be concerned with an examination of the second order properties of multiple autoregressions with random coefficients. Conditions for second order stationarity have been derived by Andel [1] in the case of multiple autoregressions with constant coefficients, and by the same author for univariate autoregressions with random coefficients [2]. The class of models to be considered here will include both these models as special cases.

Let $\{X_t\}$, where t runs through the set of integers, or a subset of this set, be a sequence of $(p \times 1)$ random vectors. Let $\mu_i = E(X_i)$, and $\gamma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)'$. Then $\{X_t\}$ is said to be second order stationary if μ_i is constant, and γ_{ij} depends on i and j only through the difference $(j-i)$. In the remainder of this paper, when we refer to stationarity, we shall mean second order stationarity.

Received February 26, 1980; revised May 16, 1980.

Key words and phrases: Multiple autoregression, random coefficient, stationarity, eigenvalues, eigenvectors, tensor product.

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0047-259X/81/020185-14$02.00/0

In what follows, the theory to be developed depends on the use of the Kronecker or tensor product of matrices, and the related properties. In particular, we shall constantly use the result that if A, B and C are three matrices for which the product ABC is defined, then

$$\operatorname{vec}(ABC) = (C' \otimes A) \operatorname{vec} B$$

where by $\operatorname{vec} A$ we mean the vector formed from A by stacking the columns of A one on top of the other from left to right. For a proof of this property, and other related properties, see Neudecker [3].

2. Stationarity Conditions for a Random Coefficient Model

Let X_t be a $p \times 1$ vector generated by a multiple autoregressive model of order n with random coefficients, i.e.,

$$X_t = \sum_{i=1}^{n} (\beta_i + B_i(t)) X_{t-i} + \varepsilon_t. \tag{2.1}$$

For model (2.1), the following assumptions are made.

(i) $\{\varepsilon_t; t = 0, \pm 1, \pm 2,...\}$ is a sequence of independent $p \times 1$ random variables with mean zero and non-negative covariance matrix G.

(ii) The $p \times p$ matrices β_i, $i = 1,..., n$ are constants.

(iii) Letting $B(t) = [B_n(t) \cdots B_1(t)]$, then $\{B(t); t = 0, \pm 1, \pm 2,...\}$ is an independent sequence of $p \times np$ matrices with zero mean and $E(B(t) \otimes B(t)) = C$. $\{B(t)\}$ is also independent of $\{\varepsilon_t\}$.

(iv) There is no non-zero constant vector z such that $z'X_t$ is purely linearly deterministic, i.e., is determined exactly as a linear function of $\{X_{t-1}, X_{t-2},...\}$. It should be noted that conditions (in terms of $\{\varepsilon_t\}$ and $\{B(t)\}$) will be found for this assumption to hold, given assumptions (i)–(iii), in the body of this paper.

One problem concerning (2.1) is to determine whether or not it is possible to give the set $\{X_{1-n},..., X_0\}$ such a second order structure that the sequence $\{X_t, t = 1, 2,...\}$ is a second order stationary sequence. This aspect has been considered by Andel [2] in the univariate case, and a generalisation is considered here. A second problem is to determine whether (2.1) represents a process $\{X_t\}$ for which X_t is measurable, with respect to the σ-field $\mathscr{F}_t$ generated by the set $\{\varepsilon_t, \varepsilon_{t-1},...\} \cup \{B(t), B(t-1),...\}$, and stationary. The two problems are obviously related, and this relationship will be made clear later in this paper.

Define the $np \times 1$ random vector Y_t by

$$Y_t' = [X_{t+1-n}', X_{t+2-n}', \dots, X_t'].$$

Then the second order properties of $\{X_t\}$ may be found from an examination of those of the sequence $\{Y_t\}$, and vice versa. Equation (2.1) may be rewritten in terms of $\{Y_t\}$ by

$$Y_t = (M + D(t))\, Y_{t-1} + \eta_t, \tag{2.2}$$

where the $np \times np$ matrix M is given by

$$M = \left[\begin{array}{c|c} 0 & I \\ \hline \beta_n \cdots\cdots & \beta_1 \end{array}\right].$$

The (1, 1) block of M is the $(n-1)p \times p$ null matrix, while the (1, 2) block is the $(n-1)p \times (n-1)p$ identity. Letting L be the $n \times 1$ vector with the only non-zero entry the nth, which is 1, $D(t) = L \otimes B(t)$, and $\eta_t = L \otimes \varepsilon_t$.

