In this paper we investigate Szilard languages of IO-grammars. First we show, similar to the proof in Moriya [Information and Control 22 (1973), 139-162], that these languages are context-sensitive. It is known that there are context-free languages $L$ such that no context-free grammar for $L$ has a context-free Szilard language. Since a context-free language $L$ is also an IO-language the question arises if there exists an IO-grammar for $L$ with a context-free Szilard language.

To solve this problem we first provide an algorithm which transforms an IO-grammar into a simple normal form. Then it is possible to construct a context-free grammar generating the Szilard language of the normal form IO-grammar. Hence every IO-language has an IO-grammar with a context-free Szilard language.

1. Introduction

Szilard languages of phrase-structure grammars are investigated in the literature as a measure of derivational complexity of grammars (Moriya, 1973). In this paper we study the Szilard languages of IO-grammars, a type of grammar, which is an extension of context-free grammars but not a phrase-structure grammar.

For the notion of a macro grammar with inside-out mode of derivation, $IO$-grammar for short, and related concepts see Duske et al. (1977) or Fischer (1968). Let $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar, where $\Pi = \{\pi_1, \ldots, \pi_n\}$ is the set of productions. A term $F(x_1, \ldots, x_{\rho(F)})$ with $F \in \mathcal{F}$, $x_i$ terms over $\Sigma, \mathcal{F}, \rho$ for $i \in [1: \rho(F)]$ is called a \textit{macro}, and the $x_i$'s are called \textit{arguments of the macro}. In contrast to the convention in Fischer (1968), 0-ary macros, i.e. macros with $\rho(F) = 0$, are written in the form $F()$.

If a macro $F(x_1, \ldots, x_{\rho(F)})$ is a subterm of a term $\phi$, it is called \textit{macro in $\phi$}. A macro $F(x_1, \ldots, x_{\rho(F)})$ in $\phi$ with $x_i \in \Sigma^*$ for $i \in [1: \rho(F)]$ is called \textit{applicable}.

Let $\phi$ and $\phi'$ be terms over $\Sigma, \mathcal{F}, \rho, \phi'$ is \textit{directly derivable} from $\phi$ with respect to $G_M$ iff there is an applicable macro $F(x_1, \ldots, x_{\rho(F)})$ in $\phi$ and a production $\pi_j : F(x_1, \ldots, x_{\rho(F)}) \rightarrow \theta$ in $\Pi$ such that $\phi'$ is obtained from $\phi$ by application of $\pi_j$. This is written as $\phi \xrightarrow{\pi_j} \phi'$.
DEFINITION 1.1. Let $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar. For every $\pi \in \Pi^*$ a relation $\Rightarrow^\pi$ on the set of terms over $\Sigma, \mathcal{F}, \rho$ is defined as follows:

1. Let $e \in \Pi^*$ be the empty word. Then $\phi \Rightarrow^e \phi'$ iff $\phi = \phi'$.
2. Let $\pi \in \Pi^*$ and $\pi_i \in \Pi$. Then $\phi \Rightarrow^\pi \phi'$ iff there is a $\phi''$ with $\phi \Rightarrow^\pi \phi''$ and $\phi'' \Rightarrow^{\pi_i} \phi'$.

We will write $\phi \Rightarrow^* \phi'$ iff there is a $\pi \in \Pi^*$ with $\phi \Rightarrow^\pi \phi'$.

DEFINITION 1.2. Let $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar. Then $S\Sigma(G_M) = \{ \pi \mid S(\pi) \Rightarrow^\pi w, w \in \Sigma^*, \pi \in \Pi^* \}$ is called the Szilard language of $G_M$.

For an IO-grammar $G_M$ a homomorphism $\psi: (\Sigma \cup \mathcal{F} \cup \mathcal{V} \cup \{( , ), \})^* \rightarrow (\mathcal{F} \cup \{ [ , ] \})^*$ is defined by

$$
\psi(X) = \begin{cases} 
[ ] & \text{if } X = ( \\
] & \text{if } X = ) \\
X & \text{if } X \in \mathcal{F} \\
e & \text{otherwise}
\end{cases}
$$

and a homomorphism $h: (\Sigma \cup \mathcal{F} \cup \mathcal{V} \cup \{( , ), [ , ] \})^* \rightarrow \mathcal{F}^*$ is defined by

$$
h(X) = \begin{cases} 
X & \text{if } X \in \mathcal{F} \\
e & \text{otherwise}.
\end{cases}
$$

The following two lemmas are given without proof.

LEMMA 1.1. Let $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar and $\phi = \xi_1 F(\sigma_1, ..., \sigma_{\rho(F)} \xi_2$ with $F \in \mathcal{F}$ and $\sigma_i \in \Sigma^*$ for $i \in [1 : \rho(F)]$ a term over $\Sigma, \mathcal{F}, \rho$. Then $\psi(\phi) = [ ] \psi(\xi_1) F[ ] \psi(\xi_2)$ with $h(\xi_i) = h(\psi(\xi_i))$ for $i = 1, 2$ holds.

LEMMA 1.2. Let $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar and $\phi$ a term over $\Sigma, \mathcal{F}, \rho$. If $\psi(\phi) = \eta_1 G[ ] \eta_2$ with $G \in \mathcal{F}$ holds, then $\phi$ has the form $\phi = \xi_1 G(\sigma_1, ..., \sigma_{\rho(G)} \xi_2$ with $\sigma_i \in \Sigma^*$ for $i \in [1 : \rho(G)]$ and $h(\eta_i) = h(\xi_i)$ for $i = 1, 2$.

