JOURNAL OF COMBINATORIAL THEORY 9, 60–71 (1970)

Tiling with Sets of Polyominoes

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Communicated by Gian-Carlo Rota

Received October 1968

ABSTRACT

The definitions and lattice hierarchy previously established for tiling regions with individual polyominoes are extended to finite sets of polyominoes. The problem of tiling the infinite plane with replicas of a finite set of polyominoes is proved to be logically equivalent to Wang's "domino problem," which is known to be algorithmically undecidable. Several different ways of extending the notion of rep-tility from single polyominoes to sets of polyominoes are discussed. Some related results of Ikeno regarding tiling with polyiamonds (shapes composed of equilateral triangles) are mentioned.

I. INTRODUCTION

In a previous article, entitled "Tiling with Polyominoes" [1] a lattice of tiling capabilities for polyominoes was established, based on the subregions of the plane which can or cannot be covered with replicas of an individual polyomino. The ability of a polyomino to tile an enlarged version of itself (the "rep-tile" property) was also fitted into this hierarchy. The discussion was not definitive in that several positions in the lattice were not proved to have separate existence, nor could they be shown to be logically equivalent to other positions in the lattice for which characteristic examples were given.

In the present article, we achieve the following results:

(a) The definitions of tiling capability, including rep-tility, are generalized from *one* polyomino to an arbitrary finite set of polyominoes.

* This research was supported in part by the Air Force Office of Scientific Research under Grant No. AFOSR-68-1555-A.

(b) The lattice hierarchy developed for the tiling capabilities of individual polyominoes is shown to hold for sets of polyominoes.

(c) For sets of polyominoes, it is possible to show the independent existence of each position in the lattice, except for exhibiting a rep-tile set which will not tile a rectangle. In fact, sets with at most *two* members suffice for the characteristic examples.

(d) The problem of tiling the infinite plane with a specified finite set of polyominoes is shown to be logically equivalent to Hao Wang's "domino problem" [2] involving tiling with sets of MacMahon squares [3] and, hence, is logically undecidable.

(e) Specifically, it is shown that the set of all (finite) MacMahon sets is tile-isomorphic to a subset of the set of all (finite) polyomino sets; and, conversely, that the set of all (finite) polyomino sets is tile-isomorphic (by a different isomorphism) to a subset of the set of all (finite) MacMahon sets.

(f) Some results obtained by Ikeno on tiling with polyiamonds (shapes made up of equilateral triangle) are mentioned.

2. The Tiling Ability of Sets

Let S be a finite set of polyomino shapes. We say that a region R can be *tiled* by S if replicas of the members of S can be used to tile R. Here we require all the replicas of all the members of S to be to the same scale (i.e., to have the same sized unit square), but we allow arbitrary rotation, reflection, and translation of the replicas, and arbitrary quantities of each shape in S (including *none* of certain shapes), in building any region geometrically similar to the specified region R.

It is clear that if a set S tiles a quadrant it surely tiles a half plane, and *a fortiori* it tiles the full plane. Similarly all the other lattice relationships described in "Tiling with Polyominoes" involving subregions of the plane hold for tiling with sets of polyominoes.

We may further define a set S of polyominoes to have the weak rep-tile property if every member of S, regarded as a region, can be tiled by the set S. We say that S has the strong rep-tile property if the regions corresponding to members of S can all be tiled to a *common scale* by members of S. It is only when the strong rep-tile property holds that the analogy with ordinary rep-tiles is properly maintained, in the sense that the rep-tilic subdivision can be iterated an arbitrary number of times. That is, if only the weak rep-tile property holds, there may be a shape A in S

which requires B and C in its first rep-tilic subdivision, where B and C do not have rep-tilic subdivisions to a common scale, in which case the rep-tilic subdivision of A cannot be iterated. For this reason we will mean the *strong rep-tile property* whenever we say that a set S has the rep-tile property.