If Y_t is measurable with respect to $\mathscr{F}_t$, then $D(t)$ and η_t will be independent of Y_{t-s}, $s = 1, 2, \dots$. Thus, applying (2.2) and the assumptions of the model,

$$E(Y_t) = ME(Y_{t-1}) \tag{2.3}$$

and

$$E(Y_t Y_t') = ME(Y_{t-1} Y_{t-1}') M' + E(D(t)\, Y_{t-1} Y_{t-1}' D'(t)) + E(\eta_t \eta_t').$$

Letting $V_{ij} = E(Y_i Y_j')$, it follows that

$$\begin{aligned} \operatorname{vec}(V_{tt}) &= (M \otimes M) \operatorname{vec} V_{t-1,t-1} \\ &\quad + E(D(t) \otimes D(t)) \operatorname{vec} V_{t-1,t-1} + \operatorname{vec} E(\eta_t \eta_t') \\ &= (M \otimes M + \tilde{C}) \operatorname{vec} V_{t-1,t-1} + \operatorname{vec}(J \otimes G), \end{aligned} \tag{2.4}$$

where $J = LL'$, and $\tilde{C} = E(D(t) \otimes D(t))$. Thus, the $\{(n-1)p(np + 1 + [(k-1)/p]) + k\}$th row of $\tilde{C}$, where $[x]$ denotes the integer part of x, is the kth row of C, for $k = 1, \dots, p^2$, and all the other rows of $\tilde{C}$ are zero.

Equation (2.4) in turn gives, for $r = 0, 1, 2, \dots$,

$$\begin{aligned} \operatorname{vec} V_{tt} &= \sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G) \\ &\quad + (M \otimes M + \tilde{C})^{r+1} \operatorname{vec} V_{t-r-1,t-r-1}. \end{aligned} \tag{2.5}$$

The following lemma establishes a simple condition for the stationarity of $\{X_t; t = 1, 2, \ldots\}$:

LEMMA 2.1. *$\{X_t; t = 1, 2, \ldots\}$ generated by* (2.1), *satisfying assumptions* (i)–(iii) *is stationary if and only if $V_{11} = V_{00}$ and $\mu_1 = \mu_0$, where $V_{ij} = E(Y_i Y_j')$ and $\mu_i = E(Y_i)$.*

Proof. The necessity is obvious. To prove the sufficiency, we use induction. Suppose $\mu_t = \mu_{t-u}$, $t = 1, \ldots, h$; $u = 1, \ldots, t$ and $V_{t,t-s} = V_{t-u,t-s-u}$, $t = s+1, \ldots, h$; $u = 1, \ldots, t-s$; $s = 0, \ldots, h$. In the case $h = 1$, these conditions reduce to $\mu_1 = \mu_0$ and $V_{11} = V_{00}$. Using (2.2), we have

$$\begin{aligned}
\mu_{h+1} &= E(Y_{h+1}) \\
&= E[(M + D(h+1))\, Y_h + \eta_{h+1}] \\
&= ME(Y_h), \qquad \text{since } D(h+1) \text{ is independent of } Y_h, \\
&= ME(Y_{h-1}), \qquad \text{by induction hypothesis,} \\
&= \mu_h
\end{aligned}$$

and, for $1 \leqslant s \leqslant h$,

$$\begin{aligned}
V_{h+1,h+1-s} &= E(Y_{h+1} Y'_{h+1-s}) \\
&= E[(\{M + D(h+1)\}\, Y_h + \eta_{h+1})\, Y'_{h+1-s}] \\
&= ME(Y_h Y'_{h+1-s}), \qquad \text{since } D(h+1) \text{ and } \eta_{h+1} \\
&\qquad\qquad \text{are independent of } Y_{h+1-s}, \\
&= ME(Y_{h-1} Y'_{h-s}), \qquad \text{by induction hypothesis,} \\
&= E(Y_h Y'_{h-s}) = V_{h,h-s},
\end{aligned}$$

while for $s = 0$,

$$\begin{aligned}
V_{h+1,h+1} &= E(Y_{h+1} Y'_{h+1}) \\
&= E[(\{M + D(h+1)\}\, Y_h + \eta_{h+1})(\{M + D(h+1)\}\, Y_h + \eta_{h+1})'],
\end{aligned}$$

so that by (2.4)

$$\begin{aligned}
\operatorname{vec} V_{h+1,h+1} &= (M \otimes M + \tilde{C}) \operatorname{vec} V_{h,h} + \operatorname{vec}(J \otimes G) \\
&= (M \otimes M + \tilde{C}) \operatorname{vec} V_{h-1,h-1} \\
&\qquad + \operatorname{vec}(J \otimes G), \qquad \text{by induction hypothesis,} \\
&= \operatorname{vec} V_{hh}
\end{aligned}$$

as required.