2. Szilard Languages of IO-Grammars are Context-Sensitive

Let $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar with $\Pi = \{ \pi_1, ..., \pi_n \}$. We now specify a context-sensitive grammar $G_{CS} = (N, T, S', P)$, called the associated context-sensitive grammar of $G_M$. $N$ and $T$ are the following disjoint unions:

$$
N = \mathcal{F} \cup \{ F_a \mid F \in \mathcal{F} \} \cup \{ c_{\pi_i} \mid i \in [1 : n] \} \cup \{ e_{\pi_i} \mid i \in [1 : n] \}
\cup \{ L, R, \upharpoonleft, D, D', S', [ , ] \} \quad \text{and}
$$

$$
T = \Pi \cup \{ d \}.
$$
$P$ consists of the following productions:

(1) $S' \rightarrow LS[ ] \uparrow R$

(2) If $\pi_i$ is the production $F(x_1, \ldots, x_\rho(F)) \rightarrow \Theta_i$ in $G_M$, then $F[\uparrow] \rightarrow c_{\pi_i} F_a D' D'$ is in $P$ for all $i \in [1: n]$.

(2a) $[\uparrow \rightarrow \uparrow[ ]$, $\uparrow[ \rightarrow \uparrow]$ $F' \uparrow \rightarrow \uparrow F$ for all $F \in \mathcal{F}$

(3) $[c_{\pi_i} \rightarrow c_{\pi_i}[ ] ]$

$F \rightarrow F$ for all $F \in \mathcal{F}$

(3a) $\rightarrow \rightarrow$ for all $F \in \mathcal{F}$

(4) $L \rightarrow L \rightarrow$ for all $i \in [1: n]$

(5) $\rightarrow [c_{\pi_i} ]$

$F \rightarrow c_{\pi_i} F$ for all $i \in [1: n]$ and $F \in \mathcal{F}$

(6) $D[ \rightarrow D$

$D \rightarrow ]D$

$DF \rightarrow FD$ for all $F \in \mathcal{F}$

(7) $DDDDR \rightarrow \uparrow Rddd$

(8) If $\pi_i$ is the production $F(x_1, \ldots, x_\rho(F)) \rightarrow \Theta_i$ in $G_M$, then $\rightarrow \psi(\Theta_i) D' D'$ is in $P$ for all $i \in [1: n]$.

(8a) $D' D' D' \rightarrow DDDD$

(9) $L \uparrow R \rightarrow ddd$

Clearly $G_{cs}$ is context-sensitive. The construction of $G_{cs}$ for $G_M$ is analog to the construction of a context-sensitive grammar for a phrase-structure grammar in Moriya (1973).

Let $S' \rightarrow LS[ ] \uparrow R = u_0 \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_m = w$ with $w \in T^*$ be a derivation according to $G_{cs}$. Let $j_0 < j_1 < \cdots < j_k$ be exactly those indices, such that $u_{j_i}$ has the form

$$\pi L u \uparrow R \delta \quad \text{with} \quad \pi \in \Pi^*, \ \delta \in d^*, \ u \in (\mathcal{F} \cup \{[ ], ]\}^*).$$

(2.1)

Obviously $j_0 = 0$ holds.

Now let $u_{j_i}, \ i \in [0: k]$, be given. There are two cases to consider:

(a) $u = e$. We can only apply production (9) to $u_{j_i}$, i.e. $u_{j_i} \xrightarrow{\pi d^* \delta} \pi d^* \delta = w$.

Thus $i = k$ and $j_k = m - 1$ holds.
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(b) \( u \neq e \). First, only productions of type (2a) are applied, until a word of the form
\[
\pi L v_1 F[^1] v_2 R \delta \quad \text{with} \quad v_1, v_2 \in (\mathcal{F} \cup \{[ , ]\})*
\]
is derived. Then a production of type (2) is applied:

(I) \( u_{i+1} \overset{2a}{\Rightarrow} \pi L v_1 F[^1] v_2 R \delta = u_{i+n_1} \)

(II) \( u_{i+n_1} \Rightarrow (\pi L v_1 e \varepsilon_{q}^* F_a D' D' v_2 R \delta \)

Now only productions of type (3) may be applied:

(III) \( u_{i+n_1+1} \overset{3}{\Rightarrow} \pi L v_1 F_a D' D' v_2 R \delta = u_{i+n_2} \)

In this situation only a production of type (4) is applicable:

(IV) \( u_{i+n_2} \Rightarrow (\pi \varepsilon_{q} L v_1 F_a D' D' v_2 R \delta \)

By means of productions of type (5) the \( \varepsilon_{q} \) can be shifted to the left of \( F_a \):

(V) \( u_{i+n_2+1} \overset{5}{\Rightarrow} \pi \varepsilon_{q} L v_1 F_a \delta = u_{i+n_3} \)

Now only a production of type (8) is applicable. Then production (8a) is applied:

(VI) \( u_{i+n_3} \Rightarrow (\pi \psi(\Theta_q) D' D' D' v_2 R \delta \)

(VII) \( u_{i+n_3+1} \Rightarrow (\pi \varepsilon_{q} L v_1 F_a D' D' D' D' v_2 R \delta = u_{i+n_4} \)

By application of productions of type (6) the \( D's \) are shifted to the right:

(VIII) \( u_{i+n_4} \overset{6}{\Rightarrow} \pi \varepsilon_{q} L v_1 D' D' D' D' R \delta = u_{i+n_5} \)

Application of production (7) yields:

(IX) \( u_{i+n_5} \Rightarrow (\pi \varepsilon_{q} L v_1 \psi(\Theta_q) v_2 \delta = u_{i+n_6} \)

LEMMA 2.1. Let the situation of (2.1) be given. If \( u_{i_0} = \pi L u \uparrow R \delta \) with \( \pi = \pi_{r_1} \pi_{r_2} \cdots \pi_{r_t} \) holds, then we have:

1. \( i = q \)
2. In step \((\Pi_{i-1})\) the production \( F[^1] \rightarrow c_{r_1} F_a D' D' \) and in step \((\Pi_{i-1})\) the production \( \tilde{c}_{a r_1} F_a \rightarrow \psi(\Theta_{r_1}) D' D' \) have been applied for \( i \in [1 : t] \).
3. \( \delta \in \delta^* \) with \(| \delta | = 3q \).

Proof. For \( q = 0 \) we have \( u_{i_0} = L S[^1] \uparrow R \) and the assertions are true.