If a set S divides (i.e., tiles) a rectangle, this rectangle can be used to tile a square, and this square can be used to tile each of the polyomino members of S. It is also easy to show that the proof that the rep-tile property implies quadrant tiling carries over from the case of individual shapes to the case of tiling with *sets*. Hence, we can reintroduce the lattice diagram from [1] in Figure 1 as the tiling hierarchy for sets of polyominoes. The two non-trivial implications are that bent strip implies strip, and that rep-tile implies quadrant. However, the proofs given in [1] carry over in both cases without need of new ideas. On the other hand,

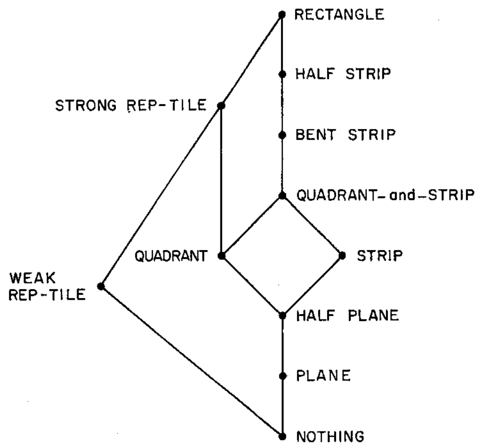

FIGURE 1. The lattice of tiling capabilities for sets of polyominoes.

the weak rep-tile property need not imply quadrant tiling, since the proof utilizes the possibility of repeated iteration of the rep-tilic subdivision process.

In Figure 2a, we see an example of a rep-tilic set of polyominoes. This example is non-trivial in that neither polyomino possesses the rep-tilic property by itself. However, this set can also tile a rectangle, as can every rep-tilic set found thus far. It should also be mentioned that no explicit example has yet been found of a set of polyominoes which possesses the weak rep-tile property but not the strong rep-tile property. If such a set exists, it is not even evident that it can tile the plane! Hence the position of the weak rep-tile property in Figure 1.

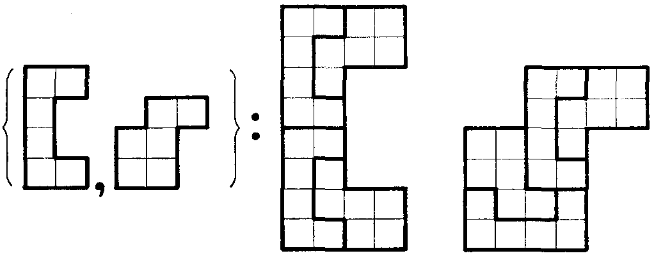

FIGURE 2a. *A rep-tilic set consisting of two hexominoes.*

In Figure 2b, we see a rep-tilic set consisting of two pentominoes, neither of which by itself is rep-tilic. Since they tile the 10 × 10 square as shown, five such squares can be used to make a replica of either of them, enlarged by a factor of 10 in each dimension. The reader is invited to investigate which pairs of pentominoes can be used to tile rectangles.

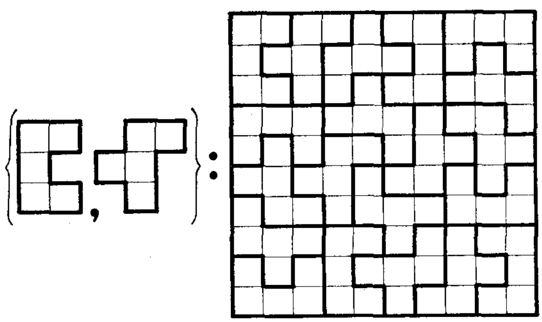

FIGURE 2b. A set of two pentominoes which tiles a square.

In Figure 3, we see an example of the phenomenon of tiling capability depending on the distinction between uniform and non-uniform scale. The set of 3 hexominoes tiles a rectangle, and is thus strongly rep-tilic. However, two of them tile the third! This set of 2 is not even weakly rep-tilic, however, because the *first* rep-tilic subdivisions of these figures would require unequal "unit squares."

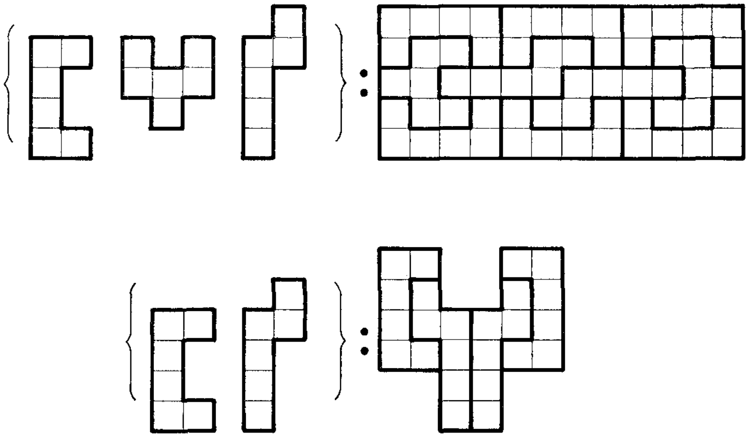

FIGURE 3. A set of three hexominoes which tiles a rectangle, where two of them can tile the third. The set of two is not rep-tilic.