Again using (2.4) we have

$$\operatorname{vec} V_{11} = (M \otimes M + \tilde{C}) \operatorname{vec} V_{00} + \operatorname{vec}(J \otimes G).$$

Hence, from the above lemma, it follows that $\{X_t; t = 1, 2,...\}$ is stationary if and only if it is possible to find a non-negative matrix V satisfying

$$\operatorname{vec} V = (M \otimes M + \tilde{C}) \operatorname{vec} V + \operatorname{vec}(J \otimes G). \tag{2.6}$$

This condition alone will suffice, for $\mu = 0$ will always satisfy the equation $\mu = M\mu$, and, from the lemma, this condition is necessary. Note that the above condition is the necessary and sufficient condition even if the matrix $(M \otimes M + \tilde{C})$ has a unit eigenvalue. This point will be discussed more fully later.

In order to determine the stationarity of the process $\{X_t; t = 0, \pm 1, \pm 2,...\}$ it will be necessary to obtain a development of Y_t in terms of $\{\eta_t, \eta_{t-1},...\}$ and $\{D(t), D(t-1),...\}$. Define the matrix product $\prod_{k=i}^{j} A_k$ by

$$\prod_{k=i}^{j} A_k = A_i A_{i+1},..., A_j, \qquad i \leqslant j.$$

Iterating equation (2.2) r times, we obtain

$$\begin{aligned} Y_t &= \eta_t + \sum_{j=1}^{r} \left\{ \prod_{k=0}^{j-1} [M + D(t-k)] \right\} \eta_{t-j} \\ &\quad + \left\{ \prod_{k=0}^{r} [M + D(t-k)] \right\} Y_{t-r-1} \\ &= W_{t,r} + R_{t,r}, \qquad \text{say}, \end{aligned} \tag{2.7}$$

where $R_{t,r} = \{\prod_{k=0}^{r} [M + D(t-k)]\} Y_{t-r-1}$. In what follows, it is important to keep in mind that the stationarity of X_t involves the convergence of $W_{t,r}$ and $R_{t,r}$ as r increases. We now prove the following lemma which will be needed in several of the subsequent theorems.

LEMMA 2.2. *If the sum* $\sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$ *converges as* $r \to \infty$, *and if* H *is positive definite, where* $\operatorname{vec} H = \operatorname{vec} G + C \sum_{j=0}^{\infty} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$, *then the matrix* M *has all its eigenvalues within the unit circle.*

Proof. Let the matrix W be defined by

$$\operatorname{vec} W = \sum_{j=0}^{\infty} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G).$$

Then W satisfies

$$\begin{aligned}\operatorname{vec} W &= (M \otimes M + \tilde{C}) \operatorname{vec} W + \operatorname{vec}(J \otimes G) \\ &= (M \otimes M) \operatorname{vec} W + (\tilde{C} \operatorname{vec} W + \operatorname{vec}(J \otimes G)) \\ &= (M \otimes M) \operatorname{vec} W + \operatorname{vec}(J \otimes H),\end{aligned}$$

where $\operatorname{vec} H = C \operatorname{vec} W + \operatorname{vec} G$, with H assumed to be positive definite. Then,

$$W = MWM' + J \otimes H.$$

Suppose M has an eigenvalue λ with corresponding left eigenvector z. If $z' = [z'_1, \ldots, z'_n]$, each z_i a $p \times 1$ vector, then

$$\begin{aligned} z' W\bar{z} &= z'MWM'\bar{z} + z'(J \otimes H)\bar{z} \\ &= |\lambda|^2 z' W \bar{z} + z'_n H \bar{z}_n. \end{aligned}$$

Now, if $z'_n H \bar{z}_n > 0$, then $|\lambda|^2 < 1$. Suppose $z'_n H \bar{z}_n = 0$, i.e., $z_n = 0$, since H is positive definite. But z is a left eigenvector of M. Thus

$$[z'_1, \ldots, z'_n] \left[\begin{array}{c|c} 0 & I \\ \hline \beta_n \cdots\cdots & \beta_1 \end{array} \right] = \lambda [z'_1, \ldots, z'_n],$$

i.e., $z'_n \beta_n = \lambda z'_1$, $z'_i + z'_n \beta_{n-i} = \lambda z'_{i+1}$, $i = 1, \ldots, n-1$.

If $\lambda \neq 0$, since $z_n = 0$, then $z_1 = z_2 = \cdots = z_{n-1} = z_n = 0$, i.e., $z = 0$. But $z \neq 0$, since z is a left eigenvector of M. Thus $z_n \neq 0$ and $z'(J \otimes H)\bar{z} > 0$, implying $|\lambda| < 1$.