If the assertions hold for \( q - 1 \) then \( u_{i_{q-1}} \) is of the form \( u_{i_{q-1}} = \pi' L u' \uparrow R \delta' \) with \( \pi' = \pi_{r_1} \pi_{r_2} \cdots \pi_{r_{q-1}} \) and \(| \delta' | = 3(q - 1) \).

\( u_{i_q} \) is derived by application of productions according to \((I_{q-1}) - (IX_{q-1})\), hence the assertions are true for \( q \). Especially \( u_{i_k} = \pi_{r_1} \pi_{r_2} \cdots \pi_{r_k} L \uparrow R \delta_{k+1} \) with \(| \delta_{k+1} | = 3 \).

THEOREM 2.1. Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar and \( G_{CS} \).
the associated context-sensitive grammar. Let situation (2.1) be given with
\[ w = \pi_{r_1} \pi_{r_2} \cdots \pi_{r_k} d^{k(k+1)} \]. Then there exists a derivation according to \( G_M \) of the form

\[ \phi_0 = S( ) \Rightarrow \phi_1 \Rightarrow \cdots \Rightarrow \phi_{k-1} \Rightarrow \phi_k \]

with \( \phi_k \in \Sigma^* \).

**Proof.** First it will be shown: There are terms \( \phi_0, \ldots, \phi_k \) over \( \Sigma, \mathcal{F}, \rho \) such that for \( q \in [0; k] \) with \( u_{j_q} = \pi_{r_1} \pi_{r_2} \cdots \pi_{r_q} L \psi(\phi_q) \uparrow R u_{j_q} \) there exists a derivation according to \( G_M \) of the form

\[ \phi_0 = S( ) \Rightarrow \phi_1 \Rightarrow \cdots \Rightarrow \phi_q \quad \text{with} \quad \psi(\phi_q) = u^{(q)} \cdot \]

For \( q = 0 \) we have \( \phi_0 = S( ) \) and \( u_{j_0} = LS[ ] \uparrow R = L \psi(\phi_0) \uparrow R \). Assume \( \phi_0, \ldots, \phi_q \) have just been defined and assume that

\[ u_{j_q} = \pi_{r_1} \cdots \pi_{r_q} L \psi(\phi_q) \uparrow R u_{j_q} \]

and

\[ u_{j_{q+1}} = \pi_{r_1} \cdots \pi_{r_q} \pi_{r_{q+1}} L \psi(\phi_{q+1}) \uparrow R u_{j_{q+1}} \]

and \( \pi_{r_{q+1}} \) is the production \( F(x_1, \ldots, x_{\rho(F)}) \rightarrow \Theta_{q+1} \). In the derivation \( u_{j_q} \Rightarrow u_{j_{q+1}} \) the productions \( F(\cdot) \rightarrow c \pi_{r_q} F_a D'D' \) and \( c \pi_{r_q} F_a \rightarrow \psi(\Theta_{q+1}) D'D' \) have been applied according to Lemma 2.1. Thus, \( \psi(\phi_q) = \eta_1 \psi(\cdot) \eta_2 \) and by Lemma 1.2 \( \phi_q = \xi_1 F(\sigma_1, \ldots, \sigma_{\rho(F)}) \xi_2 \) with \( \sigma_i \in \Sigma^* \) for \( i \in [1; \rho(F)] \) and \( \psi(\xi_i) = \eta_i \) for \( i = 1, 2 \) holds. Hence, \( F(\sigma_1, \ldots, \sigma_{\rho(F)}) \) is an applicable macro in \( \phi_q \). The application of \( \pi_{r_{q+1}} \) yields a term \( \phi_{q+1} = \xi_1 \Theta_{q+1} \xi_2 \), where \( \Theta_{q+1} \) is obtained by substitution of \( x_i \) by \( \sigma_i \) in \( \Theta_{q+1} \) for \( i \in [1; \rho(F)] \). \( u^{(q+1)} \) has the form \( \eta_1 \psi(\Theta_{q+1}) \eta_2 = \eta_1 \psi(\Theta_{q+1}) \eta_2 = \psi(\phi_{q+1}) \).

In particular \( u_{j_{k-1}} = \pi_{r_1} \cdots \pi_{r_k} L u_{j_k} \uparrow R \), i.e. \( \psi(\phi_k) = e \), holds, thus \( \phi_k \in \Sigma^* \).

Conversely we have:

**Theorem 2.2.** Let \( \phi_0 = S( ) \Rightarrow \pi_{r_1} \pi_{r_2} \cdots \pi_{r_{k-1}} \pi_{r_k} \) be a derivation according to \( G_M \) with \( \phi_k \in \Sigma^n \). Then there exists a derivation

\[ S' \Rightarrow LS[ ] \uparrow R = u_0 \Rightarrow u_1 \Rightarrow \cdots \Rightarrow u_m = w \]

according to \( G_{CS} \) with \( w = \pi_{r_1} \cdots \pi_{r_k} d^{k(k+1)} \).

**Proof.** First it will be shown: For \( q \in [0; k] \) there is a derivation \( S' \Rightarrow LS[ ] \uparrow R = u_{j_0} \Rightarrow u_{j_1} \Rightarrow \cdots \Rightarrow u_{j_q} \) according to \( G_{CS} \) with

\[ u_{j_q} = \pi_{r_1} \cdots \pi_{r_q} L \psi(\phi_q) \uparrow R u_{j_q} \].