3. CHARACTERISTIC EXAMPLES

In [1], no characteristic examples of individual polyominoes were given for the levels "Half-Plane," "Quadrant," "Quadrant-and-Strip," and "Half-Strip" in the hierarchy (Figure 1) nor for the "rep-tile" position. Using *two* polyominoes in the set, characteristic examples will now be given for the Half-Plane, the Quadrant, the Quadrant-and-Strip and the Half-Strip.

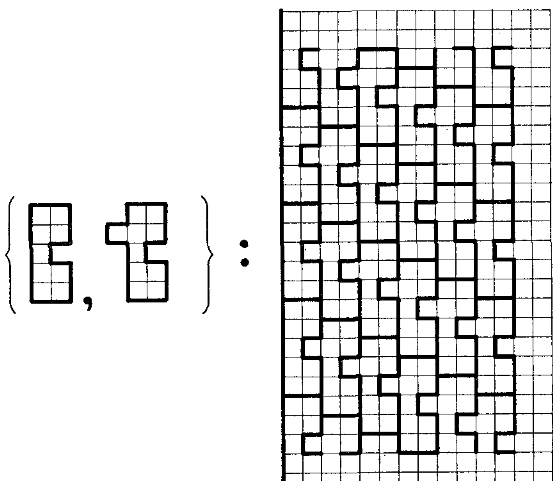

FIGURE 4. A characteristic example for the Half-Plane.

(a) *The Half-Plane.* In Figure 4, we see a pair of polyominoes which can be used to tile the half-plane, as shown. On the other hand, it is quickly verified that this set will tile neither the quadrant nor the strip because of the way the shapes are notched.

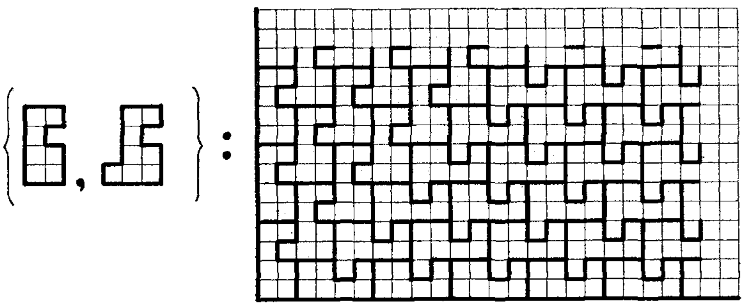

FIGURE 5. A characteristic example for the Quadrant.

(b) *The Quadrant.* In Figure 5, a tiling for the quadrant, using two polyominoes, is exhibited. Since it is easily shown that the notches prevent the formation of a strip, this example is characteristic for the quadrant.

(c) *Quadrant-and-Strip.* In both Figure 6 and Figure 7, examples are presented in which a set of two polyominoes can tile both a quadrant and a strip. (In fact, a single polyomino suffices for the strip construction in both cases.)

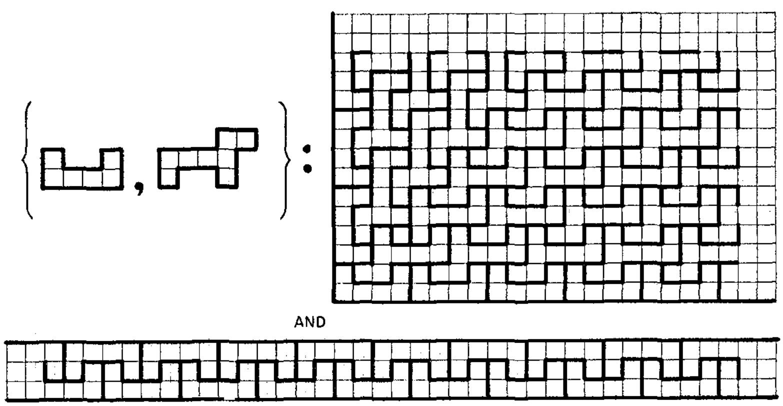

FIGURE 6. The dog-and-trough characteristic example for the Quadrant-and-Strip.