The next lemma presents a partial examination of uniqueness.

LEMMA 2.3. *If the matrix* $(M \otimes M + \tilde{C})$ *does not possess an eigenvalue equal to unity, and a stationary solution exists to Eq.* (2.1), *which is measurable with respect to* $\mathscr{F}_t$, *then this is the unique solution.*

Proof. Suppose there are two such solutions to (2.1): w_t and z_t. Let $W'_t = [w'_{t+1-n} w'_{t+2-n} \cdots w'_t]$, $Z'_t = [z'_{t+1-n} z'_{t+2-n} \cdots z'_t]$, $u_t = w_t - z_t$ and $U_t = W_t - Z_t$. If w_t and z_t are both stationary and measurable with respect to $\mathscr{F}_t$, then so is u_t, and U_t satisfies

$$U_t = (M + D(t))\, U_{t-1}$$

giving

$$\operatorname{vec} E(U_t U'_t) = (M \otimes M + \tilde{C}) \operatorname{vec} E(U_{t-1} U'_{t-1}).$$

Since $(M \otimes M + \tilde{C})$ has no unit eigenvalues, $E(U_{t-1}U'_{t-1}) = 0$ and $U_t = 0$, a.s., i.e., $w_t = z_t$, a.s.

To establish necessary conditions for the stationarity of the process satisfying (2.1), we now prove the following theorem.

THEOREM 2.1. *In order that there exist a stationary solution to* (2.1), *measurable with respect to* $\mathscr{F}_t$ *and satisfying* (i)–(iv), *it is necessary that* $\sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$ *converge as* $r \to \infty$, *and sufficient that this occur with* H *positive definite where* $\operatorname{vec} H = \operatorname{vec} G + C \sum_{j=0}^{\infty} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$.[1] *When* $(M \otimes M + \tilde{C})$ *does not have a unit eigenvalue, this latter condition is both necessary and sufficient, and there is a unique stationary solution.*

Proof. We shall first show the necessity. If X_t solves Eq. (2.1), and is stationary, then $E(Y_t Y'_t) = E(Y_{t-s} Y'_{t-s})$, $s = \pm 1, \pm 2, \ldots$. Let $V = E(Y_t Y'_t)$. Then $\operatorname{vec} H$ satisfies

$$\operatorname{vec} V = \sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G) + (M \otimes M + \tilde{C})^{r+1} \operatorname{vec} V, \qquad r = 1, 2, \ldots.$$

Let $Q_0 = J \otimes G$; $Q_j = MQ_{j-1}M' + E(D(t)\, Q_{j-1} D'(t))$, $j = 1, 2, \ldots, R_0 = V$, and $R_j = MR_{j-1}M' + E(D(t)\, R_{j-1} D'(t))$, $j = 1, 2, \ldots$. It is clear that each of Q_j, R_j, $j = 0, 1, 2, \ldots$ is non-negative definite. Also,

$$V = \sum_{j=0}^{r} Q_j + R_{r+1}, \qquad r = 0, 1, 2, \ldots.$$

Let z be any $np \times 1$ fixed vector. Then

$$z'Vz = \sum_{j=0}^{r} z'Q_j z + z'R_{r+1} z. \tag{2.8}$$

Now, $\sum_{j=0}^{r} Q_j$ is non-decreasing in r, and $z'Vz$ is non-negative. Also, $z'R_{r+1}z$ is non-negative. Since (2.8) holds for $r = 1, 2, \ldots$, it follows that $\sum_{j=0}^{r} z'Q_j z$ is bounded above by $z'Vz$, and is therefore convergent for every vector z. Thus $\sum_{j=0}^{r} Q_j$ converges, as $r \to \infty$, to a non-negative definite matrix.

Consider now the case were $(M \otimes M + \tilde{C})$ has no unit eigenvalues. Then by Lemma 2.3 the solution to 2.2 must be unique with $Y_t = \eta_t +$

[1] Of course, when G is positive definite and $\sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$ converges, H is necessarily positive definite, so that this convergence will be both necessary and sufficient for the existence of a stationary solution.

$\sum_{j=1}^{\infty}\{\sum_{k=0}^{j-1}[M+D(t-k)]\}\,\eta_{t-j}$, which follows from 2.7, and $V=E(Y_tY_t')$ is given by vec $V=\sum_{j=0}^{\infty}(M\otimes M+\tilde{C})^j\,\text{vec}(J\otimes G)$.