For \( q = 0 \) we have \( L \psi(\phi_0) \uparrow R = LS[ ] \uparrow R = u_{j_0} \). Let \( u_{j_q} = \pi_{r_1} \cdots \pi_{r_q} L \psi(\phi_q) \uparrow R u_{j_q} \) and let \( \pi_{r_{q+1}} \) be the production \( F(x_1, \ldots, x_{\rho(F)}) \rightarrow \Theta_{q+1} \).
\[ \phi_q = \xi \mathcal{F}(\sigma_1, \ldots, \sigma_{p(\mathcal{F})}) \xi \] with \( \sigma_i \in \Sigma^* \) for \( i \in [1 : p(\mathcal{F})] \) and \( \phi_{q+1} = \xi \tilde{\Theta}_{q+1} \xi \) holds, where \( \tilde{\Theta}_{q+1} \) has the form specified in the proof of Theorem 2.1. According to Lemma 1.1 we have \( \psi(\phi_q) = \eta_1 \mathcal{F}(\ ) \eta_2 \) with \( \eta_i = \psi(\xi) \) for \( i = 1, 2 \). Hence we have

\[ u_{j_k} \xrightarrow{\mathcal{F}[\ ]} \pi_{r_1} \cdots \pi_{r_k} \eta_1 \mathcal{F}(\ ) \eta_2 R^{\phi_q}. \]

Application of \( \mathcal{F}[\mathcal{F}] \rightarrow \pi_{r_1} \mathcal{F}_a D'D' \) necessarily leads to an application of \( \psi(\Theta_{q+1}) D'D' \) which finally yields an \( u_{j_{q+1}} \) of the form \( \pi_{r_1} \cdots \pi_{r_{q+1}} \eta_{t}\eta_{t+1} \mathcal{F}(\Theta_{q+1}) \eta_2 R^{\phi_{q+1}} \) with

\[ \psi(\phi_{q+1}) = \eta_1 \psi(\tilde{\Theta}_{q+1}) \eta_2 = \eta_1 \psi(\Theta_{q+1}) \eta_2 \]

From \( u_{j_k} = \pi_{r_1} \cdots \pi_{r_k} \eta_{t}\eta_{t+1} \mathcal{F}(\phi_{q+1}) R^{\phi_{q+1}} \) and \( \psi(\phi_{q+1}) = e \) the theorem follows.

**Corollary 2.1.** Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar. Then \( S_z(G_M) \) is context-sensitive.

**Proof.** Consider the homomorphism \( f: (\Pi \cup \{d\})^* \rightarrow \Pi^* \) with

\[ f(a) = \begin{cases} a & \text{if } a \in \Pi \\ e & \text{if } a = d \end{cases} \]

We have \( S_z(G_M) = f(L(G_{CS})) \), and for all \( w \in L(G_{CS}) \) the following holds: \( |f(w)| \geq |w| \). Thus, \( S_z(G_M) \) is context-sensitive (cf. Ginsburg and Greibach, 1966).

### 3. L-Normal Form for IO-Grammars

First some notations will be defined. Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar and \( \phi \) a term over \( \Sigma, \mathcal{F}, \rho \). An applicable macro \( \mathcal{F}(\sigma_1, \ldots, \sigma_{p(\mathcal{F})}) \) in \( \phi \) is called **leftmost applicable macro** or **LA-macro** for short if no applicable macro in \( \phi \) occurs to the left of it. The notations "directly left derivable" or "L-derivable" and "left derivation" are defined in an obvious way. We write \( \phi \Rightarrow \phi' \) if \( \phi' \) is derivable from \( \phi \) by application of \( \pi_i \) to the leftmost applicable macro in \( \phi \). The relation "\( \Rightarrow \)" for all \( \pi \in \Pi^* \) is analogue to the relation "\( \Rightarrow \)". We write \( \phi \Rightarrow \phi' \) if there is a \( \pi \in \Pi^* \) with \( \phi \Rightarrow_\pi \phi' \).

**Definition 3.1.** Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar. Then \( S_z(G_M) = \{ \pi \mid S(\ ) \Rightarrow w, w \in \Sigma^*, \pi \in \Pi^* \} \) is called **leftmost Szilard language** of \( G_M \).

In the following the notation of "L-normal form" for IO-grammars will be defined. Then we will show that every derivation according to an IO-grammar in L-normal form is a left derivation.
DEFINITION 3.2. An IO-grammar \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) is called an IO-grammar in L-normal form, if for all \( \pi \in \Pi \) the following holds:

\( \pi \) is either of the form

(A) \( F(x_1, \ldots, x_{\rho(F)}) \rightarrow G(x_1, \ldots, x_{\rho(F)}, H(\sigma_1, \ldots, \sigma_{\rho(H)})) \) with \( x_i \in \mathcal{V} \) for \( i \in [1: \rho(F)] \) and \( \sigma_i \in (\Sigma \cup \mathcal{V})^* \) for \( i \in [1: \rho(H)] \) or

(B) \( F(x_1, \ldots, x_{\rho(F)}) \rightarrow t \) with \( x_i \in \mathcal{V} \) for \( i \in [1: \rho(F)] \) and \( t \in (\Sigma \cup \{x_1, \ldots, x_{\rho(F)}\})^* \).

THEOREM 3.1. Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar in L-normal form. If \( S(\ ) \Rightarrow^* v \) holds with \( \pi \in \Pi^* \) and \( v \notin \Sigma^* \), then \( v \) has the form \( \xi F(r_1, \ldots, r_{\rho(F)}) \xi' \) with \( F \in \mathcal{F}, \tau_i \in \Sigma^* \) for \( i \in [1: \rho(F)] \), and \( \xi = F_1(w_1 \cdots w_{\rho(F)} \cdots F_k w_k) \) with \( F_j \in \mathcal{F}, \tau_j \in (\Sigma \cup \{\})^* \) for \( j \in [1: k] \) and \( k \geq 0 \).

Proof. The assertion holds for \( |\pi| = 0 \). Now, assume the induction hypothesis is true for all \( S(\ ) \Rightarrow^* v \) with \( |\pi| \leq m \). Let \( \pi' = \pi \tau_k \) with \( |\pi'| = m \) and \( \tau_k \in \Pi \). If \( S(\ ) \Rightarrow^* v, v \notin \Sigma^* \) holds, then there exists a \( v' \) with \( S(\ ) \Rightarrow^* v' \Rightarrow^* v \).

(1) Let \( \pi_k \) be of the form (A), then

\( v = \xi G(\tau_1, \ldots, \tau_{\rho(F)}, H(\sigma'_1, \ldots, \sigma'_{\rho(H)})) \xi' \),

where \( \sigma'_i \) is obtained by substitution of \( x_j \) by \( \tau_j \) in \( \sigma_i \) for \( i \in [1: \rho(H)] \) and \( j \in [1: \rho(F)] \).