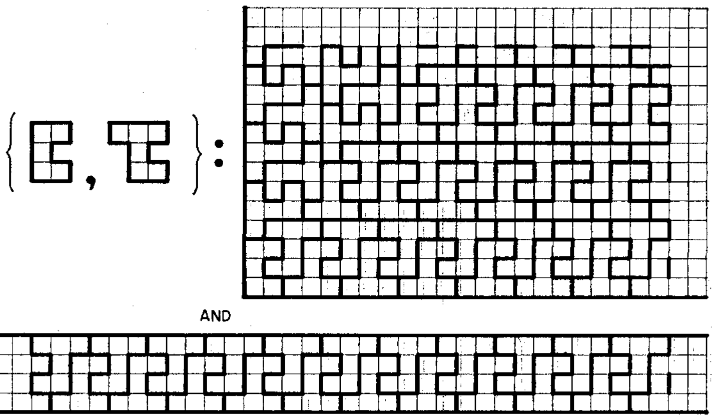

FIGURE 7. The U-pentomino and chair hexomino example for the Quadrant-and-Strip.

For the example in Figure 6, it is rather easy to verify that no bent strip construction is possible, so that this example is characteristic for the Quadrant-and-Strip case. It is believed that the example in Figure 7 is also characteristic, but the proof that it cannot tile a bent strip has not been worked out in sufficient detail.

(d) *The Half-Strip.* In Figure 8, we see a Half-Strip composed of one U-pentomino and an infinite repetition of fork-hexominoes. The inability of this set to do a rectangle is obvious, since it cannot possibly turn a third corner.

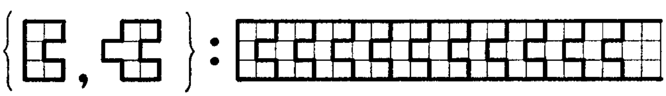

FIGURE 8. A characteristic example for the Half-Strip.

In [1], a certain hexomino was shown to tile the width-16 half-strip, without any result as to whether it can tile any rectangle. An even better example is the heptomino in Figure 9, which tiles the infinite width-6 half-strip in only one way, and thus cannot tile any 6-wide rectangle! Still, it has not been shown that this heptomino (or any other polyomino known to tile a half-strip) cannot tile *any* rectangle.

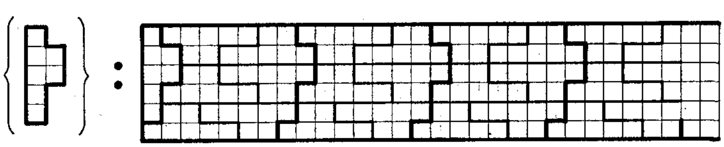

FIGURE 9. A half-strip tiling with one heptomino shape.

4. Wang's Problem and MacMahon Squares

In [3], Major MacMahon explored the rectangles and other patterns which could be made with squares having colors assigned to the four edges, with the requirement that adjacent squares must have a common color on their common border. Pictorially, MacMahon cut the square into four triangles by its diagonals, and colored these triangles rather than merely the edges. Thus, in Figure 10, we see an arrangement of the 24 tricolor MacMahon squares (distinct under the symmetries of D_4) arranged into a 4×6 rectangle with a white border.

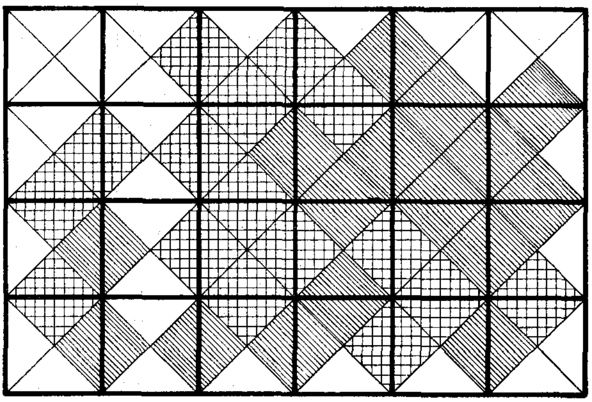

FIGURE 10. The 24 distinct tricolor MacMahon squares in a 4×6 rectangle.