Now, the matrix $H=G+E(B(t)\,VB'(t))$ is obviously non-negative definite, for every non-negative definite V. Suppose V satisfies (2.6), i.e., $V=E(Y_tY_t')$ where Y_t satisfies (2.2), and assume H is *not* positive definite. Then there is a non-trivial $p\times 1$ vector z such that $z'Hz=0$, i.e.,

$$z'\{E(\varepsilon_t\varepsilon_t')+E(B(t)\,Y_{t-1}Y_{t-1}'B'(t))\}z=0$$

and

$$E[z'(\varepsilon_t+B(t)\,Y_{t-1})]^2=0.$$

Thus $z'[\varepsilon_t+B(t)\,Y_{t-1}]=0$ a.s. But $X_t-\sum_{i=1}^{n}\beta_iX_{t-i}=\varepsilon_t+B(t)\,Y_{t-1}$ so $z'X_t=\sum_{i=1}^{n}z'\beta_iX_{t-i}$ a.s., implying that $z'X_t$ is purely linearly deterministic, contradicting assumption (iv) for the model (2.1). Hence, in order that X_t solve Eq. (2.1) and be stationary, it is also necessary that H be positive definite so that condition (iv) is satisfied. Noting that vec $H=$ vec $G+E(B(t)\otimes B(t))$ vec V, this expression and the one for vec H stated in the theorem are seen to be equivalent.

We now prove sufficiency. Suppose firstly that $\sum_{j=0}^{r}(M\otimes M+\tilde{C})^j$ vec$(J\otimes G)$ converges as $r\to\infty$. In this case, from Eq. (2.7), the term $W_{t,r}$ converges in mean square, and so in probability, as $r\to\infty$. Denoting this limit by W_t, it is then easy to see that W_t satisfies Eq. (2.2). If the $np\times np$ matrix U is given by vec $U=\sum_{j=0}^{\infty}(M\otimes M+\tilde{C})^j\,\text{vec}(J\otimes G)$, then $U=E(W_tW_t')$. Furthermore, if the matrix $H=G+E(B(t)\,UB'(t))$ is positive definite, then U satisfies

$$U=MUM'+J\otimes H. \tag{2.9}$$

Let $W_t'=[w_{t+1-n}'\,w_{t+2-n}'\cdots w_t']$, where each w_s is a $p\times 1$ random vector. Suppose there is a $p\times 1$ vector z such that $z'w_t$ is perfectly linearly predictable, i.e., $z'w_t$ is determined exactly by $\{w_{t-1},w_{t-2},\ldots\}$. Now, since W_t satisfies (2.2),

$$z'w_t=\left(z'\sum_{i=1}^{n}\beta_iw_{t-i}\right)+\left(z'\sum_{i=1}^{n}B_i(t)\,w_{t-i}+z'\varepsilon_t\right).$$

The term $z'\varepsilon_t+z'\sum_{i=1}^{n}B_i(t)\,w_{t-i}$ cannot be determined solely by a knowledge of $\{w_{t-1},w_{t-2},\ldots\}$, since w_t is measurable with respect to $\mathscr{F}_t$, and thus ε_t and $B(t)$ are independent of $\{w_{t-1},w_{t-2},\ldots\}$, unless $z'\varepsilon_t+z'\sum_{i=1}^{n}B_i(t)\,w_{t-i}$ is a constant almost surely. Since its expected value is zero, we must have $z'\sum_{i=1}^{n}B_i(t)\,w_{t-i}+z'\varepsilon_t=0$ a.s. However, the variance of this term is $z'Hz$, and H is positive definite. Thus $z=0$, and condition (iv) is satisfied.

We now prove the following corollary giving a simple sufficient condition for stationarity.

COROLLARY 2.1. *In order that there exist a unique stationary solution to* (2.1), *satisfying* (i)–(iii), *it is sufficient that all the eigenvalues of* $(M \otimes M + \tilde{C})$ *be less than unity is modulus.*

Proof. $(M \otimes M + \tilde{C})$ may be represented in Jordan canonical form as

$$(M \otimes M + \tilde{C}) = P\Lambda P^{-1},$$

where Λ has the eigenvalues of $(M \otimes M + \tilde{C})$ along its main diagonal, and zeros elsewhere, unless $(M \otimes M + \tilde{C})$ has eigenvalues of multiplicity greater than one, in which case there may be several ones in the first upper diagonal.