(2) Let \( \pi_k \) be of the form (B), then

\( v = \xi v' \xi' \) with \( v' \in \Sigma^* \) and \( \xi \neq e \) (for otherwise \( \xi' = e \) and \( v \in \Sigma^* \)).

Hence \( k > 0 \) and \( v = \xi F_k(w_k \gamma) \xi' \in \Sigma^* \) with \( \xi' \in \{\}^* \) and \( \xi = F_1(w_1 \cdots F_{k-1}(w_{k-1}) \). Since \( v \) is a term over \( \Sigma, \mathcal{F}, \rho \), we have

\( F_k(w_k \gamma) = F_k(\kappa_1, \ldots, \kappa_{\rho(F)}) \) with \( \kappa_i \in \Sigma^* \).

COROLLARY 3.1. Let \( G_M \) be an IO-grammar in L-normal form. Then every sentential form according to \( G_M \) has at most one applicable macro, and every derivation according to \( G_M \) is a left derivation.

Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar. A term \( F(\sigma_1, \ldots, \sigma_{\rho(F)}) \) over \( \Sigma \cup \mathcal{V}, \mathcal{F}, \rho \) is called a \( \mathcal{V} \)-macro. A \( \mathcal{V} \)-macro \( F(\sigma_1, \ldots, \sigma_{\rho(F)}) \) with \( \sigma_i \in (\Sigma \cup \mathcal{V})^* \) is called applicable \( \mathcal{V} \)-macro. Let \( \phi \) be term over \( \Sigma \cup \mathcal{V}, \mathcal{F}, \rho \). An applicable \( \mathcal{V} \)-macro \( F(\sigma_1, \ldots, \sigma_{\rho(F)}) \) in \( \phi \) is called leftmost applicable \( \mathcal{V} \)-macro in \( \phi \), if no applicable \( \mathcal{V} \)-macro occurs to the left of it.

Remark. Let \( \phi = \xi \eta F(\sigma_1, \ldots, \sigma_{\rho(F)}) \xi_2 \) be a term over \( \Sigma \cup \mathcal{V}, \mathcal{F}, \rho \) and \( y \notin \Sigma \cup \mathcal{V} \cup \mathcal{F} \), then \( \phi' = \xi_1 y \xi_2 \) is a term over \( \Sigma \cup (\mathcal{V} \cup \{y\}), \mathcal{F}, \rho \). Given
an IO-grammar $G_M$, the following algorithm constructs an IO-grammar
$G'_M$ in $L$-normal form, and $L(G_M) = L(G'_M)$ will be shown.

**Algorithm 3.1.**

**Input.** An IO-grammar $G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi)$ with $\Pi = \{\pi_1, ..., \pi_n\}$ and $\mathcal{V} = \{x_1, ..., x_k\}$.

**Output.** An IO-grammar $G'_M$ in $L$-normal form with $L(G_M) = L(G'_M)$.

**Method.** A sequence $G^{(1,0)}, ..., G^{(1,r_1)}, ..., G^{(n,r_n)}$ of IO-
grammars with $r_i \geq 0$ for $i \in [1:n]$ and $G^{(i,i)} = (\Sigma, \mathcal{F}_i, \mathcal{V}, \rho_i, S, \Pi_i)$ will be constructed.

1. Let $r$ be the maximal number of occurrences of $\mathcal{V}$-macros in the right
sides of productions of $\Pi$. Set $\mathcal{V} = \{x_1, ..., x_k, x_{k+1}, ..., x_{k+r}\}$, where $x_{k+1}, ..., x_{k+r}$
are new variables.

2. Set $G^{(1,0)} = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) = (\Sigma, \mathcal{F}_0, \mathcal{V}, \rho_0, S, \Pi_0)$.

3. Suppose we have constructed $G^{(i,0)}$. Now, construct a sequence
$G^{(i,1)}, ..., G^{(i,r_i)}$ of IO-grammars with $r_i \geq 0$ in the following way:

   If $\pi_i$ is of the form (A) or (B), set $r_i = 0$ and $G^{(i+1,0)} = G^{(i,0)}$ and go to (5).

   If $\pi_i : F(x_1, ..., x_{\rho(F)}) \rightarrow \Theta_i$ is not of the form (A) or (B), set
$G^{(i,1)} = (\Sigma, \mathcal{F}_i, \mathcal{V}, \rho_i, S, \Pi_i)$ with $\mathcal{F}_i = \mathcal{F}_0 \cup \{H_0\}$, where $H_0$ is a new function
symbol, $\rho_0 \mid \mathcal{F}_0 = \rho_0$ and $\rho_0(H_0) = \rho(F) + 1$. ($\rho_1 \mid \mathcal{F}_0$ is the restriction of
$\rho_1$ to $\mathcal{F}_0$). $\Pi_0 = (\Pi_0 \{\pi_0\} \cup \{\pi_0, \pi_1\}$ with

   \[\pi_0 : F(x_1, ..., x_{\rho(F)}) \rightarrow H_0(x_1, ..., x_{\rho(F)}, A(\sigma_1, ..., \sigma_{\rho(A)}))\]

   and

   \[\pi_1 : H_0(x_1, ..., x_{\rho(F)}, A(\sigma_1, ..., \sigma_{\rho(A)})) \rightarrow \Theta_1,\]

   if $\Theta_1 = \xi_1 A(\sigma_1, ..., \sigma_{\rho(A)})$ holds and $A(\sigma_1, ..., \sigma_{\rho(A)})$ is the leftmost applicable
$\mathcal{V}$-macro in $\Theta_i$.

4. Suppose we have constructed $G^{(i,j)}$ with

   \[\pi_i : H^{(i,j)}(x_1, ..., x_{\rho(F)}, ..., x_{\rho(F)+j}) \rightarrow \Theta_i\]

   The number of occurrences of $\mathcal{V}$-macros in $\Theta_i$ is $r_i - j$. If $\pi_i$ is of the form
(A) or (B), set $r_i = j$ and $G^{(i+1,0)} = G^{(i,r_i)}$ and go to (5).