H. Wang considered the problem of filling the infinite plane with replicas of a finite set of MacMahon squares with the requirement that adjacent edges have like color, and where only translation (not rotation or reflection) is allowed in using a replica of a permitted square. Wang showed [2] that the general question of whether a specified set of squares can tile the plane is algorithmically undecidable. We will now show that the question of whether an arbitrary finite set of polyominoes tiles the plane (allowing any specified group of symmetries such as rotational or dihedral symmetries, in addition to translations) is equivalent to Wang's problem, and hence also algorithmically undecidable.

The finite collections of polyominoes form a denumerable set, and can easily be placed in explicit one-to-one correspondence with the positive integers. Moreover, it is possible to give a rule whereby each cell in every distinct orientation of each polyomino in the nth set of polyominoes is assigned a separate positive integer r from 1 to R_n, where R_n is the number of "cell types" in the nth set. The reader is encouraged to formulate specific rules for these assignments, and to list the first ten or fifteen sets with their cell numberings.

The statement that the general problem of tiling the plane with sets of polyominoes is *algorithmically undecidable* means that there is no *computable function* $f(n, a, b)$ whose value is:

(i) 0 for all integer pairs (a, b) if the n-th set of polyominoes cannot be used to tile the plane;

(ii) the "cell type number" r to be used at position (a, b) of the plane, in a specific tiling of the plane with the n-th set of polyominoes.

First, suppose we are given a set of polyominoes. We then generate all the orientations (symmetries) of these figures which we intend to allow in tiling the plane. We then turn all the squares in this enlarged set of polyominoes into MacMahon squares, as follows: All outer edges of the polyominoes receive the color "0." Each interior edge in the entire set receives a unique "color," 1, 2, 3, 4..., which is the color of that edge in each of the two squares it connects. We now disconnect the polyominoes into unrotatable MacMahon squares. Note that, if we ever try to use a square, we are forced by the coloring to complete the polyomino it came from! Thus, the problem of tiling with sets of polyominoes has been mapped isomorphically *into* the problem of tiling with sets of (unrotatable) MacMahon squares.

The generation of MacMahon squares from polyominoes is indicated in Figure 11.

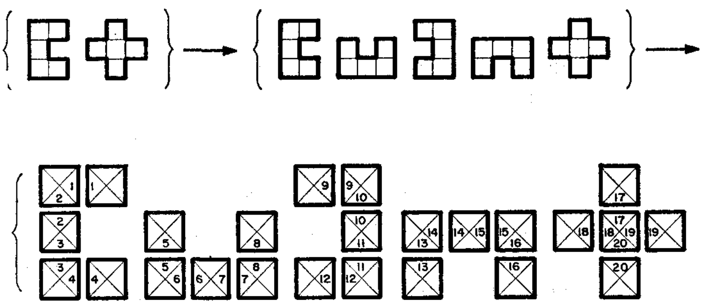

FIGURE 11. From polyominoes to MacMahon squares.

In the other direction, suppose we are given a set of N MacMahon squares, involving a total of m edge colors. Then we fabricate N polyominoes by the following modification of $(r + 6) \times (r + 6)$ squares, where $r = 1 + [\log_2 m]$. In Figure 12, we see the format for the large polyominoes. The grooves at the corners are designed to prevent rotations

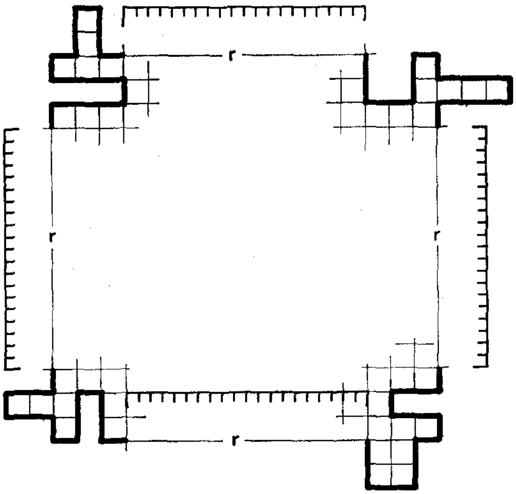

FIGURE 12. Format for the large polyominoes corresponding to MacMahon squares.

and reflections. That is, as soon as one of the large polyominoes is placed, it forces the orientations of all the others to line up. Each edge color of the MacMahon squares is represented by a binary number $< 2^r$, and the binary digits of this number are used to modify the portion of length r along the corresponding edge of the large polyomino. A 0-digit leaves the edge alone, while a 1-digit makes a one-square modification, outward along the top or right, inward along the left or bottom. Thus, the sets of MacMahon squares are mapped isomorphically *into* sets of polyominoes, insofar as tiling the plane is concerned.