Now,

$$(M \otimes M + \tilde{C})^j = P\Lambda^j P^{-1}$$

and it is well known that if the diagonal elements of Λ are less than unity in modulus, then Λ^j converges to zero at a geometric rate and

$$\lim_{r \to \infty} \sum_{j=0}^{r} \Lambda^j = (I - \Lambda)^{-1}.$$

Furthermore,

$$\begin{aligned} \lim_{r \to \infty} \sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G) &= P(I - \Lambda)^{-1} P^{-1} \operatorname{vec}(J \otimes G) \\ &= (I - P\Lambda P^{-1})^{-1} \operatorname{vec}(J \otimes G) \\ &= (I - M \otimes M - \tilde{C})^{-1} \operatorname{vec}(J \otimes G). \end{aligned}$$

Thus, using Lemmas 2.2, 2.3 and Theorem 2.1, it is seen that there exists a unique stationary solution to (2.2), given by

$$Y_t = \eta_t + \sum_{j=1}^{\infty} \left\{ \prod_{k=0}^{j-1} [M + D(t-k)] \right\} \eta_{t-j}.$$

One of the central requirements in the theory developed to date is the convergence, as $r \to \infty$, of

$$\sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G) = P \left(\sum_{j=0}^{r} \Lambda^j \right) P^{-1} \operatorname{vec}(J \otimes G), \quad (2.10)$$

where Λ is the Jordan canonical form of $(M \otimes M + \tilde{C})$. Even if $(M \otimes M + \tilde{C})$ has eigenvalues greater than or equal to unity in modulus, this does *not* preclude the right-hand side from converging. Indeed, in this case, the right-hand side converges if and only if $\text{vec}(J \otimes G)$ is orthogonal to the rows of P^{-1} corresponding to those diagonal elements of Λ which are greater than or equal to unity in modulus. It may, however, be impossible for this to occur. Such is the case for scalar models as is demonstrated in the following corollary.

COROLLARY 2.2. *In the scalar case, i.e., $p = 1$ in* (2.1), *with $G \neq 0$, in order that there exist a stationary solution to* (2.1) *measurable with respect to $\mathscr{F}_t$, it is necessary and sufficient that $(M \otimes M + \tilde{C})$ have all its eigenvalues less than unity in modulus.*

Proof. In view of Theorem 2.1 and Corollary 2.2, all that is needed to be shown is that the situation described below (2.10) does not occur. Suppose there is a left eigenvector of $(M \otimes M + \tilde{C})$, which is orthogonal to $\text{vec}(J \otimes G)$. Call this left eigenvector z', and its eigenvalue λ. Then z' is a row of P^{-1} corresponding to a diagonal element λ. Since $z' \,\text{vec}(J \otimes G) = 0$, the last entry of z is zero. But then $z'(M \otimes M + \tilde{C}) = z'(M \otimes M)$, since $\tilde{C}$ is null apart from its last row. However, the eigenvalues of $M \otimes M$ are precisely all products of eigenvalues of M, taken two at a time, and, as such, are less than or equal to unity in modulus, since the eigenvalues of M are less than unity in modulus, by Lemma 2.2. Thus $|\lambda| < 1$, and the corollary follows.

Now, if a stationary solution exists which is measurable with respect to $\mathscr{F}_t$, then the covariance matrix will satisfy

$$\text{vec}\, V = (M \otimes M)\, \text{vec}\, V + \text{vec}(J \otimes H), \tag{2.11}$$

where $H = E(B(t)\, VB'(t)) + G$. In the case where X_t satisfies (2.1), and requirement (iv), we have seen that H is positive definite, and the matrices M and $M \otimes M$ have all their eigenvalues within the unit circle. Thus $(I - M \otimes M)$ is invertible. The matrix $(I - M \otimes M)^{-1}$ plays a prominent role in the calculation of the covariance structure of stationary constant parameter autoregressions. In this case, the only dependence on $(I - M \otimes M)^{-1}$ is through p^2 of its columns, namely those columns corresponding to the p^2 elements of G in the vector $\text{vec}(J \otimes G)$. Let A be the $n^2p^2 \times p^2$ matrix formed from these columns, i.e., the kth column of A is the $\{(n-1)\, p(np + 1 + [(k-1)/p]) + k\}$th column of $(I - M \otimes M)^{-1}$. As will be shown in the following theorem, the matrix A plays a dual role in the stationarity of an autoregression with random parameters.