   If $\pi_i$ is not of the form (A) or (B), set $G^{(i+1,j+1)} = (\Sigma, \mathcal{F}_i^{j+1}, \mathcal{V}, \rho_i^{j+1}, S, \Pi_i^{j+1})$
with $\mathcal{F}_i^{j+1} = \mathcal{F}_i \cup \{H_i^{j+1}\}$, where $H_i^{j+1}$ is a new function symbol,
$\rho_i^{j+1} \mid \mathcal{F}_i^{j+1} = \rho_i^{j+1}$ and $\rho_i^{j+1}(H_i^{j+1}) = \rho_i(H_i) + 1 = \rho(F) + j + 1$. Construct $\Pi_i^{j+1}$
in the following way: First remove $\pi_i$ from $\Pi_i^{j+1}$ and then add the following two productions:
\[ \pi_i^j: H_i^j(x_1, \ldots, x_{\rho(F)}, \ldots, x_{\rho(F)+j}) \rightarrow H_i^{j+1}(x_1, \ldots, x_{\rho(F)+j}, A^j(\sigma^j_1, \ldots, \sigma^j_{\rho(A)})) \]

and

\[ \pi_i^{j+1}: H_i^{j+1}(x_1, \ldots, x_{\rho(F)+j+1}) \rightarrow \xi_i^j x_{\rho(F)+j+1} \xi_{i+1} = \Theta_i^{j+1}, \]

if \( \Theta_i^j = \xi_i^j A^j(\sigma^j_1, \ldots, \sigma^j_{\rho(A)}) \xi_{i+1}^j \) holds and \( A^j(\sigma^j_1, \ldots, \sigma^j_{\rho(A)}) \) is the leftmost applicable \( \mathcal{F} \)-macro in \( \Theta_i^j \).

The number of occurrences of \( \mathcal{F} \)-macros in \( \Theta_i^{j+1} \) is one less than in \( \Theta_i^j \).

Go to (4) with \( j := j + 1 \).

(5) If \( i = n \), set \( G'_M = G^{(n+1,0)} \). Halt. If \( i < n \), go to (3) with \( i := i + 1 \).

Obviously, \( G'_M \) is an IO-grammar in \( L \)-normal form. A similar normal form was derived in Duske et al. (1977) in a different manner. It remains to show that \( L(G_M) = L(G'_M) \) holds.

**Lemma 3.1.** Let \( G_M \) be an IO-grammar and \( S( ) \Rightarrow w \) a derivation according to \( G_M \) of a word \( w \in L(G_M) \). Then \( S( ) \Rightarrow^* w \) holds too.

**Theorem 3.2.** Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be an IO-grammar with \( \mathcal{V} = \{x_1, \ldots, x_q\} \). Let

\[ \pi: F(x_1, \ldots, x_{\rho(F)}) \rightarrow \xi_1 A(\sigma_1, \ldots, \sigma_{\rho(A)}) \xi_2 = \Theta \]

be a production of \( \Pi \) with \( q > \rho(F) \), where \( A(\sigma_1, \ldots, \sigma_{\rho(A)}) \) is the leftmost applicable \( \mathcal{V} \)-macro in \( \Theta \). Let \( G_M = (\Sigma, \mathcal{F}, \mathcal{V}, \rho, S, \Pi) \) be the IO-grammar with \( \mathcal{F} = \mathcal{F} \cup \{H\} \), where \( H \) is a new function symbol, and \( \rho(H) = \rho(F) + 1 \) and \( \Pi = (\Pi \setminus \{\pi\}) \cup \{\pi_1, \pi_2\} \) with

\[ \pi_1: F(x_1, \ldots, x_{\rho(F)}) \rightarrow H(x_1, \ldots, x_{\rho(F)}, A(\sigma_1, \ldots, \sigma_{\rho(A)})) \]

and

\[ \pi_2: H(x_1, \ldots, x_{\rho(F)}, x_{\rho(F)+1}) \rightarrow \xi_1 x_{\rho(F)+1} \xi_2. \]

Then \( L(G_M) = L(G'_M) \) holds.

**Proof.** (a) \( L(G_M) \subseteq L(G'_M) \).

Let \( w \in L(G'_M) \) and let

\[ S( ) = \phi_0 \rightarrow \phi_1 \rightarrow \phi_2 \rightarrow \cdots \rightarrow \phi_j \rightarrow \phi_{j+1} \rightarrow \cdots \rightarrow \phi_m = w \quad (3.1) \]

be a left derivation of \( w \) according to \( G_M \). If \( \pi \) is not applied, then (3.1) is a left derivation of \( w \) according to \( G_M \) too. If \( \pi_j = \pi \), then we have \( \phi_j = \gamma_j F(\eta_1, \ldots, \eta_{\rho(F)}) \gamma_2 \) with \( \eta_i \in \Sigma^* \) for \( i \in [1: \rho(F)] \) and \( \phi_{j+1} = \gamma_j \xi_j A(\sigma^j_1, \ldots, \sigma^j_{\rho(A)}) \xi_2 \gamma_2 \), where \( \xi_j, \xi_2, \sigma^j_i \) and \( \gamma_i \) for \( i \in [1: \rho(A)] \) are obtained by substitution of \( x_i \) by \( \eta_i \) in \( \Theta \).
$A(\sigma'_1, \ldots, \sigma'_{\rho(A)})$ is the leftmost applicable macro in $\phi_{j+1}$. This implies the existence of a subsequence $\phi_{j+1} \Rightarrow \phi_{j+2} \cdots \Rightarrow \phi_k$ of (3.1) with $\phi_k = \gamma_1 \xi_1 \eta_2 \xi_2 \gamma_2$, $\eta \in \Sigma^s$ and $A(\sigma'_1, \ldots, \sigma'_{\rho(A)})$\Rightarrow \phi_{j+1} \Rightarrow \cdots \Rightarrow \phi_k$ of (3.1) with $\phi_k = \gamma_1 \xi_1 \eta_2 \xi_2 \gamma_2$, $\eta \in \Sigma^s$ and $A(\sigma'_1, \ldots, \sigma'_{\rho(A)})$. Then the following derivation is possible:

$$S( ) = \phi_0 \Rightarrow \phi_1 \Rightarrow \cdots \Rightarrow \phi_j \Rightarrow \gamma_1 H(\eta_1, \ldots, \eta_{\rho(f)}, A(\sigma'_1, \ldots, \sigma'_{\rho(A)})) \Rightarrow \gamma_2 \Rightarrow \cdots \Rightarrow \phi_m = w.$$ (3.2)

This is a left derivation by means of productions of $\Pi \cup \{\pi^1, \pi^2\}$. If $\pi$ is not applied, then this derivation is a left derivation of $w$ according to $G_M$. If $\pi$ is applied, repeat the procedure.