An example of the conversion of MacMahon squares into polyominoes is shown in Figure 13.

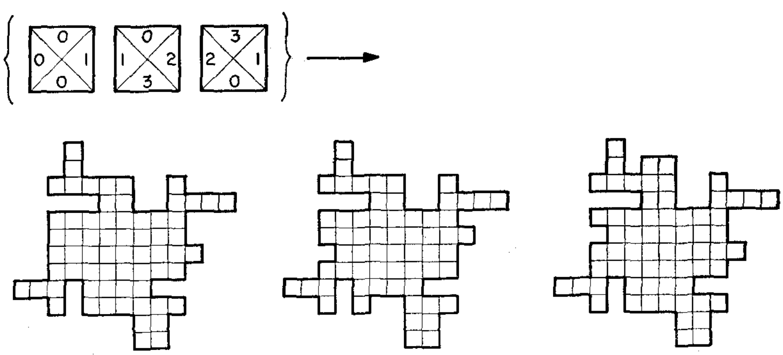

FIGURE 13. From MacMahon squares to polyominoes.

5. Tiling with Polyiamonds

Some analogous problems for tiling with figures made up of equal equilateral triangles (called "polyiamonds") have been considered by Nobuichi Ikeno of the Electrical Communication Laboratory, Nippon Telegraph and Telephone Corporation (private communication), who obtained the tiling hierarchy shown in Figure 14.

Going as far as the heptiamonds, Ikeno found individual shapes which furnish characteristic examples for every level in the hierarchy of Figure 14, with only the following four exceptions: "itself," "sextant," "one-third plane," and "half plane."

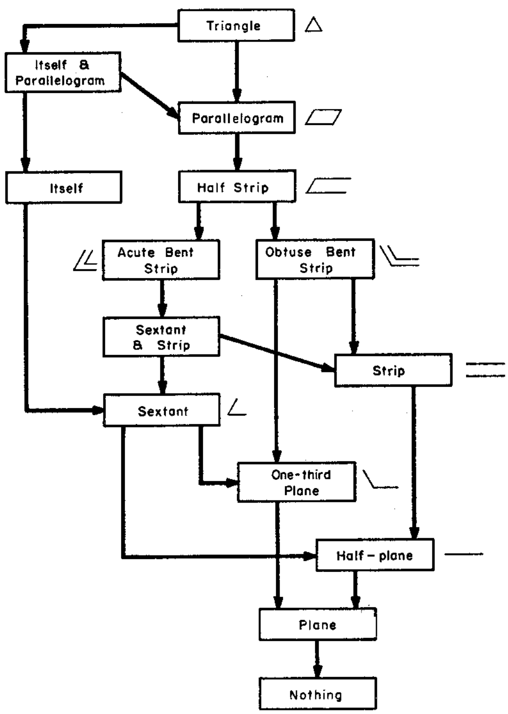

FIGURE 14. Ikeno's tiling hierarchy for polyiamonds.

6. The Regular Rep-Tile Property

A collection of shapes may be said to have the *regular rep-tile property* with index k if the replication of each shape requires exactly k copies of each distinct shape. An example with index 1, involving four distinct

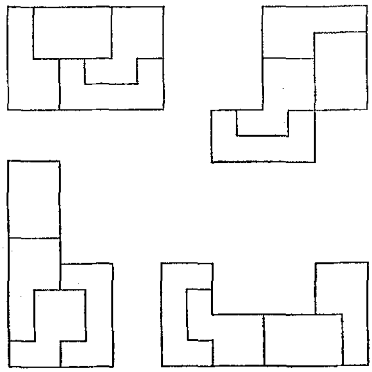

FIGURE 15. A regular rep-tile set of index 1.

hexomino shapes, is shown in Figure 15. This example appears in [4], where it is attributed to M. J. Povah of Blackburn, England. Two other examples of regular rep-tile sets of index 1 are also given in [4], one involving four octominoes and the other involving four "tetrabolos" (a *tetrabolo* being a figure composed of four isosceles right triangles).

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