THEOREM 2.2. *For the case where $(M \otimes M + \tilde{C})$ does not have a unit eigenvalue, there exists a stationary solution of* (2.1), *satisfying* (i)–(iv), *and measurable with respect to $\mathscr{F}_t$, if and only if the matrix V given by*

$$\text{vec}\, V = (I - M \otimes M - \tilde{C})^{-1}\, \text{vec}(J \otimes G) \tag{2.12}$$

is positive definite. An equivalent condition is that the eigenvalues of M be less than unity in modulus, together with the condition that the matrix H, given by

$$\operatorname{vec} H = (I - CA)^{-1} \operatorname{vec} G$$

be positive definite. The covariance matrix V is then given by

$$\operatorname{vec} V = A \operatorname{vec} H.$$

Proof. It has already been seen that Y_t will be stationary if and only if the term $\sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$ converges as $r \to \infty$, in which case V is obtained from

$$\operatorname{vec} V = \sum_{j=0}^{\infty} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G).$$

This satisfies the equation $\operatorname{vec} V = (M \otimes M + \tilde{C}) \operatorname{vec} V + \operatorname{vec}(J \otimes G)$, i.e.,

$$\operatorname{vec} V = (I - M \otimes M - \tilde{C})^{-1} \operatorname{vec}(J \otimes G),$$

the matrix $(I - M \otimes M - \tilde{C})$ being invertible since it has no zero eigenvalues. The condition on $(I - M \otimes M - \tilde{C})^{-1} \operatorname{vec}(J \otimes G)$ is thus seen to be necessary and sufficient, since $(I - M \otimes M - \tilde{C})^{-1} \operatorname{vec}(J \otimes G)$ is equal to $\sum_{j=0}^{\infty} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$ whenever it exists.

To consider the equivalent condition in terms of the matrix H, suppose that V is positive definite. Then, from the definition of H given in Eq. (2.11) and from Theorem 2.1, H will also be positive definite and M will have all its eigenvalues within the unit circle by Lemma 2.2. Conversely, if H is positive definite, and M has all its eigenvalues within the unit circle, then V will also be positive definite, being the covariance matrix of a stationary autoregression with input covariance matrix H. To see this, we express V in an alternative form, and extend the approach used in Andel [1] in the case of constant parameter autoregressions. Since M is assumed to have all its eigenvalues within the unit circle, we may write V in the form

$$\operatorname{vec} V = (I - M \otimes M)^{-1} \operatorname{vec}(J \otimes H) = \sum_{i=0}^{\infty} (M \otimes M)^i \operatorname{vec}(J \otimes H)$$

$$= \sum_{i=0}^{\infty} (M^i \otimes M^i) \operatorname{vec}(J \otimes H)$$

and so

$$V = \sum_{i=0}^{\infty} M^i (J \otimes H)(M')^i.$$

Let $z' = (z_1' \cdots z_n')$ where z_i are all $p \times 1$ vectors and $z \neq 0$. Then

$$z'Vz = \sum_{i=0}^{\infty} z'M^i(J \otimes H)(M')^i z.$$

If $z_n \neq 0$, then $z'(J \otimes H)z = z_n' H z_n > 0$, and $z'Vz > 0$. Suppose for some j, $1 \leqslant j \leqslant n-1$, that $z_j \neq 0$ but $z_{j+1} = \cdots = z_n = 0$. Then

$$z'M = (0', z_1' \cdots z_j', 0' \cdots 0')$$

and

$$z'M^{n-j} = (0' \cdots 0', z_1' \cdots z_j').$$

Then $z'M^{n-j}(J \otimes H)(M')^{n-j} z = z_j' H z_j > 0$ and $z'Vz > 0$. Finally, we derive the alternate form of H. From (2.11), V satisfies

$$\text{vec } V = (I - M \otimes M)^{-1} \text{vec}(J \otimes H) = A \text{ vec } H.$$

Thus,

$$C \text{ vec } V = CA \text{ vec } H$$

and

$$\text{vec } H = \text{vec } G + C \text{ vec } V = \text{vec } G + CA \text{ vec } H$$

so that

$$\text{vec } H = (I - CA)^{-1} \text{vec } G, \tag{2.13}$$

provided $(I - CA)$ is invertible. To see this, suppose CA has a unit eigenvalue, so that there is a non-trivial $(p \times 1)$ vector z such that

$$z'CA = z'.$$

Then, defining the $n^2p^2 \times 1$ vector w by

$$w' = z'C(I - M \otimes M)^{-1},$$

it follows that

$$w'\tilde{C} = z'C(I - M \otimes M)^{-1} \tilde{C} = z'CAC = z'C.$$

Thus

$$w'\tilde{C}(I - M \otimes M)^{-1} = z'C(I - M \otimes M)^{-1} = w',$$

i.e.,

$$w'(I - M \otimes M - \tilde{C}) = 0.$$

Since $(M \otimes M + \tilde{C})$ has no eigenvalue equal to unity, $w = 0$ and so

$$z' = z'CA = w'\tilde{C}A = 0.$$

Thus CA does not possess a unit eigenvalue and $(I - CA)$ is invertible. This completes the proof of the theorem.