(b) $L(G_M) \subseteq L(G_M)$.

Let $w \in L(G_M)$ and let

$$S( ) = \phi_0 \Rightarrow \phi_1 \Rightarrow \cdots \Rightarrow \phi_j \Rightarrow \gamma_1 H(\eta_1, \ldots, \eta_{\rho(f)}, A(\sigma'_1, \ldots, \sigma'_{\rho(A)})) \Rightarrow \gamma_2 \Rightarrow \cdots \Rightarrow \phi_m = w$$ (3.2)

be a left derivation of $w$ according to $G_M$.

If only productions from $\Pi \setminus \{\pi\}$ are applied, then (3.2) is a left derivation of $w$ according to $G_M$ too. If $\phi_{k+1}$ is obtained from $\phi_k$ by application of $\pi^2$, then we have:

$$\phi_k = \gamma_1 H(\eta_1, \ldots, \eta_{\rho(f)}, \eta) \Rightarrow \gamma_1 \xi_1 \eta_2 \gamma_2 = \phi_{k+1}.$$ (3.2)

For this occurrence of $H$ in $\phi_k$ the following holds: There is exactly one $j < k$, such that $H$ is introduced by application of $\pi_j = \pi^1$ to $\phi_j$, i.e.

$$\phi_j = \gamma_1 F(\eta_1, \ldots, \eta_{\rho(f)}) \Rightarrow \gamma_1 H(\eta_1, \ldots, \eta_{\rho(f)}, A(\sigma'_1, \ldots, \sigma'_{\rho(A)})) \Rightarrow \phi_{j+1}.$$ (3.2)

$A(\sigma'_1, \ldots, \sigma'_{\rho(A)})$ is the leftmost applicable macro in $\phi_{j+1}$, and the following holds:

$$\phi_{j+1} = \gamma_1 H(\eta_1, \ldots, \eta_{\rho(f)}, A(\sigma'_1, \ldots, \sigma'_{\rho(A)})) \Rightarrow \gamma_1 \xi_1 H(\eta_1, \ldots, \eta_{\rho(f)}, \eta) \Rightarrow \phi_{k+1}.$$ (3.2)

Then the following derivation is possible:

$$S( ) = \phi_0 \Rightarrow \phi_1 \Rightarrow \cdots \Rightarrow \phi_j \Rightarrow \gamma_1 \xi_1 A(\sigma'_1, \ldots, \sigma'_{\rho(A)}) \Rightarrow \gamma_2 \Rightarrow \cdots \Rightarrow \phi_m = w.$$ (3.2)

This is a left derivation of $w$ by means of productions from $\Pi \cup \{\pi\}$. If $\pi^1, \pi^2$ are not applied, then this derivation is a left derivation according to $G_M$. If $\pi^1, \pi^2$ are applied, repeat the procedure.
Corollary 3.3. Let $G_M$ and $G'_M$ be the IO-grammars from Algorithm 3.1. Then $L(G_M) = L(G'_M)$ holds.

Corollary 3.4. Let $G_M$ be an IO-grammar. Then there exists an IO-grammar $G'_M$ in L-normal form with $L(G_M) = L(G'_M)$.

4. Szilard Languages of IO-Grammars in L-Normal Form Are Context-Free

Let $G_M = (\Sigma, F, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar in L-normal form with $\Pi = \{\pi_1, \ldots, \pi_n\}$. The context-free grammar $G_{CF} = (\mathcal{F}, \Pi, S, P)$, where $P = \{p_1, \ldots, p_n\}$ with $p_i : F \to \pi_i H$, if

$$\pi_i : F(x_1, \ldots, x_{\rho(F)}) \to H(x_1, \ldots, x_{\rho(F)}, G(\sigma_1, \ldots, \sigma_{\rho(G)}))$$

or

$$p_i : F \to \pi_i, \quad \text{if} \quad \pi_i : F(x_1, \ldots, x_{\rho(F)}) \to t, \quad t \in (\Sigma \cup \{x_1, \ldots, x_{\rho(F)}\})^*,$$

is called the associated context-free grammar of $G_M$.

Theorem 4.1. Let $G_M = (\Sigma, F, \mathcal{V}, \rho, S, \Pi)$ be an IO-grammar in L-normal form and $G_{CF} = (\mathcal{F}, \Pi, S, P)$ the associated context-free grammar of $G_M$. Let

$$S(\ ) = \phi_0 \pi_1 \Rightarrow \phi_1 \pi_2 \Rightarrow \phi_2 \cdots \pi_k \Rightarrow \phi_k$$

be a derivation of $\phi_k$ according to $G_M$ with $h(\phi_k) = F_1 \cdots F_q$, $q \geq 0$ and $k \geq 0$. Then there exists a left derivation

$$S = \alpha_0 p_1 \Rightarrow \alpha_1 p_2 \Rightarrow \alpha_2 \cdots p_k \Rightarrow \alpha_k$$

according to $G_{CF}$ with $\alpha_k = \pi_{i_1} \cdots \pi_{i_k} F_1 \cdots F_q$.

Proof. The assertion is true for $k = 0$. Let

$$S(\ ) = \phi_0 \pi_{i_1} \Rightarrow \phi_1 \pi_{i_2} \Rightarrow \phi_2 \cdots \pi_{i_k} \Rightarrow \phi_k \pi_{i_{k+1}} \Rightarrow \phi_{k+1}$$

be a derivation according to $G_M$ with $h(\phi_k) = F_1 \cdots F_q$, $q \geq 1$. From Theorem 3.1 it follows that:

$$h(\phi_{k+1}) = F_1 \cdots F_{q-1} G^1 G^2, \quad \text{if} \quad \pi_{i_{k+1}} \text{ is of type (A)}$$

or

$$h(\phi_{k+1}) = F_1 \cdots F_{q-1}, \quad \text{if} \quad \pi_{i_{k+1}} \text{ is of type (B)}.$$

By induction hypothesis there exists a left derivation

$$S = \alpha_0 p_{i_1} \Rightarrow \alpha_1 p_{i_2} \Rightarrow \alpha_2 \cdots p_i \Rightarrow \alpha_k.$$
according to $G_{CF}$ with $\alpha_k = \pi_{i_1} \cdots \pi_{i_k} F^q \cdots F^1$. If $\pi_{i_{k+1}}$ is of type (A), $p_{i_{k+1}}$ has the form $F^q \rightarrow \pi_{i_{k+1}} G^2 G^1$. Hence the following left derivation

$$S = \alpha_0 p_{i_1} \Rightarrow \alpha_1 p_{i_2} \Rightarrow \alpha_2 \cdots p_{i_k} \Rightarrow \alpha_k p_{i_{k+1}} \Rightarrow \alpha_{k+1}$$

according to $G_{CF}$ exists with $\alpha_{k+1} = \pi_{i_1} \cdots \pi_{i_k} \pi_{i_{k+1}} G^2 G^1 F^{q-1} \cdots F^1$. If $\pi_{i_{k+1}}$ is of type (B), the reasoning is similar. On the converse we have

**Theorem 4.2.** $G_M$ and $G_{CF}$ are defined as in Theorem 4.1. Let

$$S = \alpha_0 p_{i_1} \Rightarrow \alpha_1 p_{i_2} \Rightarrow \alpha_2 \cdots p_{i_k} \Rightarrow \alpha_k$$

be a derivation according to $G_{CF}$. Then $\alpha_k = \pi_{i_1} \cdots \pi_{i_k} F^q \cdots F^1$, $q > 0$ holds, and there exists a derivation

$$S(\ ) = \phi_0 \pi_{i_1} \Rightarrow \phi_1 \pi_{i_2} \Rightarrow \phi_2 \cdots \pi_{i_k} \Rightarrow \phi_k$$

according to $G_M$ with $h(\phi_k) = F^1 \cdots F^q$.

**Proof.** The assertion holds for $k = 0$. Let

$$S = \alpha_0 p_{i_1} \Rightarrow \alpha_1 p_{i_2} \Rightarrow \alpha_2 \cdots p_{i_k} \Rightarrow \alpha_k$$

be a left derivation according to $G_{CF}$. From the induction hypothesis we have $\alpha_k = \pi_{i_1} \cdots \pi_{i_k} F^q \cdots F^1$, and there exists a derivation

$$S(\ ) = \phi_0 \pi_{i_1} \Rightarrow \phi_1 \pi_{i_2} \Rightarrow \phi_2 \cdots \pi_{i_k} \Rightarrow \phi_k$$

according to $G_M$ with $h(\phi_k) = F^1 \cdots F^q$. If $p_{i_{k+1}}$ is the production $F^q \rightarrow \pi_{i_{k+1}} G^2 G^1$, then $\alpha_{k+1} = \pi_{i_1} \cdots \pi_{i_k} \pi_{i_{k+1}} G^2 G^1 F^{q-1} \cdots F^1$. $\pi_{i_{k+1}}$ is the production

$$F^q(x_1, \ldots, x_{\rho(p_1)}) \rightarrow G^1(x_1, \ldots, x_{\rho(p)}) , G^2(\tau_1, \ldots, \tau_{\rho(G^2)})).$$

By Theorem 3.1 $\pi_{i_{k+1}}$ is applicable to $\phi_k$, i.e., there exists the derivation

$$S(\ ) = \phi_0 \pi_{i_1} \Rightarrow \phi_1 \pi_{i_2} \Rightarrow \phi_2 \cdots \pi_{i_k} \Rightarrow \phi_k \pi_{i_{k+1}} \Rightarrow \phi_{k+1}$$

according to $G_M$ with $h(\phi_{k+1}) = F^1 \cdots F^{q-1} G^1 G^2$. In case $p_{i_{k+1}}$ is of the form $F^q \rightarrow \pi_{i_{k+1}}$, the proof is similar.

Now, it follows immediately:

**Corollary 4.1.** Let $G_M$ be an IO-grammar in L-normal form and $G_{CF}$ the associated context-free grammar. Then $L(G_{CF}) = Sz(G_M)$ holds.

By Corollary 3.4 each IO-language is generated by an IO-grammar in L-normal form, hence
Corollary 4.2. For each IO-language $L$ there exists an IO-grammar $G_M$ with $L = L(G_M)$, such that $Sz(G_M)$ is context-free.

It is well known that there exists a context-free language, such that every context-free grammar generating this language has a non-context-free Szilard language. Since every context-free language is an IO-language, we can state however:

Corollary 4.3. For every context-free language $L$ there exists an IO-grammar $G_M$ with $L = L(G_M)$, such that $Sz(G_M)$ is context-free.

Thus an IO-macro-derivation mechanism for context-free languages yields the existence of simpler Szilard languages.

Remark. In Duske et al. (1977) a context free grammar $G_{CF}$ was assigned to an arbitrary IO-grammar $G_M$ in such a manner that for each production from $G_M$ there is exactly one production in $G_{CF}$, and the left derivations according to $G_M$ are in a 1-1-correspondence to the left derivations according to $G_{CF}$. This implies $Sz_L(G_M) = Sz_L(G_{CF})$.

Hence $Sz_L(G_M)$ is context-free for an arbitrary IO-grammar $G_M$.

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