COROLLARY 2.3. *In the case* $p = 1$, *there exists a unique stationary solution of* (2.1), *measurable with respect to* $\mathscr{F}_t$, *if and only if* M *has all its eigenvalues less than unity in modulus and* $C \operatorname{vec} V_w < G$, *where*

$$\operatorname{vec} V_w = (I - M \otimes M)^{-1} \operatorname{vec}(J \otimes G) = A \operatorname{vec} G.$$

Proof. Since G and H are scalars, the condition that H be positive definite is, from (2.13), equivalent to having $CA < 1$ or $CAG < G$. Let $W_t = MW_{t-1} + \eta_t$, i.e., (2.2) with $D(t) = 0$, for all t, so that (2.1) represents, in this case, a constant coefficient autoregression. Letting $V_w = E(W_t W_t')$,

$$\operatorname{vec} V_w = (I - M \otimes M)^{-1} \operatorname{vec}(J \otimes G) = A \operatorname{vec} G,$$

so that the above condition becomes, since G is scalar,

$$C \operatorname{vec} V_w < G.$$

It should be noticed that these are the same conditions derived by Andel [2] for the stationarity of $\{X_t; t = 1, 2,...\}$ generated by (2.1), for the case $p = 1$. Indeed, it follows from Lemma 2.1 that, in the case where $(M \otimes M + \tilde{C})$ does not have a unit eigenvalue, the conditions of Theorem 2.2 are also the necessary and sufficient conditions for the stationarity of $\{X_t; t = 1, 2,...\}$.

3. THE CASE WHERE $(M \otimes M + \tilde{C})$ HAS A UNIT EIGENVALUE

The results of Section 2 do not completely cover the case where $(M \otimes M + \tilde{C})$ has a unit eigenvalue. This is due to the possible lack of uniqueness of the solution, and the fact that $(I - M \otimes M - \tilde{C})$ is not invertible in this case. If the term $\sum_{j=0}^{r} (M \otimes M + \tilde{C})^j \operatorname{vec}(J \otimes G)$ converges as $r \to \infty$, there exists a solution to (2.1) which is stationary and measurable with respect to $\mathscr{F}_t$, regardless of the eigenvalues of $(M \otimes M + \tilde{C})$. There may, however, be other stationary solutions if $(M \otimes M + \tilde{C})$ has a unit eigenvalue.

It is nevertheless possible to construct an example for which Eq. (2.1) generates a process $\{X_t; t = 1, 2,...\}$ which is stationary, satisfies conditions (i)–(iv), and for which $(M \otimes M + \tilde{C})$ has a unit eigenvalue. To illustrate this, let $\{X_t; t = 1, 2, \ldots\}$ be generated by the model

$$X_t = (\beta + B(t)) X_{t-1} + \varepsilon_t, \tag{3.1}$$

where X_t and ε_t are (2×1) random vectors, $\beta = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$, $B(t) = \begin{bmatrix} 0 & 0 \\ 0 & \beta(t) \end{bmatrix}$, $\varepsilon_t = \begin{bmatrix} \delta_t \\ 0 \end{bmatrix}$, $E(\delta_t) = E(\beta(t)) = 0$, $E(\beta^2(t)) = 1$, $E(\delta_t^2) = g$, $|b| < 1$, and δ_t, $\beta(t)$ are independent.

Furthermore, let $E(X_0) = 0$ and $\operatorname{vec} E(X_0 X_0') = [g/(1 - b^2)\ 0\ 0\ c]'$, where $c > 0$. Then $\operatorname{vec} G = g[1\ 0\ 0\ 0]'$, $M = \beta$ has all its eigenvalues inside the unit circle, and

$$(M \otimes M + \tilde{C}) = \left[\begin{array}{cc|cc} b^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

having eigenvalues 0, 0, b^2 and 1. Now,

$$\begin{aligned} \operatorname{vec} E(X_1 X_1') &= (M \otimes M + \tilde{C}) \operatorname{vec} E(X_0 X_0') + \operatorname{vec}(J \otimes G) \\ &= [gb^2/(1 - b^2)\ \ 0\ \ 0\ \ c]' + [g\ \ 0\ \ 0\ \ 0]' \\ &= [g/(1 - b^2)\ \ 0\ \ 0\ \ c]' = \operatorname{vec} E(X_0 X_0'). \end{aligned}$$

Also, $E(X_1) = \beta E(X_1) = 0 = E(X_0)$. Thus, by Lemma 2.1, $\{X_t; t = 1, 2, \ldots\}$ is stationary.

References

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[3] NEUDECKER, H. (1969). Some theorems on matrix differentiation with special reference to Kronecker matrix products. *J. Amer. Statist. Assoc.* **64** 953–963.